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ON A ZERO-CROSSING PROBABILITY

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Abstract. Let $\{X(t), 0 < t < \infty\}$ be a compound Poisson process so that $E\{\exp(-sX(t))\} = \exp(-t\Phi(s))$, where $\Phi(s) = \lambda(1 - \phi(s))$, λ is the intensity of the Poisson process, and $\varphi(s)$ is the Laplace transform of the distribution of nonnegative jumps. Consider the zero-crossing probability $\theta = P\{X(t) - t = 0 \text{ for some } t, 0 < t < \infty\}$. We show that $\theta = \Phi'(\omega)$ where ω is the largest nonnegative root of the equation $\Phi(s) = s$. It is conjectured that this result holds more generally for any stochastic process with stationary independent increments and with sample paths that are nondecreasing step functions vanishing at 0.

Key words and phrases: Ballot theorem, compound Poisson process.

1. The result

Let $\{X(t), 0 < t < \infty\}$ be a separable stochastic process, with stationary independent increments and for which almost all sample paths are nondecreasing step functions vanishing at 0. We shall say that $X(t)$ crosses the line $y = t$ at t_0 if $X(t_0) = t_0$. Since the sample paths of $X(t)$ are nondecreasing step functions, $X(t)$ can only cross the line $y = t$ from above. Consider the zero-crossing probability

$$\theta = P\{X(t) \text{ crosses the line } y = t \text{ for some } t, 0 < t < \infty\}.$$

We shall write it alternatively as

$$\theta = P\{X(t) - t = 0 \text{ for some } t, 0 < t < \infty\}.$$

Let

$$E\{X(t)\} = \rho t, \qquad 0 \leq \rho \leq \infty.$$

If $\rho < 1$, it is known that $\theta = \rho$ (Takacs (1967)). In this case, the probability that there is no crossing equals the probability that $X(t)$ lies entirely below the line $y = t$. For, if $X(t)$ jumps above the line $y = t$ at some point, it must eventually cross the line to go below since $X(t)/t \to \rho < 1$ a.s. as $t \to \infty$. Therefore,

$$\theta = 1 - P\{X(t) < t \text{ for all } t, 0 < t < \infty\} .$$

The elegant, classical ballot theorem can then be used to show that

$$P\{X(t) < t \text{ for all } t, 0 < t < \infty\} = 1 - \rho .$$

It appears that similar arguments cannot be used when $\rho > 1$, and we have not found the analogous result for this case in the literature.

Let

$$E\{\exp(-sX(t))\} = \exp(-t\Phi(s)) \quad \text{for} \quad s \geq 0 ,$$

where

$$\Phi(s) = \int_{+0}^{\infty} (1 - e^{-sx}) dN(x) ,$$

and $N(x)$, $0 < x < \infty$, is a nondecreasing function for which $\lim_{x \to \infty} N(x) = 0$ and

$$\int_{+0}^{1} x dN(x) < \infty .$$

By specializing $N(x)$, we obtain different types of processes $\{X(t), 0 < t < \infty\}$. For example, for a compound Poisson process

$$N(x) = -\lambda[1 - H(x)] , \tag{1.1}$$

where λ is the intensity of the Poisson process and $H(x)$ is the distribution of jumps. Then

$$\Phi(s) = \lambda[1 - \varphi(s)] , \tag{1.2}$$

where

$$\varphi(s) = \int_{0}^{\infty} e^{-sx} dH(x) ,$$

is the Laplace transform of $H(x)$.

In this paper, we establish the following result.

THEOREM 1.1. *Let* $\{X(t), 0 < t < \infty\}$ *be a compound Poisson process with* $\Phi(s)$ *given by* (1.2). *Then*

$$\theta = \Phi'(\omega)\,,$$

where ω *is the largest nonnegative root of the equation*

$$\Phi(s) = s\,. \tag{1.3}$$

The proof is deferred to the next section.

Remark 1. Note that zero is always a root of the equation (1.3), and $\Phi'(0) = \rho$ where $E\{X(t)\} = \rho t$. If $\rho < 1$, zero is the only nonnegative root of the equation (1.3), and if $\rho > 1$, there are two nonnegative roots (Takacs (1967), Theorem 4, p. 42).

Let us consider some examples.

