Ann. Inst. Statist. Math.
Vol. 42, No. 1, 65–75 (1990)

MORE COMPARISONS OF MLE WITH UMVUE FOR EXPONENTIAL FAMILIES

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(Received December 28, 1987; revised February 6, 1989)

Abstract. Under some regularity conditions, the asymptotic expected deficiency (AED) of the maximum likelihood estimator (MLE) relative to the uniformly minimum variance unbiased estimator (UMVUE) for a given one-parameter estimable function of an exponential family is obtained. The exact expressions of the AED for normal, lognormal, inverse Gaussian, exponential (or gamma), Pareto, hyperbolic secant, Bernoulli, Poisson and geometric (or negative binomial) distributions are also derived.

Key words and phrases: Asymptotic expected deficiency, exponential family, MLE, UMVUE.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of the one-parameter exponential type having the probability density function

$$(1.1) \qquad f(x,\theta) = \exp\{\Phi_1(\theta)T(x) + \Phi_2(\theta) + d(x)\}, \qquad x \in S, \quad \theta \in \Omega$$

with respect to a fixed σ-finite measure μ (either Lebesgue or counting measure), where S is a subset of real numbers and Ω is a parameter space.

The maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE), which are considered the two most important estimators for $g(\theta)$, an estimable function of θ, are equivalent in terms of the asymptotic relative efficiency (ARE) under some regularity conditions. In order to discover which one is better, Rao (1961, 1962 and 1963) introduced several concepts of second-order efficiency, and Hodges and Lehmann (1970) gave the deficiency. In this article, we shall use the latter because of its great convenience.

Let $T_1(X_1, X_2, \ldots, X_n)$ and $T_2(X_1, X_2, \ldots, X_n)$ be two estimators of $g(\theta)$, and the measure of performance of estimators T_i is taken as the expected squared error, denoted by $V_n(T_i)$, $i = 1, 2$. We may assume that $\{V_n(T_i)\}$ are strictly monotonic decreasing sequences in the sample size n, $i = 1, 2$, and $V_n(T_i) \to 0$ as $n \to \infty$ for each i. In the problem with which we shall be concerned, $V_n(T_i)$ will typically be of the form

$$V_n(T_i) = an^{-r} + b_i n^{-(r+s)} + o(n^{-(r+s)}) \tag{1.2}$$

$i = 1, 2$, with $r > 0$.

As discussed in Hodges and Lehmann ((1970), p. 784), there exists an unique k_n such that $V_{k_n}(T_2) = V_n(T_1)$ and

$$k_n / n \to 1 \tag{1.3}$$

as $n \to +\infty$. Therefore, they defined the asymptotic expected deficiency (AED) of T_2 relative to T_1 as follows:

$$\text{AED}\,[T_2, T_1] = \lim_{n \to \infty} (k_n - n) \tag{1.4}$$

provided that the limit exists. Furthermore, they proved

$$\text{AED}\,[T_2, T_1] = \begin{cases} (b_2 - b_1)/ra & \text{if} \quad s = 1 \\ \infty & \text{if} \quad 0 < s < 1 \\ 0 & \text{if} \quad s > 1\,. \end{cases} \tag{1.5}$$

In Section 2, a simple formula for the AED of MLE for $g(\theta)$ relative to the UMVUE will be derived for the one-parameter exponential family (1.1) under some regularity conditions. We also slightly extend Theorem 3.1 of Morris (1983) (see, Lemma 2.3 below). Section 3 gives the exact expressions of the AED corresponding to normal, lognormal, inverse Gaussian, exponential (or gamma), Pareto, hyperbolic secant, Bernoulli, Poisson and geometric (or negative binomial) distributions.

2. Main result

In order to derive the main result, we put more restrictions on $f(x, \theta)$ defined as in (1.1) as follows:

(1) Ω is either the real line, or an interval on the real line.

(2) S, the set of positivity of $f(x, \theta)$, is independent of θ.

(3) $T(x)$ is not a constant function almost everywhere with respect to the fixed σ-finite measure μ.

