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COMPOSITE CONSTRUCTION OF GROUP DIVISIBLE DESIGNS

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Abstract. Two new methods of constructing group divisible designs are given. In particular, a new resolvable solution for the SR 39 is presented.

Key words and phrases: GD design, BIB design, resolvable, non-isomorphic, regular.

1. Introduction

Group divisible (GD) designs constitute the largest, simplest and perhaps the most important type of 2-associate partially balanced incomplete block (PBIB) designs. A GD design is an arrangement of v ($= mn$) treatments in b blocks such that each block contains k ($< v$) distinct treatments; each treatment is replicated r times, and the set of treatments can be partitioned into m (≥ 2) groups of n (≥ 2) treatments each, any two distinct treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. Furthermore, if $r - \lambda_1 = 0$, the GD design is said to be singular; if $r - \lambda_1 > 0$ and $rk - \lambda_2 v = 0$, it is called semi-regular (SR); and if $r - \lambda_1 > 0$ and $rk - \lambda_2 v > 0$, it is called regular (R).

Clatworthy (1973) tabulated 443 parameters' combinations of GD designs with their solutions. Since then Freeman (1976), Kageyama and Tanaka (1981), Bhagwandas and Parihar (1982), Kageyama (1985*a*, 1985*b*), Kageyama and Mohan (1985*a*, 1985*b*), Banerjee *et al.* (1985*a*, 1985*b*), Bhagwandas *et al.* (1985), Dey and Nigam (1985), Banerjee and Kageyama (1986), Sinha and Kageyama (1986), and Sinha (1987) have given several methods of constructing GD designs. From another point of view, Hanani (1975) also presented some methods of constructing GD designs.

Here two new methods of constructing GD designs are given, with the composition of a balanced incomplete block (BIB) design and a GD design. The first theorem is a generalization of the main theorem given in Sinha

(1987). In particular, a new non-isomorphic solution to SR 39 in Clatworthy (1973) is obtained. This solution is resolvable. The search for new designs has been limited to the range of parameters $r, k \leq 10$.

2. Constructions

THEOREM 2.1. *The existence of an RGD design with parameters*

$$(2.1)\qquad \begin{aligned} &v = mn, \quad b = pmn, \quad r = p(m-1), \quad k = m-1\,, \\ &\lambda_1 = 0, \quad \lambda_2 = p(m-2)/n\,, \end{aligned}$$

for positive integers m (>2), n (≥ 2) and p, and a BIB design with parameters $v' = n, b', r', k', \lambda'$, implies the existence of a GD design with parameters

$$(2.2)\qquad \begin{aligned} &v = mn, \quad b = pmnb', \quad r = p(m-1)b' + pnr', \quad k = m + k' - 1\,, \\ &\lambda_1 = pn\lambda', \quad \lambda_2 = 2pr' + pb'(m-2)/n\,. \end{aligned}$$

PROOF. Let the $v = mn$ treatments of the RGD design with (2.1) be denoted by $1, 2, \ldots, v$, and they are arranged into m groups of n treatments each as:

$$\begin{array}{cccc} 1 & 2 & \ldots & m \\ m+1 & m+2 & \ldots & 2m \\ \vdots & \vdots & & \vdots \\ (n-1)m+1 & (n-1)m+2 & \ldots & nm \end{array}.$$

Note that each block in the design with (2.1) consists of exactly one treatment from each of $m-1$ groups. Let us form m BIB designs with parameters $v' = n, b', r', k', \lambda'$, treatments being denoted by integers from each column of the above arrangement. Now, to each block of (2.1) repeated b' times, add the k' treatments from the b' blocks of the BIB design constructed from the column integers unrepresented in the block of (2.1) to get bb' $(= pmnb')$ new blocks, since $k = m - 1$. The blocks are of equal size $k' + m - 1$ $(= k)$. By this procedure, the resulting design has the same association scheme as in (2.1) and it follows from $\lambda_1 = 0$ and $k = m - 1$ that these blocks form the required GD design with parameters (2.2).

Remark. In (2.2), it holds that $r - \lambda_1 > 0$ and $rk - \lambda_2 v = pb'(k'-1)^2 \geq 0$.

We can present several applications of Theorem 2.1 by use of the existence of trivial BIB designs.

From the existence of a BIB design with parameters $v' = n = b'$, $r' = 1 = k'$, $\lambda' = 0$, we have the following.

COROLLARY 2.1. *The existence of an RGD design with* (2.1) *implies the existence of an SRGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn^2, \quad k = m, \quad r = pmn\,, \\ &\lambda_1 = 0, \quad \lambda_2 = pm\,. \end{aligned} \tag{2.3}$$

From the existence of an unreduced BIB design with parameters $v' = n$, $b' = \binom{n}{2}$, $r' = n - 1$, $k' = 2$, $\lambda' = 1$, we obtain the following.

COROLLARY 2.2. *The existence of an RGD design with* (2.1) *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn\binom{n}{2}, \quad r = p(m+1)\binom{n}{2}, \\ &k = m + 1, \quad \lambda_1 = pn, \quad \lambda_2 = p(n-1)(m+2)/2\,. \end{aligned} \tag{2.4}$$

Remark. For $n = 2$, (2.4) is the complement of (2.1).

Since there exists a BIB design with parameters $v' = n = b'$, $r' = n - 1 = k'$, $\lambda' = n - 2$, the following is obtained.

