Ann. Inst. Statist. Math.
Vol. 41, No. 1, 187-197 (1989)

NORMALIZING TRANSFORMATIONS OF SOME STATISTICS OF GAUSSIAN ARMA PROCESSES*

M. TANIGUCHI**, P. R. KRISHNAIAH AND R. CHAO

Center for Multivariate Analysis, University of Pittsburgh, Fifth Floor, Thackeray Hall, Pittsburgh, PA 15260, U.S.A.

(Received March 24, 1988; revised July 1, 1988)

Abstract. In this paper, the authors investigate Edgeworth type expansions of certain transformations of some statistics of Gaussian ARMA processes. They also investigate transformations which will make the second order part of the Edgeworth expansions vanish. Some numerical studies are made and they show that the above transformations give better approximations than the usual approximation.

Key words and phrases: Edgeworth expansion, Fisher's z-transformation, Gaussian ARMA process, maximum likelihood estimator, periodogram, quasi-maximum likelihood estimator, spectral density.

1. Introduction

In the area of multivariate analysis several authors have considered transformations of statistics which are based on functions of the elements of sample covariance matrix, and derived the Edgeworth expansions of the transformed statistics. Konishi (1978) gave a transformation of the sample correlation coefficient which extinguishes a part of the second order terms of the Edgeworth expansion. Also, Konishi (1981) discussed the transformations of a statistic based on the elements of the sample covariance matrix which extinguish the second order terms of the Edgeworth expansions. Furthermore Fang and Krishnaiah (1982) gave Edgeworth expansions of certain functions of the elements of noncentral Wishart matrix; they also obtained analogous results for functions of the elements of the sample covariance matrix when the underlying distribution is a mixture of

*This work is supported by Contract N00014-K-0292 of the Office of Naval Research and Contract F49620-85-C-0008 of the Air Force Office of Scientific Research. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

**The work of this author was done at the Center for Multivariate Analysis. His permanent address is Department of Mathematics, Hiroshima University, Naka-ku, Hiroshima 730, Japan.

multivariate distributions.

The object of this paper is to study the accuracy of Edgeworth type expansions of certain transformations of some statistics which arise in time series. The results in this paper are useful in drawing inferences of the parameters of the spectral density when the model is additive and the observations consist of the sum of noise and signal components and these signals form ARMA processes. A description of the contents of this paper is given below.

Let $\{X_t\}$ be a Gaussian ARMA process with the spectral density $f_\theta(\lambda)$, where θ is an unknown parameter. Suppose that a stretch $\boldsymbol{X}_T = (X_1, \dots, X_T)'$ of $\{X_t\}$ is available. Taniguchi (1983, 1986) gave the Edgeworth expansions of the maximum likelihood estimator $\hat{\theta}_{ML}$ and the quasi-maximum likelihood estimator $\hat{\theta}_{qML}$ of θ. Suppose that a function $g(\theta)$ is smooth with respect to θ. Also, let

$$V_T = \sqrt{TI(\theta)}\left\{g(\hat{\theta}_*) - g(\theta) - \frac{c}{T}\right\}\Big/ g'(\theta),$$

where $\hat{\theta}_*$ is the $\hat{\theta}_{ML}$ or $\hat{\theta}_{qML}$, and $I(\theta)$ is the Fisher information, and c is a constant. Then we can give the Edgeworth expansion of V_T such that

$$\text{(1.1)} \qquad P_\theta^T\{V_T \leq x\} = \Phi(x) - \frac{1}{\sqrt{T}}\phi(x)\{a_1(g', g'', c, \theta)x^2 + a_2(g', g'', c, \theta)\} + O(T^{-1}),$$

where $\Phi(x)$ and $\phi(x)$ are the standard normal distribution function and its first derivative, respectively. Then we set

$$\text{(1.2)} \qquad a_1(g', g'', c, \theta) = 0,$$

$$\text{(1.3)} \qquad a_2(g', g'', c, \theta) = 0.$$

Solving the above differential equations we can give the normalizing transformation g and the constant c which make

