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ON CHARACTERIZATION OF POWER SERIES DISTRIBUTIONS BY A MARGINAL DISTRIBUTION AND A REGRESSION FUNCTION

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Abstract. The conditional distribution of Y given $X = x$, where X and Y are non-negative integer-valued random variables, is characterized in terms of the regression function of X on Y and the marginal distribution of X which is assumed to be of a power series form. Characterizations are given for a binomial conditional distribution when X follows a Poisson, binomial or negative binomial, for a hypergeometric conditional distribution when X is binomial and for a negative hypergeometric conditional distribution when X follows a negative binomial.

Key words and phrases: Characterizations, regression function, Poisson, binomial, negative binomial, hypergeometric.

1. Introduction

Korwar (1975) considered two types of characterization problems for distributions of non-negative integer-valued random variables (r.v.) X and Y, when they follow a Poisson, binomial or negative binomial distribution. He derived characterizations of the marginal distribution of X by the conditional distribution of Y given X, and the linear regression of X on Y and characterizations of the conditional distribution of Y given X by the marginal distribution of X and the linear regression of X on Y.

Extensions and generalizations of Korwar's characterizations by conditional distributions and regression functions were considered by, among others, Dahiya and Korwar (1977), Khatri (1978*a*, 1978*b*), Cacoullos and Papageorgiou (1983, 1984), Papageorgiou (1983, 1984 and 1985) and Kyriakoussis (1988).

In this paper we are concerned with characterizing the conditional distribution of Y given $X = x$ by the marginal distribution of X and the regression function $E(X \mid Y)$ of X on Y, when X has a power series distribution with parameter θ and $E(X \mid Y)$ is of a power series form with variable θ. Korwar's characterizations of a conditional binomial when X is a Poisson, binomial or negative binomial, as well as additional characterizations of a conditional hypergeometric or negative hypergeometric, are derived as illustrative examples.

2. The main result

Denote the marginal probability functions of the r.v.'s X and Y by $p(x) = P(X = x)$ and $f(y) = P(Y = y)$, respectively. Also denote the conditional probability function of Y given $X = x$ by $p(y|x)$ and that of X given $Y = y$ by $p(x|y)$.

THEOREM 2.1. *Let X be distributed according to the Power Series Distribution*

$$p(x) = \frac{\alpha(x)\theta^x}{\sum_x \alpha(x)\theta^x}, \qquad x = 0, 1, 2, \ldots .$$

Then if the regression function of X on Y is of the form

$$E(X \mid Y = y) = m(y; \theta) = \sum_{j=0}^{r} b_j(y)\theta^j ,$$

for all θ, with $b_0(y) = y$, $y = 0, 1, 2, \ldots$ and r a positive integer or infinity, the conditional distribution of $Y \mid X$ can be determined uniquely.

PROOF. We have

$$m(y; \theta) = \sum_x x p(x|y) = \sum_x \frac{x p(y|x) p(x)}{f(y)} .$$

Hence

$$m(y; \theta) \sum_x p(y|x) p(x) = \sum_x x p(y|x) p(x) ,$$

or from the theorem's assumptions

$$m(y; \theta) \sum_x p(y|x)\alpha(x)\theta^x = \sum_x x p(y|x)\alpha(x)\theta^x . \tag{2.1}$$

Writing

$$\sum_x p(y|x)\alpha(x)\theta^x = g(y;\theta)\,, \tag{2.2}$$

equation (2.1) becomes

$$\frac{g'(y;\theta)}{g(y;\theta)} = \frac{m(y;\theta)}{\theta}\,,$$

where $g'(y;\theta) = (\partial/\partial\theta)g(y;\theta)$. Consequently,

$$g(y;\theta) = c(y)\exp\left\{\int \frac{m(y;\theta)}{\theta}\,d\theta\right\},$$

where $c(y)$ is only a function of y. Hence

$$g(y;\theta) = c(y)\theta^y \exp\left\{\sum_{j=1}^{r} b_j(y)\theta^j/j\right\}. \tag{2.3}$$

However, from the exponential Bell polynomials, $B_n = B_n(b_1, b_2, \ldots, b_n)$, which may be defined by their exponential generating function as

$$\sum_{n=0}^{\infty} B_n\theta^n/n! = \exp\left[\phi(\theta) - \phi(0)\right],$$

where

$$\phi(\theta) = \sum_{j=0}^{\infty} b_j\theta^j/j!, \qquad B_0 = 1\,, \tag{2.4}$$

equation (2.3) becomes

$$g(y;\theta) = c(y)\theta^y \sum_{n=0}^{\infty} B_n(y)\theta^n/n!\,,$$

where

$$B_n(y) = B_n(b_1(y), 1!b_2(y), \ldots, (\mathrm{n}-1)!b_n(y)), \qquad n = 1, 2, \ldots, \quad B_0(y) = 1\,.$$