Example 1. $H(x) = 0$ for $x < a$, $= 1$ for $x \geq a$ so that $X(t)$ is a scaled Poisson process. Then $E\{X(t)\} = a\lambda t$, $\Phi(s) = \lambda(1 - e^{-as})$ and $\Phi'(s) = a\lambda e^{-as}$. So if $a\lambda < 1$, $\theta = a\lambda$ and if $a\lambda > 1$, $\theta = a\lambda e^{-a\omega}$ where $0 < \omega = \lambda(1 - e^{-a\omega})$. Alternatively, θ can be expressed as the smallest nonnegative root (in t) of the equation

$$te^{-t} = a\lambda e^{-a\lambda}\,. \tag{1.4}$$

For the Poisson process where $a = 1$, this result was obtained by a different method in Nair *et al.* (1986).

Example 2. $H(x)$ is geometric, i.e., $H(x) = \sum_{n \leq x} h_n$ where $h_n = p^n(1 - p)$, $n = 0, 1, 2, \ldots$. Then $E\{X(t)\} = \lambda pt/(1 - p)$, $\Phi(s) = \lambda p(1 - e^{-s})/(1 - pe^{-s})$ and $\Phi'(s) = \lambda p(1 - p)e^{-s}/(1 - pe^{-s})^2$. So if $\lambda p < (1 - p)$, $\theta = \lambda p/(1 - p)$ and if $\lambda p > (1 - p)$, $\theta = \lambda p(1 - p)e^{-\omega}/(1 - pe^{-\omega})^2$ where $0 < \omega = \lambda p(1 - e^{-\omega})/(1 - pe^{-\omega})$.

Example 3. $H(x)$ is gamma, i.e., $H(x) = \int_0^x \beta^\gamma y^{\gamma-1} e^{-\beta y} dy/\Gamma(\gamma)$. Then $E\{X(t)\} = \lambda\gamma t/\beta$, $\Phi(s) = \lambda\{1 - [\beta/(\beta + s)]^\gamma\}$ and $\Phi'(s) = \lambda\gamma\beta^\gamma/(\beta + s)^{\gamma+1}$. So if $\lambda\gamma < \beta$, $\theta = \lambda\gamma/\beta$ and if $\lambda\gamma > \beta$, $\theta = \lambda\gamma\beta^\gamma/(\beta + \omega)^\gamma$ where $0 < \omega = \lambda\{1 - [\beta/(\beta + \omega)]^\gamma\}$. Alternatively, θ is the smallest nonnegative root of the equation

$$\frac{(\lambda\gamma/t)^{\gamma/(\gamma+1)}}{(\lambda + \lambda\gamma/t)} = \frac{\beta^{\gamma/(\gamma+1)}}{(\lambda + \beta)}\,. \tag{1.5}$$

The special case $\gamma = 1$ corresponds to the exponential distribution. Then $\Phi(s) = \lambda s/(\beta + s)$ and we get $\theta = \lambda/\beta$ if $\lambda < \beta$ and $\theta = \beta/\lambda$ if $\lambda > \beta$.

Theorem 1.1 provides a general way to compute θ for the compound Poisson process both when $\rho < 1$ and when $\rho > 1$. As remarked earlier, it is known that when $\rho < 1$, the result in Theorem 1.1 holds for general processes with stationary independent increments and with sample paths that are nondecreasing step functions vanishing at 0. We conjecture that Theorem 1.1 holds for such general processes even when $\rho > 1$. In the following examples, we obtain θ assuming this conjecture holds.

Example 4. $\{X(t), 0 < t < \infty\}$ is a Gamma process, i.e., $N(x) = -\int_x^\infty y^{-1}e^{-\beta y}dy$. Then $E\{X(t)\} = t/\beta$, $\Phi(s) = \log(1 + s/\beta)$ and $\Phi'(s) = (\beta + s)^{-1}$. So if $\beta > 1$, $\theta = \beta^{-1}$ and if $\beta < 1$, $\theta = (\beta + \omega)^{-1}$ where $0 < \omega = \log(1 + \omega/\beta)$. Alternatively, θ is the smallest nonnegative root of the equation

$$t^{-1}e^{-1/t} = \beta e^{-\beta}. \tag{1.6}$$

Note the similarity between (1.6) and (1.4).