It is well-known that the complete sufficient statistic for θ is given by:

$$(2.1) \qquad Z = \sum_{i=1}^{n} T(X_i)/n$$

and the probability density function of Z is of the form:

$$(2.2) \qquad h(z, \theta) = \exp\{n\Phi_1(\theta)z + n\Phi_2(\theta) + d_n^*(z)\}$$

with respect to the fixed σ-finite measure μ, where $d_n^*(z)$ depends on z and n.

Let $g(\theta)$ be a given estimable function of θ. If an UMVUE for $g(\theta)$ exists, then it must be a function of Z and the sample size n, let it be $U_n(Z)$. From the application point of view, usually $U_n(Z) \to g(\theta)$ a.e. as $n \to \infty$, for example Church and Harris (1970), Downton (1973) and Chao (1981), etc.; furthermore, each member of the function $U_n(z)$ has derivatives of all orders. Therefore, in this article, we shall assume that $U_n^{(i)}(Z) = O(1)$ almost everywhere with respect to the measure μ, for all $i = 1, 2, \ldots$. For convenience, we drop the subscript n and denote $U(Z)$ to be the UMVUE for $g(\theta)$.

We assume that the following regularity conditions hold throughout the rest of this article unless specified:

(1) z is an interior point of Ω.

(2) $\theta\Phi_1'(\theta) + \Phi_2'(\theta) = 0$ and $\Phi_1'(\theta) > 0$ for all $\theta \in \Omega$.

(3) The density $h(z, \theta)$ can be differentiated with respect to θ under the integral with respect to z any number of times.

(4) The functions $U(z)$ and $g(\theta)$ admit a convergent Taylor series for all interior points of Ω.

Note that the approaches presented by Morris's (1982, 1983) (concentrated in natural exponential families and quadratic variance function) will certainly be able to unify some statistical inferences such as the unbiased estimation, Bhattacharyya and Cramér-Rao lower bounds, as well as the calculation of the AED of MLE for $g(\theta)$ relative to UMVUE, but with very unpleasant expression for some cases. In order to obtain a simple and concise formulation of the AED as presented in (2.3), we really need the regularity condition (2); that is a key condition in this paper. It is also noted that the regularity condition (2) gives $E(T(X)) = \theta$ and $\mathrm{Var}(T(X)) = [\Phi_1'(\theta)]^{-1}$ where X is a random sample from (1.1).

The result derived in this paper can be used to obtain more AED than that found in Morris's (1982, 1983); for example, our paper can obtain the AED of some distributions with cubic variance function which cannot be obtained by Morris's papers. We consider inverse Gaussian distribution in detail in Section 3.

The regularity condition (3) implicitly assumes that the functions $\Phi_i(\theta)$, $i = 1, 2$, are analytic on the interior points of Ω. And the estimable function $g(\theta)$ in the condition (4) is actually analytic when the probability density functions are the natural exponential family (NEF) with the natural parameter θ; for more detail, see Abbey and David (1970) or Theorem 9 of Chapter 2 in Lehmann (1986).

The regularity condition (2) can be used to prove that the log likelihood function L is strongly unimodal, and that its MLE for θ is unique; the proof for the uniqueness of the MLE can be found in Barndorff-Nielsen (1978) for an exponential family. Now, the invariant property of the MLE (Zehna (1966)) implies that $g(Z)$ is the MLE for $g(\theta)$ if Z is the MLE of θ. We have the following result.

THEOREM 2.1. *Under the regularity conditions* (1)–(4), *the AED of the MLE $g(Z)$ for $g(\theta)$ relative to the UMVUE $U(Z)$ for the exponential family* (1.1) *is given by*

$$\text{(2.3)} \qquad \text{AED}\,[g(Z), U(Z)] = V(\theta)\left\{g'''(\theta)/g'(\theta) + \frac{1}{4}[g''(\theta)/g'(\theta)]^2\right\} + V'(\theta)g''(\theta)/g'(\theta)$$

where $V(\theta) = [\Phi_1'(\theta)]^{-1}$.

