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ON THE RATE OF CONVERGENCE OF SPATIAL BIRTH-AND-DEATH PROCESSES

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Abstract. Sufficient conditions for geometrical fast convergence of general spatial birth-and-death processes to equilibrium are established.

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1. Introduction

Let S be an arbitrary set (e.g., a subset of $\mathbb{R}^d$). A spatial birth-and-death process on S is, loosely speaking, a continuous time Markov chain with states in the space of all finite point configurations in S, and so that a transition can only be a birth of a new point or a death of an existing point (for a formal and more general description and definition, see Preston (1977) and Section 2 in the present paper). The object of this paper is to study the rate of convergence to equilibrium for such processes.

Spatial birth-and-death processes are interesting for many reasons. Obviously, they might be used as models for many dynamic spatial phenomena (cf. Møller and Sørensen (1989)). Their relevance in spatial statistics lies in their close relationship to Gibbs processes (Preston (1977)) and their use in simulation of spatial point patterns (Kelly and Ripley (1976) and Ripley (1977)). Simulated realizations can be seen in Ripley (1977, 1981), Diggle (1983) and Baddeley and Møller (1989).

The present paper is organized as follows. Unique existence and convergence of spatial birth-and-death processes are discussed in Section 2. In Section 3 we give sufficient conditions for geometrical convergence of a general spatial birth-and-death process to equilibrium, to the best of my knowledge, similar results have been given only in the special case of a hard core birth-and-death process (see Lotwick and Silverman (1981)). The results in Sections 2 and 3 are related to well-known results for simple birth-and-death processes (i.e., when the process counts only the number of individuals alive).

2. Short diversion into spatial birth-and-death processes

In this section we briefly discuss some results for spatial birth-and-death processes. Most of the results can be found in Preston (1977), but they are included in the present section for the sake of completeness and since they provide a better understanding of the conditions and results in Section 3.

We start with a description of a spatial birth-and-death process. For $n = 0, 1, \dots$, suppose $(\Omega_n, \mathscr{F}_n)$ is a measure space such that the Ω_n's are disjoint sets and Ω_0 consists of a single element denoted 0. Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ and let $\mathscr{F}$ be the σ-algebra on Ω generated by $(\mathscr{F}_n)$, $n \geq 0$. Now, assume there is a continuous time homogeneous Markov chain $\{X(t): t \geq 0\}$ with state space Ω and such that if $X(t) = x \in \Omega_n$, then an immediate transition after time t can only be to a state in either Ω_{n+1} or Ω_{n-1}. Then $\{X(t): t \geq 0\}$ is called a spatial birth-and-death process. This process is characterized by two measurable functions $\beta, \delta: \Omega \to [0, \infty)$ with $\delta(0) = 0$, and probability kernels $K_\beta^{(n)}: \Omega_n \times \mathscr{F}_{n+1} \to [0, 1]$, $n \geq 0$, and $K_\delta^{(n)}: \Omega_n \times \mathscr{F}_{n-1} \to [0, 1]$, $n \geq 1$ as follows. Suppose $X(t) = x \in \Omega_n$ and let τ be the waiting time to the first transition after time t. Then, given that $X(t) = x$, τ is exponential distributed with mean $1/\alpha(x)$ (taking $1/0 = \infty$) where $\alpha(x) = \beta(x) + \delta(x)$, $X(t + \tau) \in \Omega_{n+1}$ with probability $\beta(x)/\alpha(x)$, and

$$K_\beta^{(n)}(x, F) = P(X(t+\tau) \in F \mid X(t) = x,\ X(t+\tau) \in \Omega_{n+1}) , \tag{2.1}$$

$$K_\delta^{(n)}(x, F) = P(X(t+\tau) \in F \mid X(t) = x,\ X(t+\tau) \in \Omega_{n-1}) . \tag{2.2}$$

The results in this paper do not depend on the choice of the probability kernels $K_\beta^{(n)}$ and $K_\delta^{(n)}$. The functions β and δ are called the birth rate and the death rate, respectively.

The above-introduced terminology might be illuminated by the following construction of $(\Omega, \mathscr{F})$ from a given measure space $(S, \mathscr{B})$. Let $\mathscr{B}_n = \mathscr{B} \otimes \cdots \otimes \mathscr{B}$ be the product σ-algebra on $S_n = S \times \cdots \times S$ (n times), Ω_n the set of all point configurations $\{x_1, \dots, x_n\}$ in S with n (not necessarily distinct) points, and $\mathscr{F}_n$ the σ-algebra induced by the mapping $\omega_n: S_n \to \Omega_n$ defined by $\omega_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$. Then transitions $\Omega_n \to \Omega_{n+1}$ or $\Omega_n \to \Omega_{n-1}$ correspond to either adding a new point (a "birth") or deleting an existing point (a "death"), respectively. Especially, if S consists of a single point, Ω_n may be identified with $\{n\}$, so the process $\{X(t): t \geq 0\}$ is a usual birth-and-death process having the non-negative integers as state space. We call such a process a simple birth-and-death process and denote the birth rate by $\beta_n = \beta(\{n\})$ and the death rate by $\delta(n) = \delta(\{n\})$. Simple birth-and-death processes have been studied by a number of authors (see, e.g., van Doorn (1981) and the references therein).