$$\text{(1.4)} \qquad P_\theta^T\{V_T \leq x\} = \Phi(x) + O(T^{-1}).$$

Some interesting examples will be given. Suppose that $\{X_t\}$ is an ARMA (p, q) process with spectral density

$$\text{(1.5)} \qquad f_\theta(\lambda) = \frac{\sigma^2}{2\pi} \frac{\prod_{j=1}^{q}(1 - \beta_j e^{i\lambda})(1 - \beta_j e^{-i\lambda})}{\prod_{j=1}^{p}(1 - \alpha_j e^{i\lambda})(1 - \alpha_j e^{-i\lambda})},$$

where $\alpha_1,\ldots,\alpha_p$, $\beta_1,\ldots,\beta_q$ are real numbers such that $|\alpha_j| < 1$, $j = 1,\ldots,p$, $|\beta_j| < 1, j = 1,\ldots, q$. Suppose that α_k is an unknown parameter (i.e., $\theta = \alpha_k$), and that $\alpha_1,\ldots,\alpha_{k-1}$, $\alpha_{k+1},\ldots,\alpha_p$, $\beta_1,\ldots,\beta_q$ are known parameters. Then for the maximum likelihood estimator $\hat{\alpha}_{k,ML}$ of α_k, the differential equations (1.2) and (1.3) lead to

$$g(\alpha_k) = \frac{1}{2} \log \{(1 + \alpha_k)/(1 - \alpha_k)\}\,, \tag{1.6}$$

which is Fisher's z-transformation and gives the normalizing transformation. As for the maximum likelihood estimators $\hat{\beta}_{k,ML}$ $(k = 1,\ldots, q)$ and $\hat{\sigma}^2_{ML}$ of β_k $(k = 1,\ldots, q)$ and σ^2, respectively, it will be shown that

$$g(\beta_k) = \beta_k \;\; (k = 1,\ldots, q) \qquad \text{and} \qquad g(\sigma^2) = \{\sigma^2\}^{1/3}\,,$$

give the normalizing transformations of $\hat{\beta}_{k,ML}$ $(k = 1,\ldots, q)$ and $\hat{\sigma}^2_{ML}$, respectively. Also, for the quasi-maximum likelihood estimators the normalizing transformations will be given. In Section 4, the results of some numerical studies will be given.

2. Preliminaries

We introduce $\mathscr{D}_\Delta$ and $\mathscr{D}_{\text{ARMA}}$, spaces of functions on $[-\pi,\pi]$ defined by

$$\mathscr{D}_\Delta = \left\{ f\colon f(\lambda) = \sum_{u=-\infty}^{\infty} a(u) \exp(-iu\lambda),\; a(u) = a(-u),\right.$$
$$\left. \sum_{u=-\infty}^{\infty} |u|\; |a(u)| < \infty \right\},$$

$$\mathscr{D}_{\text{ARMA}} = \left\{ f\colon f(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\sum_{j=0}^{q} a_j e^{ij\lambda}\right|^2}{\left|\sum_{j=0}^{p} b_j e^{ij\lambda}\right|^2}\,,\; (\sigma^2 > 0) \right\}.$$

In this latter expression p and q are positive integers, and $A(z) = \sum_{j=0}^{q} a_j z^j$ and $B(z) = \sum_{j=0}^{p} b_j z^j$ are both bounded away from zero for $|z| \le 1$.

We make the following assumptions.

ASSUMPTION 1. $\{X_t\}$ is a Gaussian stationary process with the spectral density $f_\theta(\lambda) \in \mathscr{D}_{\text{ARMA}}$, $\theta \in \operatorname{Int} \Theta \subset \boldsymbol{R}^1$, and mean 0, where Θ is a compact set of $\boldsymbol{R}^1$.

ASSUMPTION 2. The spectral density $f_\theta(\lambda)$ is continuously three times differentiable with respect to θ, and the derivatives $\partial f_\theta/\partial\theta$, $\partial^2 f_\theta/\partial\theta^2$ and $\partial^3 f_\theta/\partial\theta^3$ belong to $\mathscr{D}_\Delta$.

