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ON THE CONNECTEDNESS OF PARTIALLY BALANCED BLOCK DESIGNS*

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Abstract. The necessary and sufficient conditions for m-associate partially balanced block (PBB) designs to be connected are given. This generalizes the criterion for m-associate partially balanced incomplete block (PBIB) designs, which has originally been established by Ogawa, Ikeda and Kageyama (1984, *Proceedings of the Seminar on "Combinatorics and Applications"*, 248–255, Statistical Publishing Society, Calcutta).

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1. Introduction

Let $\boldsymbol{B}_i$ $(i = 0, 1, \ldots, m)$ be the $v \times v$ association matrices with the elements $b^i_{\mathscr{L}\mathscr{B}} = b^i_{\mathscr{B}\mathscr{L}}$, where $b^i_{\mathscr{L}\mathscr{B}} = 1$ if $\mathscr{L}$ and $\mathscr{B}$ are i-th associates, and $b^i_{\mathscr{L}\mathscr{B}} = 0$ otherwise $(\mathscr{L}, \mathscr{B} = 1, 2, \ldots, v)$. They are: being symmetric and linearly independent, $\sum_{i=0}^{m} \boldsymbol{B}_i = \mathbf{1}\,\mathbf{1}'$, $\boldsymbol{B}_i\mathbf{1} = n_i\mathbf{1}$ and $\boldsymbol{B}_j\boldsymbol{B}_k = \sum_{i=0}^{m} p^i_{jk}\boldsymbol{B}_i$, where $\mathbf{1}$ is the vector of ones and $\boldsymbol{B}_0$ is the unit matrix $\boldsymbol{I}$ of order v. The numbers v, n_i and p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ are called the parameters of the association scheme: all must be nonnegative integers. It is well known (cf. Bose and Mesner (1959)) that:

$$n_j n_k = \sum_{i=0}^{m} p^i_{jk} n_i \,, \tag{1.1}$$

$$\sum_{k=1}^{m} p^i_{jk} = n_j, \quad \text{if} \quad i \neq j \,, \tag{1.2}$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij} \,, \tag{1.3}$$

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for $i, j, k = 0, 1, \ldots, m$. Moreover, if A is any subset of $\{1, 2, \ldots, m\}$ and $j \notin A$, then

$$\boldsymbol{B}_j \left(\sum_{k \in A} \boldsymbol{B}_k \right) = \sum_{i=1}^{m} \left(\sum_{k \in A} p_{jk}^{i} \right) \boldsymbol{B}_i . \tag{1.4}$$

Let z_{lj} denote the l-th latent root of the matrix $\boldsymbol{P}_j = (p_{ij}^{k})$ $(i, k = 0, 1, \ldots, m)$. It is known (cf. Bose and Mesner (1959)) that

$$z_{lj} z_{lk} = \sum_{i=0}^{m} p_{jk}^{i} z_{li}, \qquad l, j, k = 0, 1, \ldots, m , \tag{1.5}$$

$n_i \geq z_{li}$, $z_{0j} = n_j$, $z_{00} = z_{10} = \cdots = z_{m0} = 1$ and the matrix

$$\boldsymbol{Z} = \begin{bmatrix} z_{00} & z_{10} & \cdots & z_{m0} \\ z_{01} & z_{11} & \cdots & z_{m1} \\ \vdots & \vdots & & \vdots \\ z_{0m} & z_{1m} & \cdots & z_{mm} \end{bmatrix},$$

of order $m + 1$ is nonsingular. Note that z_{lj} is also the l-th latent root of the matrix $\boldsymbol{B}_j$.

Let $\boldsymbol{N}$ denote a $v \times b$ incidence matrix of a partially balanced block (PBB) design based on an association scheme with association matrices $\boldsymbol{B}_i$ in which i-th treatment $(i = 1, 2, \ldots, v)$ occurs r_i times and j-th block $(j = 1, 2, \ldots, b)$ is size k_j. The $\boldsymbol{C}$-matrix of this design can be written as (cf. Kageyama (1974), p. 582)

$$\boldsymbol{C} = \sum_{i=0}^{m} a_i \boldsymbol{B}_i , \tag{1.6}$$

where $a_i \leq 0$, for $i = 1, 2, \ldots, m$. The latent roots of matrix (1.6) are of the form:

$$\mu_0 = \sum_{i=0}^{m} a_i n_i = 0, \qquad \mu_l = \sum_{i=0}^{m} a_i z_{li}, \qquad l = 1, 2, \ldots, m . \tag{1.7}$$

Here, the block design is called connected if and only if (iff) the rank of its $\boldsymbol{C}$-matrix is exactly $v - 1$. Then the PBB design is connected iff $\mu_l > 0$ for all $l \in \{1, 2, \ldots, m\}$. Otherwise, if there exists $l \in \{1, 2, \ldots, m\}$ such that $\mu_l = 0$, then PBB design is called disconnected.