Suppose two measurable functions $\beta, \delta\colon \Omega \to [0, \infty)$ and probability kernels $K_\beta^{(n)}\colon \Omega_n \times \mathscr{F}_{n+1} \to [0, 1]$, $n \geq 0$, and $K_\delta^{(n)}\colon \Omega_n \times \mathscr{F}_{n-1} \to [0, 1]$, $n \geq 1$ are given. In order to ensure the unique existence of a spatial birth-and-death process $\{X(t)\colon t \geq 0\}$ with birth rate β, death rate δ, and $K_\beta^{(n)}$ and $K_\delta^{(n)}$ equal to the right hand side of (2.1) and (2.2), respectively, conditions on β and δ must be imposed. This is due to the possibility that an infinite number of transitions can occur in a finite time with a positive probability. In fact, the process exists uniquely if and only if Kolmogorov's backward equations for the process have a unique solution. It was Preston's idea to compare these backward equations with those of a simple birth-and-death process with rates

$$\beta_n = \sup_{x \in \Omega_n} \beta(x), \qquad \delta_n = \inf_{x \in \Omega_n} \delta(x) \qquad (\delta_0 = 0)\,, \tag{2.3}$$

assuming $\beta_n < \infty$ for all n. Using a coupling argument, Preston (1977) proved that if there exists such a simple process $\{x(t)\colon t \geq 0\}$ and this is unique, then the spatial birth-and-death process $\{X(t)\colon t \geq 0\}$ exists uniquely and for all $t \geq 0$ and $m = 0, 1, \ldots$ holds

$$\sum_{n=0}^{m} Q_t(x, \Omega_n) \geq \sum_{n=0}^{m} q_t(j, n)\,, \tag{2.4}$$

whenever $x \in \Omega_i, j \geq i \geq 0$, where

$$Q_t(x, F) = P(X(t) \in F \mid X(0) = x), \qquad F \in \mathscr{F}\,, \tag{2.5}$$

$$q_t(j, n) = P(x(t) = n \mid x(0) = j)\,, \tag{2.6}$$

(cf. Preston (1977), Proposition 6.1 and formulae (6.1)–(6.3)). Reuter and Ledermann (1953) (see also Karlin and McGregor (1957*a*)) give sufficient conditions for the unique existence of the simple process: there exists $n_0 \geq 1$ such that either

$$\beta_n = 0 \quad \text{for all} \quad n \geq n_0 \tag{2.7}$$

or

$$\beta_n > 0 \quad \text{for all} \quad n \geq n_0 \quad \text{and} \quad \sum_{n=n_0}^{\infty} w_n = \infty\,, \tag{2.8}$$

holds, where

$$w_n = \frac{1}{\beta_n} + \frac{\delta_n}{\beta_n \beta_{n-1}} + \cdots + \frac{\delta_n \cdots \delta_{n_0+1}}{\beta_n \cdots \beta_{n_0}} + \frac{\delta_n \cdots \delta_{n_0}}{\beta_n \cdots \beta_{n_0}} .$$

In the following it is therefore assumed that either (2.7) or (2.8) holds.

Now, let γ be an arbitrary initial distribution of the spatial birth-and-death process, so

$$Q_t(F) = \int Q_t(x, F)\gamma(dx), \qquad F \in \mathscr{F}$$

is the distribution of $X(t)$. The following theorem gives sufficient conditions under which $\lim_{t\to\infty} Q_t(x, \cdot)$ and hence $\lim_{t\to\infty} Q_t(\cdot)$ exists.

THEOREM 2.1. *Suppose that $\delta_n > 0$ for all $n \geq 1$ and that one of the following conditions holds:*

(2.9) *there exists $n_0 \geq 0$ such that $\beta_n = 0$, for all $n > n_0$.*

(2.10) *$\beta_n > 0$ for all $n \geq 1$ and we have*

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\beta_1 \cdots \beta_{n-1}}{\delta_1 \cdots \delta_n} < \infty$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\delta_1 \cdots \delta_n}{\beta_1 \cdots \beta_n} = \infty$.

Then for all $F \in \mathscr{F}$,

$$v(F) = \lim_{t\to\infty} Q_t(x, F) \tag{2.11}$$

exists and does not depend on $x \in \Omega$, and v is the unique invariant probability measure (or equilibrium distribution) for the spatial birth-and-death process; i.e.,

$$v(F) = \int Q_t(x, F) v(dx) , \tag{2.12}$$

for all $F \in \mathscr{F}$.

We conclude this section with some remarks on Theorem 2.1.

(i) The theorem is due to Preston ((1977), Proposition 7.1 and Theorem 7.1); note that in the case where $\beta_0 = 0$, the condition (a) in (2.10) seems to have disappeared in Preston's paper. The proof of Theorem 2.1 is based on well-known properties of simple birth-and-death processes and the fact that the simple process with birth rate β_n and death rate δ_n converges more slowly to the state 0 than the spatial process (cf. (2.4)).

(ii) The theorem includes both the case where the state 0 is a reflecting barrier, i.e., $\beta_0 > 0$, and the case where 0 is absorbing, i.e., $\beta_0 = 0$. In the latter case, the process is called a truncated spatial birth-and-death process.