ASSUMPTION 3. If $\theta_1 \neq \theta_2$, then $f_{\theta_1} \neq f_{\theta_2}$ on a set of positive Lebesgue measure.

ASSUMPTION 4.

$$I(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta}\log f_\theta(\lambda)\right\}^2 d\lambda > 0 .$$

Suppose that a stretch $\boldsymbol{X}_T = (X_1,\ldots,X_T)'$ of the series $\{X_t\}$ is available. Let Σ_T be the covariance matrix of $\boldsymbol{X}_T$. The likelihood function based on $\boldsymbol{X}_T$ is given by

$$L(\theta) = (2\pi)^{-T/2}|\Sigma_T|^{-1/2}\exp\left(-1/2\,\boldsymbol{X}_T'\Sigma_T^{-1}\boldsymbol{X}_T\right). \tag{2.1}$$

We define the maximum likelihood estimator $\hat\theta_{ML}$ of θ by a value which maximizes $L(\theta)$ with respect to $\theta \in \Theta$. Now we denote

$$\begin{aligned} J(\theta) = &-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}^3\{f_\theta(\lambda)^{-3}\}d\lambda \\ &+\frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial^2}{\partial\theta^2} f_\theta(\lambda)\right\}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}\{f_\theta(\lambda)\}^{-2}d\lambda ,\end{aligned}$$

and

$$K(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}^3\{f_\theta(\lambda)\}^{-3}d\lambda .$$

Then, from Taniguchi (1983, 1986) we have,

LEMMA 2.1. *Under Assumptions* 1–4,

$$\begin{aligned} &P_\theta^T\{\sqrt{TI(\theta)}\,(\hat\theta_{ML}-\theta)\le x\} \\ &\quad = \Phi(x) - \phi(x)\left\{\frac{\alpha_1}{\sqrt{T}} + \frac{\gamma_1}{6\sqrt{T}}(x^2-1)\right\} + O(T^{-1}) ,\end{aligned}$$

where

$$\alpha_1 = -\frac{J(\theta)+K(\theta)}{2I(\theta)^{3/2}}, \quad \gamma_1 = -\frac{3J(\theta)+2K(\theta)}{I(\theta)^{3/2}},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\exp -\frac{x^2}{2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt .$$

As another estimator of θ we use the quasi-maximum likelihood estimator $\hat{\theta}_{qML}$ which maximizes the following quasi-likelihood

$$l_T(\theta) = -\frac{1}{2}\sum_{j=0}^{T-1} \{ \log f_\theta(\lambda_j) + I_T(\lambda_j)/f_\theta(\lambda_j)\} ,$$

with respect to θ, where $\lambda_j = 2\pi j/T$, and

$$I_T(\lambda_j) = \frac{1}{2\pi T}\left| \sum_{t=1}^{T} X_t e^{-it\lambda_j}\right|^2 .$$

We set down

$$B(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial\theta} f_\theta(\lambda)\right\} b_\theta(\lambda)\{f_\theta(\lambda)\}^{-2} d\lambda ,$$

$$b_\theta(\lambda) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} |n|\gamma(n)e^{in\lambda} ,$$

where $\gamma(n) = E_\theta(X_t X_{t+n})$. From Taniguchi (1983) we can get the following lemma.