It is known (cf. Kageyama (1974), p. 584) that a PBB design with a constant block size based on an association scheme of m associate classes is a partially balanced incomplete block (PBIB) design based on the same association scheme. Thus, a PBIB design is a special case of PBB designs with $\lambda_i = -ka_i$ $(i = 1, 2, \ldots, m)$, where the block size k and λ_i are parameters of PBIB designs.

The main purpose of this note is to give necessary and sufficient conditions for connectedness of PBB designs. This generalizes the criteria of Ogawa *et al.* (1984), Saha and Kageyama (1984) and Baksalary and Tabis (1987), pertaining to PBIB designs.

2. Results

The following is a general disconnectedness criterion for PBB designs.

THEOREM 2.1. *An m-associate PBB design with* $a_1 = a_2 = \cdots = a_s = 0 > a_{s+1}, a_{s+2}, \ldots, a_m$ *is disconnected iff there exists a nonempty subset* A_* *of* $\{1, 2, \ldots, s\}$ *such that*

$$p^i_{jk} = 0, \quad \forall i \in A_*, \quad \forall j, k \in \{1, \ldots, m\} - A_* . \tag{2.1}$$

PROOF OF NECESSITY. If a PBB design is disconnected, then $\mu_l = 0$, for some $l \in \{1, \ldots, m\}$. From this and (1.7) and from $n_0 - z_{l0} = 0$ we have $\sum_{i=1}^{m} a_i(n_i - z_{li}) = 0$, which implies that $n_i = z_{li}$ for $i \in \{s+1, \ldots, m\}$. Let's define $A_* = \{i \in \{1, 2, \ldots, s\} : n_i > z_{li}\}$ and $A_{**} = \{i \in \{1, 2, \ldots, s\} : n_i = z_{li}\}$. From the nonsingularity of matrix $\boldsymbol{Z}$ follows nonemptiness of set A_*, however, set A_{**} can be empty. Note that A_{**} is empty iff $A_* = \{1, 2, \ldots, s\}$, and that $\{1, 2, \ldots, m\} - A_* = \{s+1, \ldots, m\} \cup A_{**}$. In general one can write that $n_i = z_{li}$ for $i \in \{s+1, \ldots, m\} \cup A_{**}$. From (1.1) and (1.5) we obtain $\sum_{i=0}^{m} p^i_{jk}(n_i - z_{li}) = 0$ for $j, k \in \{s+1, \ldots, m\} \cup A_{**}$, since $n_i = z_{li}$ for $i \notin A_*$, therefore one continues $\sum_{i \in A_*} p^i_{jk}(n_i - z_{li}) = 0$ which leads, in particular, to (2.1).

PROOF OF SUFFICIENCY. From (2.1) and (1.2) we have

$$\sum_{k \in A_*} p^i_{jk} = n_j, \quad \forall j \in \{s+1, \ldots, m\}, \quad \forall i \in A_* , \tag{2.2}$$

and (2.1) and (1.3) give $\sum_{k \in A_*} p^i_{jk} = 0$ for $i \in \{1, 2, \ldots, m\} - A_* \cup \{0\}$. This and (1.4) lead to $(\boldsymbol{B}_j - n_j\boldsymbol{I}) \sum_{k \in A_*} \boldsymbol{B}_k = \boldsymbol{0}$.

Hence and from the obvious equality $\boldsymbol{C} = \sum_{j=0}^{m} a_j(\boldsymbol{B}_j - n_j\boldsymbol{I})$ we obtain

$$C \sum_{k \in A_*} \boldsymbol{B}_k = \boldsymbol{0}\,. \tag{2.3}$$

Since $r\left(\sum_{k \in A_*} \boldsymbol{B}_k\right) \geq 2$ it follows from (2.3) that $r(\boldsymbol{C}) \leq v - 2$ ($v > 2$ being a natural condition that should be added somewhere), which completes the proof.

The connectedness criterion is obtained as a consequence of Theorem 2.1.

THEOREM 2.2. *An m-associate PBB design with* $a_1 = a_2 = \cdots = a_s = 0 > a_{s+1}, a_{s+2}, \ldots, a_m$ *is connected iff for every nonempty subset* A_* *of* $\{1, 2, \ldots, s\}$ *such that* $p^i_{jk} > 0$ *for some* $i \in A_*$, *and* $j, k \in \{1, 2, \ldots, m\} - A_*$.

Note that the necessity part of the above theorem follows from the sufficiency part of Theorem 2.1. Similarly the sufficiency part follows from the necessity part of Theorem 2.1.