(iii) The invariant distribution for the simple birth-and-death process with birth rate β_n and death rate δ_n exists and is unique under the conditions of Theorem 2.1, and its density with respect to the counting measure is given by $\rho\pi_n$, $n = 0, 1, 2, \ldots$, where the π_n are the *potential coefficients* defined by

$$\pi_0 = 1, \qquad \pi_n = \beta_0 \cdots \beta_{n-1} / \delta_1 \cdots \delta_n \, , \tag{2.13}$$

for $n \geq 1$ and where $\rho = 1 \Big/ \sum_{n=0}^{\infty} \pi_n$ (Karlin and McGregor (1957*b*, 1965)). Combining (2.4) with (2.11), it follows that the invariant distribution for the simple process dominates the invariant distribution for the original process in the following sense:

$$\sum_{n=m}^{\infty} \nu(\Omega_n) \leq \rho \sum_{n=m}^{\infty} \pi_n \quad \text{for all} \quad m \geq 0 \, . \tag{2.14}$$

(iv) The conditions in (2.10) guarantee that the backward and forward equations for the simple process have a unique solution (Karlin and McGregor (1957*a*)), and in this case and when $\beta_0 > 0$, (a) holds iff the simple process is positive recurrent (Karlin and McGregor (1957*b*)). As remarked by Preston (1977), it would be better in the transient case to compare the original process with a simple birth-and-death process with birth rate $\tilde{\beta}_n = \inf_{x \in \Omega_n} \beta(x)$ and death rate $\tilde{\delta}_n = \sup_{x \in \Omega_n} \delta(x)$. Notice also that the conditions (2.7) and (2.9) are identical; and under (a), condition (2.8) with $n_0 = 1$ and condition (b) in (2.10) are equivalent. Thus, under the conditions of the theorem, the spatial birth-and-death process exists and is uniquely defined.

3. On the rate of convergence to equilibrium of general spatial birth-and-death processes

Let the situation be as in the previous section and consider a spatial birth-and-death process on S. Theorem 3.1 and Corollary 3.1 give new results for the rate of convergence of the process in general. Comments on the theorem and its corollary are appended below. The proof of the theorem is given at the end of this section.

THEOREM 3.1. *Suppose the conditions of Theorem* 2.1 *hold, and let* γ *and* κ *be any probability measures on* $(\Omega, \mathscr{F})$ *which satisfy the following conditions both for* $\chi = \gamma$ *and* $\chi = \kappa$:

$$\chi\left(\bigcup_{n=0}^{n_0} \Omega_n\right) = 1 \qquad \text{if} \quad (2.9)\ \text{holds}, \tag{3.1}$$

$$\sum_{n=2}^{\infty} \chi(\Omega_n)\sqrt{\frac{\delta_1 \cdots \delta_n}{\beta_1 \cdots \beta_{n-1}}} < \infty \qquad \text{if} \quad (2.10)\ \text{holds}. \tag{3.2}$$

Then there exist real constants $c > 0$ *and* $0 < r < 1$ *such that*

$$\sup_{F \in \mathscr{F}} \left| \int Q_t(x, F)\gamma(dx) - \int Q_t(y, F)\kappa(dy) \right| \le cr^t. \tag{3.3}$$

Moreover, in the truncated case (i.e., when (2.9) *holds)* c *and* r *can be chosen independently of* γ *and* κ.

COROLLARY 3.1. *Let the situation be as in Theorem* 3.1, *and let* ν *be the invariant distribution for the spatial birth-and-death process.*

If (2.9) *holds then there exist* $c > 0$ *and* $0 < r < 1$ *such that*

$$\sup_{F \in \mathscr{F}} \left| \nu(F) - \int Q_t(x, F)\gamma(dx) \right| \le cr^t, \tag{3.4}$$

for all initial distributions γ *which satisfy* (3.1).

If (2.10) *holds with* $\beta_0 = 0$ *and* γ *satisfies* (3.2), *then there exists* $c > 0$ *such that*

$$\int Q_t(x, 0)\gamma(dx) = 1 + O(e^{-ct}). \tag{3.5}$$

Suppose (2.10) *holds with* $\beta_0 > 0$ *and* γ *satisfies* (3.2). *If*

$$\sum_{n=2}^{\infty} \sqrt{\frac{\beta_1 \cdots \beta_{n-1}}{\delta_1 \cdots \delta_n}} < \infty \tag{3.6}$$

and

$$\beta_n \le \delta_{n+1} \qquad \text{for all sufficiently large} \quad n \tag{3.7}$$

hold, then there exists $c > 0$ *such that*

$$\sup_{F \in \mathscr{F}} \left| \nu(F) - \int Q_t(x, F)\gamma(dx) \right| = O(e^{-ct}). \tag{3.8}$$

PROOF OF COROLLARY 3.1. Assume Theorem 3.1 is true. Since ν satisfies (3.1) when (2.9) holds, (3.4) follows from (3.3). If $\beta_0 = 0$, then $\nu(0) = 1$, so (3.5) is seen to be equivalent to (3.3). Suppose (2.10), (3.6) and (3.7) hold and $\beta_0 > 0$. By (2.13) and (2.14)

$$\sum_{n=m}^{\infty} \nu(\Omega_n) \le \rho \sum_{n=m}^{\infty} \pi_n ,$$

for all $m \ge 0$ and (3.6) and (3.7) give

$$\sum_{n=2}^{\infty} \pi_n / \sqrt{\pi_n} < \infty, \qquad 1/\sqrt{\pi_n} \le 1/\sqrt{\pi_{n+1}} ,$$

for all sufficiently large n. Combining these results, we obtain $\sum_{n=2}^{\infty} \nu(\Omega_n)/\sqrt{\pi_n} < \infty$, i.e., ν satisfies (3.2). Hence (3.8) follows from (3.3). □

Remarks. (i) For truncated spatial birth-and-death processes it is, of course, only interesting to consider initial distributions which satisfy (3.1). Lotwick and Silverman (1981) proved (3.4) in the special case of a hard core birth-and-death process. In fact, their proof applies as well for a general truncated spatial birth-and-death process (see Lemma 3.5 below).