In order to prove this theorem, we need some lemmas.

LEMMA 2.1. *Under the regularity conditions* (2) *and* (3), *we have*

$$\text{(2.4)} \qquad \partial\mu_i/\partial\theta = -i\mu_{i-1} + n\Phi_1'(\theta)\mu_{i+1}$$

$\mu_0 = 1$ *and* $\mu_1 = 0$, $i = 1, 2, 3, \ldots$ *where* $\mu_i = E(Z-\theta)^i$ *are the central moments of* Z.

The proof of Lemma 2.1 is easy and omitted here; it gives the recursive relationship among the central moments of Z. For example, we have

$$\text{(2.5)} \qquad \mu_2 = [n\Phi_1'(\theta)]^{-1}, \quad \mu_3 = -\Phi_1''(\theta)[\Phi_1'(\theta)]^{-3}\cdot n^{-2},$$
$$\mu_4 = 3[n\Phi_1'(\theta)]^{-2} + \{-\Phi_1'''(\theta)/\Phi_1'(\theta) + 3[\Phi_1''(\theta)/\Phi_1'(\theta)]^2\}\cdot[n\Phi_1'(\theta)]^{-3}$$

and in general

$$\mu_{2i-1} = O(n^{-i}) \quad \text{and} \quad \mu_{2i} = O(n^{-i}), \quad i = 1, 2, \ldots .$$

Now, under the regularity conditions (1) and (4), the MLE $g(Z)$ has a convergent Taylor expansion,

(2.6)
$$g(Z) = g(\theta) + \sum_{i=1}^{+\infty} g^{(i)}(\theta)(Z-\theta)^i/i!$$

and combining Lemma 2.1 and the following relation

$$\text{MSE}\,[g(Z)] = \text{Var}\,[g(Z)] + \frac{1}{4}[g''(\theta)]^2[n\Phi_1'(\theta)]^{-2} + O(n^{-3}),$$

we have

LEMMA 2.2. *Under the regularity conditions* (1)–(4), *the mean squared error of the MLE* $g(Z)$ *for* $g(\theta)$ *is given by*

(2.7)
$$\begin{aligned}\text{MSE}\,[g(Z)] = {} & [g'(\theta)]^2[n\Phi_1'(\theta)]^{-1} \\ & + \{g'(\theta)g'''(\theta) + \frac{3}{4}[g''(\theta)]^2 - g'(\theta)g''(\theta)\Phi_1''(\theta)/\Phi_1'(\theta)\} \\ & \cdot [n\Phi_1'(\theta)]^{-2} + O(n^{-3}).\end{aligned}$$

Next we find the variance of the UMVUE for $g(\theta)$ as follows

LEMMA 2.3. *Under the regularity conditions* (1)–(4), *the mean squared error* (*or variance*) *of the UMVUE* $U(Z)$ *for* $g(\theta)$ *is given by*

(2.8)
$$\begin{aligned}\text{MSE}\,[U(Z)] = {} & [g'(\theta)]^2[n\Phi_1'(\theta)]^{-1} \\ & + \frac{1}{2}[g''(\theta)]^2[n\Phi_1'(\theta)]^{-2} + O(n^{-3}).\end{aligned}$$

PROOF. Using the conditions (1)–(4), we obtain

$$U(Z) = U(\theta) + \sum_{i=1}^{\infty} U^{(i)}(\theta)(Z-\theta)^i/i!$$

and the unbiasedness of $U(Z)$ yields

$$g(\theta) = U(\theta) + \sum_{i=1}^{\infty} U^{(i)}(\theta)\mu_i/i!$$

for all $\theta \in \Omega$, and the conditions (2), (3) and Lemma 2.1 give

$$\begin{aligned}g'(\theta) = {} & U'(\theta) + \sum_{i=1}^{\infty} U^{(i+1)}(\theta)\mu_i/i! \\ & + \sum_{i=1}^{\infty} U^{(i)}(\theta)[-i\mu_{i-1} + n\Phi_1'(\theta)\mu_{i+1}]/i!\end{aligned}$$