Example 5. $\{X(t), 0 < t < \infty\}$ is a generalized stable process, i.e., $N(x) = -p\int_x^\infty y^{-p-1}e^{-\beta y}dy/\Gamma(1+p)$, $0 < p < 1$. Then $E\{X(t)\} = p\beta^{p-1}$ and $\Phi(s) = (s+\beta)^p - \beta^p$. So $\theta = p\beta^{p-1}$ if $p\beta^{p-1} < 1$ and $= p(\beta^p + \omega)^{(p-1)/p}$ where $0 < \omega = (\omega + \beta)^{-p} - \beta^p$ if $p\beta^{p-1} > 1$. Alternatively, θ is the smallest nonnegative root of the equation

$$(t/p)^{p/(p-1)} - (t/p)^{1/(p-1)} = \beta^p - \beta. \tag{1.7}$$

The case $\beta = 0$ corresponds to the stable process. In this case, $E\{X(t)\} \equiv \infty$, $\Phi(s) = s^p$ and $\theta = p$ is the only solution.

Remark 2. For Example 2, we have not found a convenient expression for θ, analogous to equations (1.4)–(1.7).

2. Proof

To prove Theorem 1.1, we need the following version of Bürmann's theorem (see Whittaker and Watson (1927), p. 128).

THEOREM 2.1. (Bürmann): *Suppose $f(z)$ and $g(z)$ are infinitely differentiable at $z = 0$, $g(0) \neq 0$ and let $v = u/g(u)$. Then*

$$f(u) = f(0) + \sum_{j=1}^{\infty} \frac{v^j}{j!} \{D_x^{j-1}[f'(x)g^j(x)]\}|_{x=0} ,$$

where $D_x^j(f) = d^j f / dx^j$.

COROLLARY 2.1. *By differentiating* $f(u)$ *with respect to* $v = u/g(u)$ *we get*

$$\partial f / \partial v = f'(0)g(0) + \sum_{j=1}^{\infty} \frac{v^j}{j!} \{D_x^j[f'(x)g(x)g^j(x)]\}|_{x=0} . \tag{2.1}$$

PROOF OF THEOREM 1.1 FOR THE LATTICE CASE. We first consider the case where the distribution $H(x)$ in (1.1) is supported on the lattice $\{a, 2a, \ldots\}$ for some $a > 0$. Let K be distributed as $H(x)$, $K_1, K_2, \ldots$ be i.i.d. copies of K,

$$\begin{aligned} h_n &= P\{K = na\}, & n &= 1, 2, \ldots \quad \text{and} \\ h_n^{(j)} &= P\{K_1 + \cdots + K_j = na\}, & n &= j, j+1, \ldots . \end{aligned} \tag{2.2}$$

Note that $X(t)$ can equal t only at the values $t = na$, $n = 1, 2, \ldots$. Now if

$$U(1) = 1 + \sum_{n=1}^{\infty} P\{X(na) = na\} , \tag{2.3}$$

then from Feller ((1968), Chapter 13),

$$\theta \equiv P\{X(t) = t \text{ for some } t > 0\} = 1 - 1/U(1) . \tag{2.4}$$

So from (2.2) and (2.3)

$$\begin{aligned} U(1) &= 1 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{(na\lambda)^j}{j!} e^{-na\lambda} h_n^{(j)} \\ &= 1 + \sum_{j=1}^{\infty} \frac{(a\lambda)^j}{j!} \left\{ D_x^j \left[\sum_{n=j}^{\infty} e^{n(x-a\lambda)} h_n^{(j)} \right] \right\} \Bigg|_{x=0} . \end{aligned}$$

Since

$$\sum_{n=j}^{\infty} e^{-sna} h_n^{(j)} = \left[\sum_{n=1}^{\infty} e^{-sna} h_n \right]^j = \varphi^j(s) ,$$

where $\varphi(s) = Ee^{-sK}$, we see that

$$U(1) = 1 + \sum_{j=1}^{\infty} \frac{(a\lambda)^j}{j!} \{D_x^j[\varphi^j(\lambda - x/a)]\}|_{x=0} . \tag{2.5}$$