(ii) Condition (3.2) is trivially satisfied when the support of χ is bounded in the sense that $\chi(\Omega_n) = 0$ for all sufficiently large n. This holds in particular if χ is a point measure, i.e.,

$$\sup_{F \in \mathscr{F}} |Q_t(x, F) - Q_t(y, F)| = O(e^{-ct}) ,$$

for $x, y \in \Omega$. Furthermore, if either the invariant distribution ν satisfies (3.2) or if (3.6) and (3.7) hold, then

$$\sup_{F \in \mathscr{F}} |\nu(F) - Q_t(x, F)| = O(e^{-ct}) ,$$

for $x \in \Omega$. Observe also that (a) in (2.10) holds under (3.6). In fact, conditions (3.6) and (3.7) are often satisfied in applications (cf. Møller (1987)).

(iii) As noted in the proof of Corollary 3.1, the results (3.3) and (3.5) are equivalent when 0 is an absorbing state and (2.10) holds. Especially, for each $x \in \Omega$ there exists $c > 0$ such that

$$(3.9) \qquad Q_t(x, 0) = 1 + O(e^{-ct}) \quad \text{when} \quad \beta_0 = 0 .$$

(iv) In the proof of Theorem 3.1 we compare the spatial birth-and-death process with a simple birth-and-death process with birth rate β_n and death rate δ_n. Under (2.10) it is well-known that the simple process is exponential ergodic, i.e., there exists a $c > 0$ such that

$$q_t(m,n) = \rho\pi_n + O(e^{-ct}), \tag{3.10}$$

for all m, n with $m, n \geq 0$ if $\beta_0 > 0$ and $m \geq 0$, $n \geq 1$ if $\beta_0 = 0$, where we have used a notation as in (2.6) and (2.13) (see Callaert (1971, 1974) and van Doorn (1981). In fact, for a simple process which is either recurrent and with $\beta_0 > 0$ or where the state 0 is certain and absorbing, condition (2.10) is necessary and sufficient for exponential ergodicity). Moreover, the simple process is irreducible on $\{0, 1, \ldots\}$ when $\beta_0 > 0$ and $\beta_n, \delta_n > 0$ for all $n \geq 1$, so if (3.10) holds for a particular choice of m, n and with c replaced by $c_{mn} > 0$, then there exists $c > 0$ such that (3.10) holds for all $m, n \geq 0$ (since exponential ergodicity is a solidarity property for an irreducible Markov chain with discrete state space (Kingman (1963))). Combining this with Callaert ((1974), Theorems 1 and 3) it follows that under the conditions of (2.10), the result (3.9) implies (3.10) when $\beta_0 = 0$; and if $\beta_0 > 0$ and (3.6) holds we have again a stronger result than (3.10), since by (3.8)

$$\sup_{A \subset \{0,1,\ldots\}} \left| q_t(m,A) - \rho \sum_{n \in A} \pi_n \right| = O(e^{-ct}),$$

for $m \geq 0$. Observe also that the conditions (3.2) and (3.6) are equivalent for a simple process with $\beta_0 > 0$.

In the remaining part of this section we prove Theorem 3.1. It follows from the next lemma that it suffices to consider the waiting time until two independent simple processes are both in the state 0.

LEMMA 3.1. *Let $\{X(t)\colon t \geq 0\}$ and $\{Y(t)\colon t \geq 0\}$ be two identical distributed spatial birth-and-death processes with birth rate β and death rate δ. The initial distributions of the two processes are denoted by Θ respective Φ. Similarly, let $\{x(t)\colon t \geq 0\}$ and $\{y(t)\colon t \geq 0\}$ be two simple birth-and-death processes with the same birth rate $\beta_n = \sup_{x \in \Omega_n} \beta(x)$ and the same death rate $\delta_n = \inf_{x \in \Omega_n} \sum_{\eta \in x} \delta(x)$ and with initial distributions θ and φ given by*

$$\theta(n) = \Theta(\Omega_n), \quad \varphi(n) = \Phi(\Omega_n) \quad \textit{for all} \quad n \geq 0\,.$$

Furthermore, define the stopping times

$$T = \inf\{t \geq 0\colon X(t) = Y(t) = 0\},$$

$$\tau = \inf\{t \geq 0\colon x(t) = y(t) = 0\}$$

(*taking* $\inf \varnothing = \infty$). *Suppose the backward equations for the simple processes have a unique solution (recall that this is the case under the conditions of Theorem* 3.1). *Then for all* $F \in \mathscr{F}$,

$$\frac{1}{2}\left|\int Q_t(x, F)\Theta(dx) - \int Q_t(y, F)\Phi(dy)\right|$$
$$\leq P_{\Theta\times\Phi}(T > t) \leq P_{\theta\times\varphi}(\tau > t)\,,$$

where under the probability measure $P_{\Theta\times\Phi}$ *the spatial processes* $\{X(t)\colon t \geq 0\}$ *and* $\{Y(t)\colon t \geq 0\}$ *are independently distributed, and where under the probability measure* $P_{\theta\times\varphi}$ *the simple processes* $\{x(t)\colon t \geq 0\}$ *and* $\{y(t)\colon t \geq 0\}$ *are independently distributed.*