Necessary and sufficient conditions of connectedness occurring in Theorem 2.2 are very helpful in the construction of PBB designs when the incidence matrix is looked for. One can even state hypothetical connectedness (maybe nonexistent) of PBB designs depending on the given set of values a_i and p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$. We can arrive in this respect at interesting conclusions which can be exemplified by Corollary 2.3. Of course in a situation where incidence matrix of PBB design is known, Theorem 2.2 can not be applied since the extremely simple characterization of connectedness is as follows:

The block design is connected, if it is possible to construct a chain of treatments $i = i_0, i_1, i_2, \ldots, i_n = j$ such that every consecutive pair of treatments in the chain occurs together in a block.

Theorem 2.2 asserts, in particular, the following

COROLLARY 2.1. *An m-associate PBIB design with* $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 0 < \lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_m$ *is connected iff for every nonempty subset* A_* *of* $\{1, 2, \ldots, s\}$ *such that* $p^i_{jk} > 0$ *for some* $i \in A_*$, *and some* $j, k \in \{1, 2, \ldots, m\} - A_*$.

This corollary is an alternative formulation of the criteria established by Ogawa *et al.* (1984), Saha and Kageyama (1984) and Baksalary and Tabis (1987).

If the matrix $\boldsymbol{Z}$ is known, very useful is

COROLLARY 2.2. *An m-associate PBB design with* $a_1 = a_2 = \cdots = a_s = 0 > a_{s+1}, a_{s+2}, \ldots, a_m$ *is disconnected iff there exists* $l \in \{1, 2, \ldots, m\}$ *such* *that $n_i = z_{li}$ for every $i \in \{s+1,\dots,m\}$. This design is connected iff for every $l \in \{1,2,\dots,m\}$ there exists $i \in \{s+1,\dots,m\}$ such that $n_i > z_{li}$.*

Proof of disconnectedness is contained in proof of Theorem 2.1, proof of connectedness is analogical.

This criterion is then applied to three-associate PBB designs based on the rectangular association scheme with $v = mn$ symbols introduced by Vartak (1955). For this scheme matrix $\boldsymbol{Z}$ is of the following form:

$$\boldsymbol{Z} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ n-1 & -1 & n-1 & -1 \\ m-1 & m-1 & -1 & -1 \\ (n-1)(m-1) & 1-m & 1-n & 1 \end{bmatrix}.$$

COROLLARY 2.3. *Three-associate PBB designs, based on the rectangular association scheme with $v = mn$ symbols, are connected except for the following cases*:

(i) $a_1 = a_2 = 0$ *and* $m = n = 2$,
(ii) $a_1 = a_3 = 0$,
(iii) $a_2 = a_3 = 0$.

PROOF. In the case of (i), we have $\{s+1,\dots,m\} = \{3\}$ and $z_{l3} = (n-1)(m-1) = n_3 = 1$ for $l = 3$ and $m = n = 2$. In the case of (ii), where $\{s+1,\dots,m\} = \{2\}$, we have $z_{l2} = n_2 = m-1$ for $l = 1$. In the case of (iii), we have $\{s+1,\dots,m\} = \{1\}$ and $z_{l1} = n_1 = n-1$ for $l = 2$.

Finally, we give an example of an application of Theorem 2.1.

COROLLARY 2.4. *Two-associate PBB designs are disconnected only in the following cases*:

(i) $a_1 = 0 > a_2$ *and* $p^1_{22} = 0$,
(ii) $a_2 = 0 > a_1$ *and* $p^2_{11} = 0$.

The proof of (i) follows from Theorem 2.1 with $m = 2$ and $A_* = \{1\}$. The proof of (ii) is similar to the above, after interchanging the first associates with the second associates.

None of the following designs fulfil condition (i): group divisible (GD), triangular with $n \geq 5$, Latin square type $L_i(s)$, cyclic and the design based on partial geometries. Condition (ii) is fulfilled by each GD design with $v = mn$ (m (≥ 2) groups of n (≥ 2) treatments each), when $a_2 = 0 > a_1$.

REFERENCES

Baksalary, J. K. and Tabis, Z. (1987). A note on connectedness of PBIB designs, *Canad. J. Math.*, **15**, 147–150.

Bose, R. C. and Mesner, D. M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs, *Ann. Math. Statist.*, **30**, 21–38.

Kageyama, S. (1974). Reduction of associate classes for block designs and related combinatorial arrangements, *Hiroshima Math. J.*, **4**, 527–618.

Ogawa, J., Ikeda, S. and Kageyama, S. (1984). Connectedness of PBIB designs with applications, *Proc. Seminar on "Combinatorics and Applications" in Honour of Professor S. S. Shrikhande on His 65th Birthday*, December 14–17, 1982, Indian Statistical Institute, 248–255, Statistical Publishing Society, Calcutta, India.

Saha, G. M. and Kageyama. S. (1984). Connectedness-conditions of PBIB designs, *J. Japan Statist. Soc.*, **14**, 169–178.

Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs, *Ann. Math. Statist.*, **26**, 420–438.