Explicit expressions for $B_n = B_n(b_1, b_2, \ldots, b_n)$, as functions of $b_1, b_2, \ldots, b_n$, are given in Kendall and Stuart ((1969), p. 69) and David and Barton ((1962), p. 42) where the role of the cumulants k_i is played here by the b_i as defined in (2.4) and the Bell polynomials themselves are the moments μ_i'. From equation (2.2), we have

$$\sum_{x=0}^{\infty} p(y|x)\alpha(x)\theta^x = c(y) \sum_{x=y}^{\infty} B_{x-y}(y)\theta^x/(x-y)!\,.$$

Equating coefficients of θ^x, we obtain

$$\begin{aligned} p(y|x) &= c(y)\,\frac{1}{\alpha(x)}\,\frac{B_{x-y}(y)}{(x-y)!}\,, \qquad y = 0, 1, 2, \ldots, x, \quad x = 0, 1, 2, \ldots\,, \\ p(y|x) &= 0 \qquad\qquad\qquad\qquad\qquad \text{for} \quad x < y\,. \end{aligned} \tag{2.5}$$

To determine the coefficients $c(y)$, note first that, from (2.5), for a given x, y ranges from zero to x, so that in turn

$$\sum_{i=0}^{x} p(i|x) = 1, \qquad x = 0, 1, 2, \ldots\,;$$

equivalently,

$$p(y|y) = 1 - \sum_{i=0}^{y-1} p(i|y)\,.$$

From the relation (2.5), we have

$$c(y) = \alpha(y) - \sum_{i=0}^{y-1} c(i)\,\frac{B_{y-i}(i)}{(y-i)!}\,, \qquad y = 1, 2, \ldots\,, \tag{2.6}$$

and

$$c(0) = \alpha(0)\,, \tag{2.7}$$

because $p(0|0) = 1$.

From (2.6) and (2.7) the remaining coefficients $c(1), c(2), \ldots$ can be determined uniquely.

Conversely, if the conditional distribution of $Y|X = x$ and the marginal distribution of X are known, then the joint distribution of (X, Y) is determined; therefore the marginal distribution of Y, the conditional distribution of $X \mid Y$ and the regression $E(X \mid Y)$ can also be determined.

Characterizations of some well-known distributions are given as illustrative examples when the r.v. X follows a Poisson, binomial or negative binomial distribution. For clarity these results are summarized in Table 1.

Table 1. Bell polynomials and conditional distributions.

Marginal Distribution $P(X=x)$	Regression $E(X \mid Y=y)=m(y;\theta)$	Exponential Bell Polynomials $B_n(y)$	Conditional Distribution $P(Y \mid X=x)$
$e^{-\theta}\theta^x/x!$	$y+\theta q$	q^n	$\binom{x}{y}p^y q^{x-y}$
$\theta>0$	for all $\theta>0$, $y=0,1,2,\ldots$ and some $0<q<1$	$n=0,1,2,\ldots$	$y=0,1,2,\ldots,x,\ p=1-q$
$\binom{v}{x}\alpha^x(1-\alpha)^{v-x}$	$(y+vq\theta)/(1+q\theta)$	$n!\binom{v-y}{n}q^n$	$\binom{x}{y}p^y q^{x-y}$
$0<\alpha<1,\ v=0,1,2,\ldots$	for all $\theta=\alpha/(1-\alpha)$, $y=0,1,2,\ldots$ and some $0<q<1-\alpha$	$n=0,1,2,\ldots,\ v-y$, $y\le v$	$y=0,1,2,\ldots,x,\ p=1-q$
$\binom{-v}{x}(-1)^x\alpha^v(1-\alpha)^x$	$(y+vq\theta)/(1-q\theta)$	$n!\binom{-y-v}{n}(-1)^n q^n$	$\binom{x}{y}p^y q^{x-y}$
$0<\alpha<1,\ v>0$	for all $\theta=1-\alpha$, $y=0,1,2,\ldots$ and some $0<q<1$	$n=0,1,2,\ldots$	$y=0,1,2,\ldots,x,\ p=1-q$
$\binom{N}{x}p^x(1-p)^{N-x}$	$y+(N-v)\theta/(\theta+1)$	$n!\binom{N-v}{n}$	$\binom{x}{y}\binom{N-x}{v-y}\Big/\binom{N}{v}$
$0<p<\frac{1}{2},\ N=0,1,\ldots$	for all $\theta=p/(1-p)$, $y=0,1,\ldots$ and some positive integer $v<N$	$n=0,1,2,\ldots,\ N-v$	$y=0,1,2,\ldots,\min\{x,v\}$
$\binom{-N}{x}(-1)^x p^N(1-p)^x$	$y+(N-v)\theta/(1-\theta)$	$n!\binom{-N+v}{n}(-1)^n$	$\binom{-v}{y}\binom{-N+v}{x-y}\Big/\binom{-N}{x}$
$0<p<1,\ N>0$	for all $\theta=1-p$, $y=0,1,2,\ldots$ and for some positive integer $v<N$	$n=0,1,2,\ldots$	$y=0,1,2,\ldots,x$

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