$$= n\Phi_1'(\theta)\sum_{i=1}^{\infty} U^{(i)}(\theta)\mu_{i+1}/i!$$

for all interior points θ of Ω. Note that $U(Z)$ depends on Z and the sample size n. Since the regularity conditions give $U^{(i)}(Z) = O(1)$ for all $i = 1, 2, \ldots$, we have the following relations:

$$E[U(Z) - U(\theta)]^2 = [U'(\theta)]^2\mu_2 + U'(\theta)U''(\theta)\mu_3$$
$$+\left\{\frac{1}{4}[U''(\theta)]^2 + \frac{1}{3}U'(\theta)U'''(\theta)\right\}\mu_4 + O(n^{-3}),$$

$$[U(\theta) - g(\theta)]^2 = \frac{1}{4}[U''(\theta)]^2\mu_2^2 + O(n^{-3}),$$

$$U'(\theta) = g'(\theta) + \{U''(\theta)\Phi_1''(\theta)[\Phi_1'(\theta)]^{-1} - U'''(\theta)\}$$
$$\cdot[2n\Phi_1'(\theta)]^{-1} + O(n^{-2}),$$

$$U''(\theta) = g''(\theta) + O(n^{-1}), \qquad U'''(\theta) = g'''(\theta) + O(n^{-1}).$$

Finally, the identity

$$\mathrm{MSE}\,[U(Z)] = E[U(Z) - U(\theta)]^2 + [U(\theta) - g(\theta)]^2$$

gives the desired result and the proof is complete.

Note that Lemma 2.3 can be used for the quadratic variance function (QVF) case, which Morris ((1983), formula (3.7)) considered for normal, exponential (or gamma), hyperbolic secant, binomial, Poisson and negative binomial distributions.

PROOF OF THEOREM 2.1. Lemmas 2.2, 2.3 and Hodges and Lehmann's (1970) result (1.2) and (1.5) with $r = s = 1$ yield the desired result.

The fact pointed out in the introduction can be obtained as follows.

COROLLARY 2.1. *Under the regularity conditions* (1)–(4), *the asymptotic relative efficiency of the MLE for* $g(\theta)$ *relative to the UMVUE is equal to one.*

Finally, we note that the regularity conditions (3) and (4) may be slightly weakened; for example, the Taylor expansion of the remainder term may be such that the UMVUE has up to a term smaller order than $O(n^{-2})$, but in some sense Lemma 2.3 gives a sufficient extension of

Theorem 3.1 of Morris (1983) for large sample size n.

3. The AED for some distributions

Abbey and David (1970) considered the Koopman-Darmois class of exponential densities and developed a method for obtaining the UMVUE of $g(\theta)$ without explicit knowledge of any unbiased estimator of $g(\theta)$. Morris (1982) gives a detailed insight into the natural exponential families with quadratic variance functions (NEF-QVF), and defined the family of elementary distributions as having unit magnitude ± 1 in the leading coefficient of the variance function, namely: the normal $N(\theta, 1)$, exponential, hyperbolic secant (HS), Bernoulli, geometric and Poisson distributions. We note that these six elementary distributions satisfy the second regularity condition given in Section 2. Therefore, we can easily obtain the AED for these six elementary distributions from Theorem 2.1. Note that Theorem 2.1 is true without restriction on quadratic variance functions.

Furthermore, it can be shown that normal $N(0, \theta)$, gamma, lognormal, inverse Gaussian and the one-parameter Pareto distribution also satisfy the second regularity condition. Baxter (1980) considered the transformation $Y = \log(X/k)$ and found that the one-parameter Pareto distribution is equivalent to the one-parameter exponential distribution; hence the Pareto and exponential distribution have the same AED value. It is also noted that $N(0, \theta)$, $\Gamma(1/2, \theta)$ and $\log N(0, \theta)$ have the same AED value.