COROLLARY 2.3. *The existence of an RGD design with* (2.1) *implies the existence of an RGD design* (*provided* $n \geq 3$) *with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn^2, \quad r = pn(m + n - 2), \quad k = m + n - 2\,, \\ &\lambda_1 = pn(n-2), \quad \lambda_2 = p(2n + m - 4)\,. \end{aligned} \tag{2.5}$$

In particular, if $n = 2$, the resulting design is an SRGD design the same as (2.3).

Remark. When $n = 3$, (2.4) and (2.5) yield the same values of parameters.

Since a single block containing all the treatments can be regarded as a BIB design with parameters $v' = n = k'$, $b' = 1 = r' = \lambda'$, we have the following.

COROLLARY 2.4. *The existence of an RGD design with* (2.1) *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn, \quad r = p(m+n-1), \quad k = m+n-1\,, \\ &\lambda_1 = pn, \quad \lambda_2 = p(m+2n-2)/n\,. \end{aligned} \tag{2.6}$$

Remark. For $n = 2$, (2.6) is the complement of (2.1), and (2.4) and (2.6) yield the same parameter values. Corollary 2.4 has been reported as the main result in Sinha (1987).

THEOREM 2.2. *The existence of an RGD design with parameters* (2.1) *and a BIB design with parameters* $v' = m-1, b', r', k', \lambda'$, *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmnb', \quad r = pnb' + p(m-1)r', \quad k = n + k'\,, \\ &\lambda_1 = pnb', \quad \lambda_2 = 2r'p + \lambda' p(m-2)/n\,, \end{aligned} \tag{2.7}$$

for some positive integer p.

PROOF. We start with the same GD design as in (2.1). Now, we form b BIB designs with parameters $v' = m-1, b', r', k', \lambda'$ and their treatments are from each of the b blocks of (2.1) since $k = m-1$. Now, a block B of (2.1) is replaced by b' blocks of the BIB design formed from the contents of the block B, and to these blocks are added n treatments from the column unrepresented in B. Then we get $pmnb'$ new blocks, each of size $k = k' + n$. It follows that these blocks form a GD design with the required parameters (2.7) and the same association scheme as in (2.1). Thus the proof is completed.

By use of the obvious existence of BIB designs, we can present special cases of Theorem 2.2. The existence of a BIB design with parameters $v' = m-1 = b'$, $r' = 1 = k'$, $\lambda' = 0$ yields the following.

COROLLARY 2.5. *The existence of an RGD design with* (2.1) *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn(m-1), \quad r = p(m-1)(n+1), \quad k = n+1\,, \\ &\lambda_1 = pn(m-1), \quad \lambda_2 = 2p\,. \end{aligned} \tag{2.8}$$

The existence of an unreduced BIB design with parameters $v' = m-1$, $b' = \binom{m-1}{2}$, $r' = m-2$, $k' = 2$, $\lambda' = 1$ yields the following.

COROLLARY 2.6. *The existence of an RGD design with* (2.1) *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn\binom{m-1}{2}, \quad r = p(m-1)(m-2)(n+2)/2\,, \\ &k = n+2, \quad \lambda_1 = pn\binom{m-1}{2}, \quad \lambda_2 = p(m-2)(2n+1)/2\,. \end{aligned} \tag{2.9}$$

Remark. When $n = 2$ and $m = 3$, (2.9) is the complement of (2.1).

The existence of a BIB design with parameters $v' = m - 1 = b'$, $r' = m - 2 = k'$, $\lambda' = m - 3$ yields the following.

COROLLARY 2.7. *The existence of a GD design with* (2.1) *implies the existence of an RGD design with parameters*

$$\begin{aligned} &v = mn, \quad b = pmn(m-1), \quad r = p(m-1)(m+n-2)\,, \\ &k = m+n-2, \quad \lambda_1 = pn(m-1), \quad \lambda_2 = p(m-2)(2n+m-3)/n\,. \end{aligned} \tag{2.10}$$

Remark. When $m = 4$, (2.9) and (2.10) yield the same values of parameters. Taking a BIB design with parameters $v' = m - 1 = k'$, $b' = 1 = r' = \lambda'$, we get the design with the same parameters as in (2.6).

3. Designs with $r, k \leq 10$

The methods described in Theorems 2.1 and 2.2 are useful for the combinatorial construction of GD designs, but they may produce designs with relatively large parameter values. In the range r, $k \leq 10$ of much practical value, three examples are here taken up. The references to design numbers are from Clatworthy (1973).

(1) The application of Corollaries 2.6 and 2.2 to designs R 18 and R 144 gives designs R 96 and R 175, respectively.

(2) The application of Corollary 2.1 to the design R 54 gives a new and resolvable solution to the design SR 39 with parameters $v = 8$, $b = 16$, $r = 8$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 4$, $m = 4$, $n = 2$ as follows:

$$[(1,2,3,4)(5,6,7,8)]\;[(1,2,4,7)(3,5,6,8)]\;[(1,3,4,6)(2,5,7,8)]$$
$$[(1,2,3,8)(4,5,6,7)]\;[(1,3,6,8)(2,4,5,7)]\;[(2,3,5,8)(1,4,6,7)]$$
$$[(1,6,7,8)(2,3,4,5)]\;[(1,2,7,8)(3,4,5,6)]\,.$$

It is obvious that the present solution is non-isomorphic, as all its blocks are distinct in comparison to the reported solutions in Clatworthy (1973) which are obtained by duplicating solutions of the design SR 36.

Additional Remark. It is known (see Cheng (1981)) that a GD design with $\lambda_2 = \lambda_1 + 1$ is A-, D- and E-optimum among PBIB designs with the same parameters v, b, r, k. Here we obtain a series of optimum GD designs from Corollary 2.4. However, only a few designs are available.

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