LEMMA 2.2. *Under Assumptions* 1–4,

$$\begin{aligned} &P_\theta^T\{\sqrt{TI(\theta)}\,(\hat{\theta}_{qML}-\theta) \le x\} \\ &\quad = \Phi(x) - \phi(x)\left\{ \frac{\alpha_1'}{\sqrt{T}} + \frac{\gamma_1'}{6\sqrt{T}}(x^2-1)\right\} + O(T^{-1}) , \end{aligned}$$

where

$$\alpha_1' = -\frac{B(\theta)}{I(\theta)^{1/2}} - \frac{J(\theta)+K(\theta)}{2I(\theta)^{3/2}} ,$$

$$\gamma_1' = -\frac{3J(\theta)+2K(\theta)}{I(\theta)^{3/2}} .$$

3. Normalizing transformations

In this section we seek transformations of $\hat{\theta}_{ML}$ and $\hat{\theta}_{qML}$ which make the second-order terms of the Edgeworth expansions vanish. Let $g(\theta)$ be a three times continuously differentiable function. Note that

$$(3.1)\quad V_T^{(*)} = \sqrt{TI(\theta)}\left\{ g(\hat{\theta}_*) - g(\theta) - \frac{c}{T} \right\} \Big/ g'(\theta)$$

$$= \sqrt{TI(\theta)}\left\{ (\hat{\theta}_* - \theta)g'(\theta) + \frac{1}{2}(\hat{\theta}_* - \theta)^2 g''(\theta) - \frac{c}{T} \right\} \Big/ g'(\theta)$$

$$+ \text{higher order terms},$$

where $\hat{\theta}_*$ is $\hat{\theta}_{ML}$ or $\hat{\theta}_{qML}$, and c is a constant. Then from (3.1) and Lemmas 2.1 and 2.2, it is not difficult to show the following theorem (see Taniguchi (1986)).

THEOREM 3.1. *Under Assumptions* 1–4,

$$(3.2)\quad P_\theta^T\left[\sqrt{TI(\theta)}\left\{ g(\hat{\theta}_{ML}) - g(\theta) - \frac{c}{T} \right\} \Big/ g'(\theta) \le x \right]$$

$$= \Phi(x) - \phi(x)\left[\frac{1}{6\sqrt{T}}\left\{ -\frac{3J(\theta) + 2K(\theta)}{I(\theta)^{3/2}} + \frac{3g''(\theta)}{g'(\theta)I(\theta)^{1/2}} \right\} x^2 \right.$$

$$\left. + \frac{1}{\sqrt{T}}\left\{ -\frac{K(\theta)}{6I(\theta)^{3/2}} - \frac{cI(\theta)^{1/2}}{g'(\theta)} \right\} \right] + O(T^{-1}).$$

COROLLARY 3.1. *Under Assumptions* 1–4, *if* $g_0(\theta)$ *and* c_0 *satisfy*

$$(3.3)\quad g_0''(\theta)/g_0'(\theta) = \frac{3J(\theta) + 2K(\theta)}{3I(\theta)},$$

and

$$(3.4)\quad c_0 = -\frac{K(\theta)g_0'(\theta)}{6I(\theta)^2},$$

then

$$(3.5)\quad P_\theta^T\left[\sqrt{TI(\theta)}\left\{ g_0(\hat{\theta}_{ML}) - g_0(\theta) - \frac{c_0}{T} \right\} \Big/ g_0'(\theta) \le x \right] = \Phi(x) + O(T^{-1}).$$

We seek the function $g_0(\cdot)$ and the constant c_0 which satisfy (3.3) and (3.4). For ARMA (p, q) process with the spectral density (1.5), we have

$$I(\alpha_j) = \frac{1}{1-\alpha_j^2}, \quad K(\alpha_j) = \frac{6\alpha_j}{(1-\alpha_j^2)^2}, \quad J(\alpha_j) = \frac{-2\alpha_j}{(1-\alpha_j^2)^2}, \quad (j = 1,\dots,p),$$

$$I(\beta_j) = \frac{1}{1-\beta_j^2}, \quad K(\beta_j) = \frac{-6\beta_j}{(1-\beta_j^2)^2}, \quad J(\beta_j) = \frac{4\beta_j}{(1-\beta_j^2)^2}, \quad (j = 1,\dots,q), \tag{3.6}$$

$$I(\sigma^2) = \frac{1}{2\sigma^4}, \quad K(\sigma^2) = \frac{1}{\sigma^6}, \quad J(\sigma^2) = -\frac{1}{\sigma^6},$$

(see Taniguchi (1983)). Let $\hat{\alpha}_{k,ML}$, $\hat{\beta}_{k,ML}$ and $\hat{\sigma}^2_{ML}$ be the maximum likelihood estimators of α_k, β_k and σ^2, respectively.