PROOF. The proof of the first inequality is analogous to the proof of Lemma 2.6 in Pitman (1974): By the strong Markov property of the process $Z(t) = (X(t), Y(t))$,

$$P_{\Theta\times\Phi}(X(t) \in F, T \leq t) = P_{\Theta\times\Phi}(Y(t) \in F, T \leq t)\,,$$

and hence

$$\left|\int Q_t(x, F)\Theta(dx) - \int Q_t(x, F)\Phi(dy)\right|$$
$$= |P_{\Theta\times\Phi}(X(t) \in F) - P_{\Theta\times\Phi}(Y(t) \in F)|$$
$$= |P_{\Theta\times\Phi}(X(t) \in F, T > t) - P_{\Theta\times\Phi}(Y(t) \in F, T > t)|$$
$$\leq 2P_{\Theta\times\Phi}(T > t)\,.$$

The last inequality in Lemma 3.1 is a consequence of Preston ((1977), Proposition 6.1 and (6.1)–(6.3)), since we get

$$P_{\Theta\times\Phi}(T > t \mid X(0) = x,\ Y(0) = y)$$
$$\leq P_{\theta\times\varphi}(\tau > t \mid x(0) = m,\ y(0) = n)\,,$$

for all $x \in \Omega_m$ and $y \in \Omega_n$.

LEMMA 3.2. *Theorem* 3.1 *is true in the truncated case, i.e., when* (2.9) *holds.*

PROOF. With a notation as in Lemma 3.1, let z be the double simple process $z = \{z(t)\colon t \geq 0\}$ where $z(t) = (x(t), y(t))$. Then for all $(m, n) \neq (0, 0)$,

$$
\begin{aligned}
&P_{\theta\times\varphi}\,(\text{first transition for } z \text{ occurs before time } t \text{ and}\\
&\qquad\qquad\qquad\qquad\qquad \text{is a death } |\, z(0)=(m,n))\\
&= [1-\exp\{-(\alpha_m+\alpha_n)t\}]\,\frac{\delta_m+\delta_n}{\alpha_m+\alpha_n}\\
&\geq [1-\exp\{-\tilde{\alpha}_{m+n}t\}]\,\frac{\tilde{\delta}_{m+n}}{\tilde{\alpha}_{m+n}}\\
&= K(m+n,t), \qquad \text{say}\,,
\end{aligned}
$$

where $\alpha_m=\beta_m+\delta_m$, $\tilde{\beta}_m=\max\limits_{i,j:\,i+j=m}(\beta_i+\beta_j)$, $\tilde{\delta}_m=\min\limits_{i,j:\,i+j=m}(\delta_i+\delta_j)$ and $\tilde{\alpha}_m=\tilde{\beta}_m+\tilde{\delta}_m$. Since $\tilde{\delta}_m>0$ for all $m\geq 1$, we have $K(m,t)>0$ for all $m\geq 1$. The remaining part of the proof is now analogous to the proof of Theorem A in Lotwick and Silverman (1981). □

Not surprisingly, the proof of Theorem 3.1 is much easier in the truncated case than in the case of (2.10). In the latter case we shall use the spectral representation of the transition probabilities $q_t(m,n)$ of the simple birth-and-death process with birth rate β_n and death rate δ_n. We start by recalling this spectral representation when (2.20) holds and $\beta_n>0$, $\delta_n>0$ for all $n\geq 1$.

For $\beta_0>0$ define the polynomials $R_n(u)$, $n=0,1,\ldots$ for all $u\geq 0$ by the recurrence relations

$$
\begin{aligned}
&R_0(u)=1\,,\\
&-uR_0(u)=-(\beta_0+\delta_0)R_0(u)+\beta_0R_1(u)\,,\\
&-uR_n(u)=\delta_nR_{n-1}(u)-(\beta_n+\delta_n)R_n(u)+\beta_nR_{n+1}(u), \qquad n\geq 2\,.
\end{aligned}
$$

(In the literature "R_n" is usually denoted by "Q_n", but since "Q_t" denotes the transition kernels of the spatial birth-and-death process, we use the present notation in order to avoid confusion.) When $\beta_0=0$ we define the polynomials $R_n^*(u)$, $n=0,1,\ldots$, $u\geq 0$ as above but with β_n replaced by $\beta_n^*=\beta_{n+1}$ and δ_n replaced by $\delta_n^*=\delta_{n+1}$. Moreover, let π_n^* denote the potential coefficient defined with respect to β_n^*, δ_n^*, i.e.,

$$
\begin{aligned}
&\pi_0^*=1\,,\\
&\pi_n^*=\beta_0^*\cdots\beta_{n-1}^*/\delta_1^*\cdots\delta_n^*=\beta_1\cdots\beta_n/\delta_2\cdots\delta_{n+1}, \qquad n\geq 1\,.
\end{aligned}
$$