The exponent of the Pareto, lognormal distributions are non-linear in the random variable. In contrast to the lognormal distribution, we may term the Pareto distribution a "log-exponential" distribution. The same argument can be applied similarly to the non-linear transformations of the exponential family in which the exponent is linear in the random variable (Morris (1982)). Finally, we note that the random walk distribution (Johnson and Kotz (1970)) has the same cubic variance function as that of the inverse Gaussian distribution.

The AED of the MLE for $g(\theta)$ relative to the UMVUE corresponding to each distribution discussed above will be presented as in the end of this section. The table presented there could be used to assess the AED of the MLE relative to the UMVUE which are used to estimate some functions of unknown parameter such as probability functions, hazard functions and reliability functions (see, for example, Church and Harris (1970), Downton (1973), Kelley *et al.* (1976), Tong (1977) and Chao (1981, 1982)), etc. In this paper, we only give an example to address the issue of positivity of the AED as follows.

We consider the application of the result to the problem of estimating $R = P(Y < X)$ which has been extensively studied in reliability and its related fields by Church and Harris (1970), Downton (1973), Kelley *et al.* (1976), Tong (1977) and Chao (1981, 1982). This problem is of importance in the following physical situation. Suppose that X is the strength of a component which is subjected to a stress Y. The component fails when and only when $X \leq Y$. In the usual case, we assume that the distribution of stresses Y is known (Church and Harris (1970)), hence we consider the following two cases:

Case 1. (Both X and Y are normally distributed) We assume that X is distributed as a normal distribution with unknown mean θ and known variance and that Y has a standard normal distribution. Then, Church and Harris (1970) and Downton (1973) gave

$$R = \Phi(\theta/\sqrt{2}) \tag{3.1}$$

where $\Phi(\cdot)$ is the standard normal distribution. The UMVUE $\hat{R}_U$ and MLE $\hat{R}_M$ for R are, respectively, as follows:

$$\hat{R}_U = \Phi(\sqrt{n}\,\bar{X}/\sqrt{2n-1}) \quad \text{and} \quad \hat{R}_M = \Phi(\bar{X}/\sqrt{2}) \tag{3.2}$$

where $\bar{X} = (X_1 + \cdots + X_n)/n$ and $X_1, X_2, \ldots, X_n$ are random samples of size n. Note that

$$\hat{R}_U > \hat{R}_M \tag{3.3}$$

if and only if $\bar{x} > 0$ for all sample size n. Now, using Table 1 with $g(\theta) = \Phi(\theta/\sqrt{2})$, we get

$$\text{AED}\,[\hat{R}_M, \hat{R}_U] = (5\theta^2 - 8)/16 \tag{3.4}$$

and conclude that:

(1) For $|\theta| < (8/5)^{1/2}$ ($\doteq 1.26$), it is approximately equivalent to $0.19 < R < 0.81$, the MLE is better than the UMVUE and the minimal AED in this range is equal to -0.5, when $\theta = 0$ ($R = 0.5$). Therefore, the MLE can only save a maximal 0.5 observations.

(2) For $|\theta| > (8/5)^{1/2}$ ($R > 0.81$ or $R < 0.19$), the UMVUE is superior to the MLE. However, since in many applications R should be nearly unity, the UMVUE is always superior to the MLE.

Table 2 shows the values of AED corresponding to some specific values of θ (or R).

Case 2. (Both X and Y are exponentially distributed) We assume that X is distributed as an exponential distribution with unknown parameter θ, and that Y has an exponential distribution with known parameter 1. These assumptions are adopted by Kelley *et al.* (1976) and Chao (1982). They gave

Table 1. The AED of MLE for $g(\theta)$ relative to UMVUE.