Case 1. If $\theta = \alpha_k$, then

$$g_0(\alpha_k) = \frac{1}{2} \log \{(1+\alpha_k)/(1-\alpha_k)\},$$

and

$$c_0 = -\frac{\alpha_k}{(1-\alpha_k^2)}.$$

That is

$$P_\alpha^T\left[\sqrt{T(1-\alpha_k^2)}\left\{\frac{1}{2}\log\{(1+\hat{\alpha}_{k,ML})/(1-\hat{\alpha}_{k,ML})\} - \frac{1}{2}\log\{(1+\alpha_k)/(1-\alpha_k)\} + \frac{\alpha_k}{T(1-\alpha_k^2)}\right\} \le x\right] = \Phi(x) + O(T^{-1}). \tag{3.7}$$

Case 2. If $\theta = \beta_k$, then

$$g_0(\beta_k) = \beta_k,$$

and

$$c_0 = \beta_k .$$

That is

$$\text{(3.8)} \qquad P_\beta^T\left[\sqrt{\frac{T}{1-\beta_k^2}}\left(\hat{\beta}_{k,ML} - \beta_k - \frac{\beta_k}{T}\right) \le x\right] = \Phi(x) + O(T^{-1}) .$$

Case 3. If $\theta = \sigma^2$, then

$$g_0(\sigma^2) = \{\sigma^2\}^{1/3} ,$$

and

$$c_0 = -\frac{2}{9}\sigma^{2/3} .$$

That is

$$\text{(3.9)} \qquad P_{\sigma^2}^T\left[\frac{3}{\sqrt{2}}\frac{\sqrt{T}}{\sigma^{2/3}}\left\{(\hat{\sigma}_{ML}^2)^{1/3} - (\sigma^2)^{1/3} + \frac{2\sigma^{2/3}}{9T}\right\} \le x\right] = \Phi(x) + O(T^{-1}) .$$

As for the quasi-maximum likelihood estimators, we similarly have the following theorem.

THEOREM 3.2. *Under Assumptions* 1–4,

$$\text{(3.10)} \qquad P_\theta^T\left[\sqrt{TI(\theta)}\left\{g(\hat{\theta}_{qML}) - g(\theta) - \frac{c}{T}\right\}\Big/ g'(\theta) \le x\right]$$

$$= \Phi(x) - \phi(x)\left[\frac{1}{\sqrt{T}}\left\{-\frac{B(\theta)}{I(\theta)^{1/2}} - \frac{K(\theta)}{6I(\theta)^{3/2}} - \frac{cI(\theta)^{1/2}}{g'(\theta)}\right\} + \frac{1}{6\sqrt{T}}\left\{-\frac{3J(\theta)+2K(\theta)}{I(\theta)^{3/2}} + \frac{3g''(\theta)}{g'(\theta)I(\theta)^{1/2}}\right\}x^2\right]$$

$$+ O(T^{-1}) .$$

COROLLARY 3.2. *Under Assumptions* 1–4, *if* $g_1(\theta)$ *and* c_1 *satisfy*

$$\text{(3.11)} \qquad g_1''(\theta)/g_1'(\theta) = \frac{3J(\theta)+2K(\theta)}{3I(\theta)} ,$$

and

$$c_1 = -g_1'(\theta)\left\{\frac{B(\theta)}{I(\theta)} + \frac{K(\theta)}{6I(\theta)^2}\right\}, \tag{3.12}$$

then

$$P_\theta^T\left[\sqrt{TI(\theta)}\left\{g_1(\hat{\theta}_{qML}) - g_1(\theta) - \frac{c_1}{T}\right\}\Big/ g_1'(\theta) \le x\right] \tag{3.13}$$
$$= \Phi(x) + O(T^{-1}).$$