Then it has been shown by Karlin and McGregor (1957*a*) that under (2.10) the polynomials R_n (or R_n^*) are orthogonal with respect to a unique measure ψ (or ψ^*) on $[0,\infty)$, that is

$$(3.11) \qquad \pi_n \int_0^\infty R_m(u) R_n(u) \psi(du) = 1\{m = n\} \qquad \text{if} \quad \beta_0 > 0 ,$$

$$(3.12) \qquad \pi_n^* \int_0^\infty R_m^*(u) R_n^*(u) \psi(du) = 1\{m = n\} \qquad \text{if} \quad \beta_0 = 0 ,$$

and the spectral representation of $q_t(m, n)$ is given by

$$(3.13) \quad q_t(m, n) = \pi_n \int_0^\infty e^{-ut} R_m(u) R_n(u) \psi(du) \qquad \text{if} \quad \beta_0 > 0 ,$$

$$(3.14) \quad q_t(m+1, n+1) = \pi_n^* \int_0^\infty e^{-ut} R_m^*(u) R_n^*(u) \psi^*(du) \qquad \text{if} \quad \beta_0 = 0 ,$$

for all $m, n \geq 0$. Furthermore,

$$(3.15) \qquad q_t(m+1, 0) = \delta_1 \int_0^t q_s(m+1, 1) ds \qquad \text{if} \quad \beta_0 = 0 ,$$

for all $m \geq 0$. Finally, we recall that under (3.12) there exist real constants $u_1 > 0$ and $u_1^* > 0$ such that

$$(3.16) \qquad \psi((0, u_1)) = 0 \quad \text{and} \quad \psi(0) = \rho \in (0, 1) \qquad \text{if} \quad \beta_0 > 0 ,$$

$$(3.17) \qquad \psi^*([0, u_1^*)) = 0 \qquad \text{if} \quad \beta_0 = 0$$

(Callaert (1971, 1974) and van Doorn (1981)).

Now, to prove (3.3) in Theorem 3.1, assume without loss of generality that γ satisfies (3.2) and κ is the point measure $\kappa(F) = 1\{0 \in F\}$. Let $\gamma_n = \gamma(\Omega_n)$, i.e., (3.2) states that

$$(3.18) \qquad \sum_{n=1}^{\infty} \gamma_n / \sqrt{\pi_n} < \infty \qquad \text{if} \quad \beta_0 > 0 ,$$

$$(3.19) \qquad \sum_{n=1}^{\infty} \gamma_n / \sqrt{\pi_{n-1}^*} < \infty \qquad \text{if} \quad \beta_0 = 0 .$$

By Lemma 3.1 it suffices to prove that (2.10) and (3.18)–(3.19) imply that

$$(3.20) \qquad \sum_{m=1}^{\infty} P(\tau > t \mid x(0) = m, y(0) = 0) \gamma_m = O(e^{-ct}) ,$$

for some $c > 0$, where the simple processes $\{x(t): t \geq 0\}$ and $\{y(t): t \geq 0\}$ are independent. Below we exhibit the conditional moments of the waiting time τ with respect to

$$G_m(t) = P(\tau \leq t \mid x(0) = m, y(0) = 0), \qquad m \geq 1 .$$

LEMMA 3.3. *Assume* (2.10) *holds. Then* $G_m(\cdot)$ *is a distribution on* $[0,\infty)$ *and the moments*

$$a_m^{(k)} = \int_0^\infty t^k G_m(dt), \qquad m, k = 1, 2, \ldots ,$$

exist and are given by

$$a_m^{(k)} = k!\delta_1 \int_{u_1^*}^\infty \frac{R_{m-1}^*(u)}{u^{k+1}} \psi^*(du) \quad \text{when} \quad \beta_0 = 0 , \tag{3.21}$$

$$a_m^{(k)} = k! \sum_{j=1}^{k} c_{k-j} M_j^{(m)} \quad \text{when} \quad \beta_0 > 0 , \tag{3.22}$$

where

$$M_j^{(m)} = \frac{1}{\rho}\int_{u_1}^\infty \frac{1 - R_m(u)}{u^j} \psi(du) + \frac{1}{\rho^2}\int_{u_1}^\infty\int_{u_1}^\infty \frac{1 - R_m(u)}{(u+v)^j} \psi(du)\psi(dv)$$

$$c_0 = 1 ,$$

$$c_p = \sum_{\substack{i_1,\ldots,i_p \in \{0,1,\ldots,p\} \\ i_1 + 2i_2 + \cdots + pi_p = p}} \frac{(i_1 + \cdots + i_p)!}{i_1! \cdots i_p!} K_1^{i_1} \cdots K_p^{i_p}, \qquad p \geq 1 ,$$

and where

$$K_j = \frac{2}{\rho}\int_{u_1}^\infty \frac{\psi(du)}{u^j} + \frac{1}{\rho^2}\int_{u_1}^\infty\int_{u_1}^\infty \frac{\psi(du)\psi(dv)}{(u+v)^j} .$$

PROOF. First, let $\beta_0 = 0$. Since

$$G_m(t) = q_t(m, 0) = 1 - \sum_{n=1}^\infty q_t(m, n) ,$$

it follows from (3.14) that $\lim_{t\to\infty} G_m(t) = 1$, so $G_m(\cdot)$ is a distribution function on $[0,\infty)$. By (3.14), (3.15) and (3.17),

$$G_m(t) = \delta_1 \int_{u_1^*}^\infty \frac{1 - e^{-ut}}{u} R_{m-1}^*(u)\psi^*(du) . \tag{3.23}$$

Define the Laplace transform

$$\chi_m^*(s) = \int_0^\infty e^{-st} G_m(dt) ,$$

for $s \geq 0$. By (3.23)

$$\chi_m^*(s) = \delta_1 \int_{u_1^*}^{\infty} \frac{1}{1 + (s/u)} \frac{R_{m-1}^*(u)}{u} \psi^*(du),$$

and hence for all small $s > 0$,

$$\chi_m^*(s) = \delta_1 \sum_{n=0}^{\infty} (-s)^k \int_{u_1^*}^{\infty} \frac{R_{m-1}^*(u)}{u^{k+1}} \psi^*(du),$$

whereby (3.21) follows.