Distribution		$f(x,\theta)$	AED
Bernoulli		$\theta^x(1-\theta)^{1-x}$	$(1-2\theta)g_1(\theta)+\theta(1-\theta)g_2(\theta)$*
Geometric		$\theta^{-1}(1-\theta^{-1})^{x-1}$	$(2\theta-1)g_1(\theta)+\theta(\theta-1)g_2(\theta)$
Poisson		$e^{-\theta}\theta^x/x!$	$g_1(\theta)+\theta g_2(\theta)$
Normal	$N(\theta,1)$	$(2\Pi)^{-1/2}\exp\{-(x-\theta)^2/2\}$	$g_2(\theta)$
	$N(0,\theta)$	$(2\Pi\theta)^{-1/2}\exp\{-x^2/2\theta\}$	$4\theta g_1(\theta)+2\theta^2 g_2(\theta)$
gamma		$[\Gamma(\alpha)]^{-1}\left(\frac{\alpha}{\theta}\right)^{\alpha}x^{\alpha-1}\exp\{-\alpha x/\theta\}$	$2\theta g_1(\theta)/\alpha+\theta^2 g_2(\theta)/\alpha$
exponential		$\theta^{-1}\exp\{-x/\theta\}$	$2\theta g_1(\theta)+\theta^2 g_2(\theta)$
hyperbolic secant		$\frac{\exp\{\tan^{-1}(\theta)\cdot x\}}{2\cosh(\Pi x/2)(1+\theta^2)^{1/2}}$	$2\theta g_1(\theta)+(1+\theta^2)g_2(\theta)$
lognormal	$LN(\theta,1)$	$(x\sqrt{2\Pi})\exp\{-(\log x-\theta)^2/2\}$	$g_2(\theta)$
	$LN(0,\theta)$	$(x\sqrt{2\Pi\theta})^{-1}\exp\{-(\log x)^2/2\theta\}$	$4\theta g_1(\theta)+2\theta^2 g_2(\theta)$
inverse Gaussion	$IG(\theta,1)$	$(2\Pi x^3)^{-1/2}\exp\left\{-\frac{(x-\theta)^2}{2\theta^2 x}\right\}$	$3\theta^2 g_1(\theta)+\theta^3 g_2(\theta)$
	$IG(1,\theta)$	$(2\Pi\theta x^3)^{-1/2}\exp\{-(x-1)^2/2x\}$	$4\theta g_1(\theta)+2\theta^2 g_2(\theta)$

*$g_1=g''/g'$, $g_2=g_1^2/4+g'''/g'$.

Table 2.

R:	0.900	0.950	0.975	0.990	0.999	1.000
θ:	1.810	2.330	2.770	3.300	4.384	∞
AED:	0.520	1.200	1.900	2.900	5.510	∞

(3.5) $$R=\theta/(1+\theta),\qquad \hat{R}_M=\bar{X}/(1+\bar{X})$$

and

$$\hat{R}_U=\frac{(n-1)!(-1)^{n-1}}{(n\bar{X})^{n-1}}\left[\sum_{i=0}^{n-1}\frac{(-1)^i(n\bar{X})^i}{i!}-e^{-n\bar{X}}\right],$$

where $\bar{X}$ is the sample mean, $\hat{R}_M$ and $\hat{R}_U$ are the MLE and UMVUE for R, respectively. Table 1 with $g(\theta)=\theta/(1+\theta)$ gives

(3.6) $$\text{AED}\,[\hat{R}_M,\hat{R}_U]=-4R+7R^2,$$

where $0<R<1$, and we conclude that: For $0<R<4/7$, the MLE is better than the UMVUE and the minimal AED in this range is equal to $-4/7$, when $R=2/7$. Therefore, the MLE can save a maximal 0.57 observations. For $R>4/7$, the UMVUE is superior to the MLE and the maximal AED is approximately equal to 3, when R is nearly one. Similarly, R should be close to unity in the practical situation. Therefore, the UMVUE is always superior to the MLE in this case. This conclusion exactly coincides with the work given by Chao (1982). She provides a simple and satisfactory approximation formulas for s-bias and MSE of the MLE $\hat{R}_M$, even when the sample size n is small. Finally, we note that the sample size must be integral. The main reason that the UMVUE is better than the MLE in the example is that it is never worse, in the sense that the MLE nowhere achieves (asymptotically) an MSE no larger than that of the UMVUE with smaller sample size.

Acknowledgements

The authors would like to thank the referees for very careful reading of the original version, which resulted in substantial improvements.

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