By (3.3) and (3.11) we can see $g_0(\theta) = g_1(\theta)$. Thus for the quasi-maximum likelihood estimators, the same transformations as the maximum likelihood estimators give the normalizing transformation. Also we can seek the constant c_1 which satisfies (3.12). Since evaluations of $B(\theta)$ are difficult for general ARMA (p, q) processes, we consider the ARMA $(1, 1)$ process with the spectral density

$$f_\theta(\lambda) = \frac{\sigma^2}{2\pi}\frac{|1 - \beta e^{i\lambda}|^2}{|1 - \alpha e^{i\lambda}|^2}.$$

Thus we have

$$B(\alpha) = \frac{(\alpha - \beta)(1 - 2\alpha\beta + \beta^2)}{(1 - \alpha^2)(1 - \alpha\beta)(1 - \beta^2)},$$

$$B(\beta) = \frac{(\beta - \alpha)(1 + \beta^2 - 2\alpha\beta - \alpha^2 + 3\alpha^2\beta^3 - 2\alpha\beta^3)}{(1 - \beta^2)^2(1 - \alpha\beta)(1 - \alpha^2)},$$

$$B(\sigma^2) = -\frac{(\alpha - \beta)^2}{\sigma^2(1 - \alpha^2)(1 - \beta^2)},$$

(see Taniguchi (1983)), so we can give the explicit forms of $c_1 = -g'(\theta)\cdot\{B(\theta)/I(\theta) + K(\theta)/6I(\theta)^2\}$ for $\theta = \alpha, \beta$ and σ^2.

4. Numerical comparisons

In this section we give some numerical comparisons related to the approximation (3.7) in an autoregressive process $X_t = \alpha X_{t-1} + \varepsilon_t$, where ε_t are i.i.d. $N(0, \sigma^2)$. Let

$$\tilde{\alpha}_{ML} = (1 - T^{-1})\frac{\sum_{t=1}^{T-1} X_t X_{t+1}}{\sum_{t=2}^{T-1} X_t^2}.$$

Table 1. Values of $L(\alpha, x)$, $R(\alpha, x)$ and $M(\alpha, x)$ for $T = 300$, $\alpha = -0.9(0.3)0.9$.

α \ x	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0
$L(-0.9, x)$	.0217	.0524	.0822	.0907	.0767	.0818	.0885	.0575	.0512
$R(-0.9, x)$	.0051	.0176	.0278	.0403	.0327	.0288	.0257	.0012	.0040
$M(-0.9, x)$	.0195	.0438	.0504	.0431	.0338	.0450	.0647	.0413	.0406
$L(-0.6, x)$	.0109	.0250	.0252	.0279	.0241	.0290	.0239	.0201	.0156
$R(-0.6, x)$	.0005	.0070	.0034	.0065	.0078	.0084	.0053	.0021	.0022
$M(-0.6, x)$	.0091	.0188	.0126	.0103	.0079	.0140	.0155	.0149	.0124
$L(-0.3, x)$	.0049	.0106	.0092	.0167	.0083	.0190	.0109	.0097	.0096
$R(-0.3, x)$	.0001	.0040	.0014	.0055	.0018	.0122	.0007	.0027	.0054
$M(-0.3, x)$	.0041	.0086	.0046	.0077	.0018	.0148	.0069	.0083	.0086
$L(0.0, x)$	.0004	.0001	.0066	.0052	.0137	.0101	.0037	.0049	.0030
$R(0.0, x)$	.0006	.0011	.0064	.0052	.0137	.0099	.0041	.0055	.0038
$M(0.0, x)$	.0006	.0009	.0060	.0058	.0137	.0093	.0047	.0053	.0030
$L(0.3, x)$	.0076	.0075	.0073	.0066	.0145	.0041	.0070	.0066	.0075
$R(0.3, x)$	.0044	.0004	.0016	.0011	.0072	.0016	.0021	.0003	.0039
$M(0.3, x)$	.0068	.0061	.0033	.0008	.0072	.0008	.0026	.0038	.0067
$L(0.6, x)$	.0156	.0173	.0283	.0220	.0180	.0207	.0260	.0206	.0139
$R(0.6, x)$	.0003	.0044	.0081	.0042	.0004	.0011	.0038	.0058	.0027
$M(0.6, x)$	.0116	.0107	.0203	.0090	.0002	.0037	.0142	.0158	.0119
$L(0.9, x)$	.0444	.0665	.0839	.0724	.0856	.0859	.0742	.0530	.0205
$R(0.9, x)$	.0030	.0043	.0143	.0194	.0411	.0327	.0248	.0206	.0069
$M(0.9, x)$	.0348	.0497	.0549	.0378	.0428	.0383	.0442	.0418	.0201