Next, let $\beta_0 > 0$ and define the Laplace transforms

$$\varphi_m(s) = \int_0^{\infty} e^{-st} G_m(dt), \qquad m \geq 1, \tag{3.24}$$

$$\chi_m(s) = \int_0^{\infty} e^{-st} q_t(m,0) q_t(0,0) dt, \qquad m \geq 0, \tag{3.25}$$

for $s \geq 0$. By a standard enumeration of paths it is found that

$$q_t(m,0)q_t(0,0) = \int_0^t q_{t-s}(0,0)^2 G_m(ds). \tag{3.26}$$

From (3.24)–(3.26), we get

$$\chi_m(s) = \varphi_m(s)\chi_0(s). \tag{3.27}$$

By (3.13) and (3.15), $q_t(0,0) > 0$ for all $t \geq 0$, i.e., $\chi_0(s) > 0$ for all $s > 0$. Hence combining (3.13), (3.16), (3.25) and (3.27), we obtain

$$1 - \varphi_m(s) = \frac{\frac{1}{\rho}\int_{u_1}^{\infty} \frac{s}{u+s} \{1 - R_m(u)\}\psi(du)}{1 + \frac{2}{\rho}\int_{u_1}^{\infty} \frac{s}{u+s}\psi(du) + \frac{1}{\rho^2}\int_{u_1}^{\infty}\int_{u_1}^{\infty} \frac{s}{u+v+s}\psi(du)\psi(dv)}$$
$$+ \frac{\frac{1}{\rho^2}\int_{u_1}^{\infty}\int_{u_1}^{\infty} \frac{s}{u+v+s}\{1 - R_m(u)\}\psi(du)\psi(dv)}{1 + \frac{2}{\rho}\int_{u_1}^{\infty} \frac{s}{u+s}\psi(du) + \frac{1}{\rho^2}\int_{u_1}^{\infty}\int_{u_1}^{\infty} \frac{s}{u+v+s}\psi(du)\psi(dv)},$$

for all $s > 0$. Since $s/(u+s) \to 0$ boundedly on $0 < u < \infty$ as $s \to 0$, we have for all small $s > 0$

$$
\begin{aligned}
\varphi_m(s) - 1 = &\left\{ \sum_{j=1}^{\infty} (-s)^j \left[\frac{1}{\rho} \int_{u_1}^{\infty} \frac{1 - R_m(u)}{u^j} \psi(du) \right.\right. \\
&\left.\left. + \frac{1}{\rho^2} \int_{u_1}^{\infty} \int_{u_1}^{\infty} \frac{1 - R_m(u)}{(u+v)^j} \psi(du)\psi(dv) \right] \right\} \\
&\times \left\{ \sum_{i=0}^{\infty} (-1)^i \left[\frac{2}{\rho} \int_{u_1}^{\infty} \frac{s}{u+s} \psi(du) \right.\right. \\
&\left.\left. + \frac{1}{\rho^2} \int_{u_1}^{\infty} \int_{u_1}^{\infty} \frac{s}{u+v+s} \psi(du)\psi(dv) \right]^i \right\} \\
= &\; N_1 \times N_2, \quad \text{say},
\end{aligned}
$$

where

$$
N_1 = \sum_{j=1}^{\infty} (-s)^j M_j^{(m)}
$$

and

$$
\begin{aligned}
N_2 &= \sum_{i=0}^{\infty} \left[\sum_{j=1}^{\infty} (-s)^j \left(\frac{2}{\rho} \int_{u_1}^{\infty} \frac{\psi(du)}{u^j} + \frac{1}{\rho^2} \int_{u_1}^{\infty} \int_{u_1}^{\infty} \frac{\psi(du)\psi(dv)}{(u+v)^j} \right) \right]^i \\
&= \sum_{i=0}^{\infty} \left[\sum_{j=1}^{\infty} (-s)^j K_j \right]^i \\
&= \sum_{p=0}^{\infty} (-s)^p c_p .
\end{aligned}
$$

Hence, for all small $s > 0$,

$$
\varphi_m(s) = 1 + \sum_{k=1}^{\infty} (-s)^k \sum_{j=1}^{k} c_{k-j} M_j^{(m)} . \tag{3.28}
$$

Thus $\varphi_m(0) = 1$, i.e., $\int_0^{\infty} G_m(dt) = 1$ by (3.24), so $G_m(\cdot)$ is a distribution function on $[0, \infty)$. Finally, (3.22) follows from (3.24) and (3.28). □

LEMMA 3.4. *Let the situation be as in Lemma* 3.3. *Then there exist real constants $a > 0$ and $b > 0$ such that for all $m, k = 1, 2, \dots$,*

$$
a_m^{(k)} \le k!ab^k / \sqrt{\pi_{m-1}^{*}} \qquad \textit{when} \quad \beta_0 = 0 , \tag{3.29}
$$

$$
a_m^{(k)} \le k!ab^k (1 + 1/\sqrt{\pi_m}) \quad \textit{when} \quad \beta_0 > 0 . \tag{3.30}
$$