It is known (Fujikoshi and Ochi (1984)) that

$$P_\alpha^T\{\sqrt{TI(\alpha)}\,(\hat{\alpha}_{ML} - \alpha) \le x\} - P_\alpha^T\{\sqrt{TI(\alpha)}\,(\tilde{\alpha}_{ML} - \alpha) \le x\} = o(T^{-1}) .$$

Thus henceforth we use $\tilde{\alpha}_{ML}$ in place of the exact maximum likelihood estimator $\hat{\alpha}_{ML}$. Let

$$L(\alpha, x) = \left| P_\alpha^T\left[\sqrt{\frac{T}{1-\alpha^2}}\,(\tilde{\alpha}_{ML} - \alpha) \le x \right] - \Phi(x) \right| ,$$

$$R(\alpha, x) = \left| P_\alpha^T\left[\sqrt{T(1-\alpha^2)} \left\{ \frac{1}{2} \log \{(1+\tilde{\alpha}_{ML})/(1-\tilde{\alpha}_{ML})\} - \frac{1}{2} \log \{(1+\alpha)/(1-\alpha)\} + \frac{\alpha}{T(1-\alpha^2)} \right\} \le x \right] - \Phi(x) \right| ,$$

and

$$M(\alpha, x) = \left| P_\alpha^T\left[\sqrt{T(1-\tilde{\alpha}_{ML}^2)} \left\{ \frac{1}{2} \log \{(1+\tilde{\alpha}_{ML})/(1-\tilde{\alpha}_{ML})\} - \frac{1}{2} \log \{(1+\alpha)/(1-\alpha)\} + \frac{\tilde{\alpha}_{ML}}{T(1-\tilde{\alpha}_{ML}^2)} \right\} \le x \right] - \Phi(x) \right| .$$

Here the probabilities $P^T(\cdot)$ are computed by 5000 trials simulation. Table 1 gives the values of $L(\alpha, x)$, $R(\alpha, x)$ and $M(\alpha, x)$ for $T = 300$, $\alpha = -0.9(0.3)0.9$ and $x = -2.0(0.5)2.0$. From the table we observe that the transformations proposed by us give better approximations than the usual normal approximations even if the normalizing factor is estimated.

REFERENCES

Fang, C. and Krishnaiah, P. R. (1982). Asymptotic joint distributions of the elements of sample covariance matrix, *Statistics and Probability: Essays in Honor of C. R. Rao*, 249–262, North-Holland, New York-Amsterdam.

Fujikoshi, Y. and Ochi, Y. (1984). Asymptotic properties of the maximum likelihood estimate in the first order autoregressive process, *Ann. Inst. Statist. Math.*, **36**, 119–128.

Konishi, S. (1978). An approximation to the distribution of the sample correlation coefficient, *Biometrika*, **65**, 654–656.

Konishi, S. (1981). Normalizing transformations of some statistics in multivariate analysis, *Biometrika*, **68**, 647–651.

Taniguchi, M. (1983). On the second order asymptotic efficiency of estimators of Gaussian ARMA processes, *Ann. Statist*, **11**, 157–169.

Taniguchi, M. (1986). Third order asymptotic properties of maximum likelihood estimators for Gaussian ARMA processes, *J. Multivariate Anal.*, **18**, 1–31.