PROOF. Observe first that Cauchy–Schwartz's inequality and (3.11)–

(3.12) give for $j, k, m = 0, 1, \ldots,$

$$\left| \int_{u_1^*}^{\infty} \frac{R_m^*(u)}{u^{k+1}} \psi^*(du) \right| \le \frac{1}{\sqrt{\pi_m^*} u_1^{*k+1}} \quad \text{when} \quad \beta_0 = 0 , \tag{3.31}$$

$$\left| \int_{u_1}^{\infty} \frac{R_m(u)}{u^j} \psi(du) \right| \le \frac{1}{\sqrt{\pi_m} u_1^j} \quad \text{when} \quad \beta_0 > 0 , \tag{3.32}$$

$$\left| \int_{u_1}^{\infty} \int_{u_1}^{\infty} \frac{R_m(u)}{(u+v)^j} \psi(du)\psi(dv) \right| \le \frac{1}{\sqrt{\pi_m}(2u_1)^j} \quad \text{when} \quad \beta_0 > 0 . \tag{3.33}$$

Now, for $\beta_0 = 0$, (3.21) and (3.31) give (3.29) with $a = \delta_1 / u_1^*$ and $b = 1/u_1^*$.

Suppose $\beta_0 > 0$. Then, since $R_0(u) = 1$ and $\pi_0 = 1$, (3.32) and (3.33) give

$$K_j \le \left(\frac{2}{\rho} + \frac{1}{2^j \rho^2} \right) \frac{1}{u_1^j} \le d_1 / u_1^j ,$$

for all $j \ge 1$, where $d_1 = 2/\rho + 1/2\rho^2 > 5/2$. Hence by definition of c_p,

$$c_p \le (d_1/u_1)^p \sum_{\substack{i_1, \ldots, i_p \in \{0, 1, \ldots, p\} \\ i_1 + 2i_2 + \cdots + pi_p = p}} \frac{(i_1 + \cdots + i_p)!}{i_1! \cdots i_p!} \le \left(\frac{2d_1}{u_1} \right)^p , \tag{3.34}$$

for all $p \ge 0$, since the sum in (3.34) is equal to 2^{p-1} for $p \ge 1$. Furthermore, by (3.32) and (3.33),

$$M_j^{(m)} \le \left(\frac{1}{\rho} + \frac{1}{2^j \rho^2} \right) \left(1 + \frac{1}{\sqrt{\pi_m}} \right) \frac{1}{u_1^j} \le \left(1 + \frac{1}{\sqrt{\pi_m}} \right) d_2 / u_1^j , \tag{3.35}$$

for all $j \ge 1$, where $d_2 = (1/\rho + 1/2\rho^2)$. Combining (3.34) and (3.35), we get

$$\begin{aligned} \sum_{j=1}^{k} c_{k-j} M_j^{(m)} &\le \left(1 + \frac{1}{\sqrt{\pi_m}} \right) \frac{d_2}{u_1^k} \frac{(2d_1)^k - 1}{2d_1 - 1} \\ &\le \left(1 + \frac{1}{\sqrt{\pi_m}} \right) d_2 \left(\frac{2d_1}{u_1} \right)^k , \end{aligned}$$

for all $m, k \ge 1$, since $2d_1 > 5$. Now, (3.30) follows from (3.22) with $a = d_2$ and $b = 2d_1/u_1$. □

Finally, we prove that (3.20) holds under (2.10) and (3.18)–(3.19). Suppose (2.10) holds. If $\beta_0 = 0$, then (3.29) gives

$$t^k(1 - G_m(t)) \leq \int_0^\infty s^k G_m(ds) \leq k!ab^k/\sqrt{\pi_{m-1}^*} ,$$

$k, m = 1, 2, \ldots$, and hence with $a_0 = \max\{a, 1\}$,

$$t^k(1 - G_m(t)) \leq k!a_0 b^k(1 + 1/\sqrt{\pi_{m-1}^*}) ,$$

$k = 0, 1, \ldots, m = 1, 2, \ldots$, i.e.,

$$e^{t/2b}(1 - G_m(t)) \leq 2a_0(1 + 1/\sqrt{\pi_{m-1}^*}), \qquad m = 1, 2, \ldots .$$

Similarly, if $\beta_0 > 0$, then (3.30) gives

$$e^{t/2b}(1 - G_m(t)) \leq 2a_0(1 + 1/\sqrt{\pi_m}), \qquad m = 1, 2, \ldots .$$

Hence, when $\gamma_m \geq 0$ and $\sum_{m=0}^{\infty} \gamma_m = 1$,

$$\sum_{m=0}^{\infty} (1 - G_m(t))\gamma_m \leq K e^{-t/2b} , \tag{3.36}$$

where

$$K = 2a_0\left(1 + \sum_{m=1}^{\infty} \gamma_m/\sqrt{\pi_{m-1}^*}\right) \quad \text{if} \quad \beta_0 = 0 ,$$

$$K = 2a_0\left(1 + \sum_{m=1}^{\infty} \gamma_m/\sqrt{\pi_m}\right) \quad \text{if} \quad \beta_0 > 0 .$$

If (3.18)–(3.19) hold, then $K < \infty$ and (3.20) holds by (3.36). Thus the proof of Theorem 3.1 is completed.

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