Equating (2.5) with (2.1) by setting $v = a\lambda$, $f'(x) = 1/g(x)$ and $g(x) = \varphi(\lambda - x/a)$ and observing from Bürmann's theorem that

$$a\lambda = u/\varphi(\lambda - u/a) , \tag{2.6}$$

we get

$$U(1) = 1 + \partial f/\partial(a\lambda) - f'(0)g(0) = \partial f/\partial(a\lambda) ,$$

since $f'(x)g(x) \equiv 1$. So

$$\begin{aligned} U(1) &= \frac{\partial f/\partial u}{\partial(a\lambda)/\partial u} \\ &= \{1 + [u\varphi'(\lambda - u/a)]/[a\varphi(\lambda - u/a)]\}^{-1} \\ &= [1 + \lambda\varphi'(\lambda - u/a)]^{-1} , \end{aligned}$$

where the last equality follows from (2.6). From (2.4) we now get

$$\theta = -\lambda\varphi'(\lambda - u/a) . \tag{2.7}$$

Making a change of variable $s = \lambda - u/a$ and observing that $\Phi(s) = \lambda(1 - \varphi(s))$, we can reexpress (2.7) as $\theta = \Phi'(s)$ and (2.6) as $\Phi(s) = s$ as stated in Theorem 1.1.

To complete the proof for the lattice case, note from Remark 1 that if $\rho < 1$, $\omega = 0$ is the only nonnegative solution of the equation $\Phi(s) = s$. If $\rho > 1$, the equation has two nonnegative roots, but $\Phi'(0) = \rho > 1$, so that the only admissible root is $\omega > 0$.

Remark 3. It has been brought to our attention that the result for the general lattice case can be proved alternatively, and perhaps more elegantly, using the method of so-called associated random walks.

Remark 4. In the above, we have assumed that $P\{K = 0\} = 0$. This presents no loss of generality since the result for the case $P\{K = 0\} = \alpha > 0$ can be obtained by considering $\{X^*(t), 0 < t < \infty\}$ with intensity $\lambda^* = \lambda(1 - \alpha)$ and jump distribution $H^*(x) = [H(x) - H(0)]/(1 - \alpha)$.

PROOF FOR THE NONLATTICE CASE. The idea is to approximate $X(t)$

by $X_n(t) = ([nX(t)] + 1)/n$ where $[\cdot]$ is the greatest integer function.

First we show that we can restrict attention to the crossings of $X(t)$ and t in the interval $(0, c_n)$ where $c_n \to \infty$ with $c_n = o(n)$ as $n \to \infty$. We show this for the case $\rho > 1$; the proof for $\rho < 1$ is similar. Let τ satisfy $1 < \tau < \rho$.

$$\begin{aligned}
&\lim_{n\to\infty} P\{X(t) \leq t \text{ for some } t \geq c_n\} \\
&\qquad \leq \lim_{n\to\infty} P\{X(c_n) \leq c_n\tau\} \\
&\qquad\quad + \lim_{n\to\infty} P\{X(t) - X(c_n) \leq (t - c_n) + c_n(1-\tau) \text{ for some } t \geq c_n\}\,.
\end{aligned}$$

It follows from the law of large numbers that the first term equals 0 since $X(c_n)/c_n \to \rho > \tau$ a.s. as $n \to \infty$. The second term also equals 0 since: (i) $\{X(t) - X(c_n), t > c_n\}$ has the same distribution as $\{X(t), t > 0\}$, (ii) $\inf_{0<t<\infty} (X(t) - t)$ is a.s. finite for $\rho > 1$ and (iii) $c_n(1-\tau) \to -\infty$ as $n \to \infty$.

Now let $M =$ number of times $X(t)$ equals t for $0 < t < c_n$ and let $M_n =$ number of times $X_n(t)$ equals t for $0 < t < c_n$. We show $\lim_{n\to\infty} P\{M \neq M_n\} = 0$. Let $0 < T_1 < \cdots < T_J$ be the points at which the jumps of $X(t)$ (and $X_n(t)$) occur for $t \in (0, c_n)$. Then

$$\text{(2.9)} \qquad \begin{aligned}
P\{M \neq M_n\} \leq\ & P\left\{ X(T_j) < T_j \leq X(T_j) + \frac{1}{n}, j = 1,\ldots, J \right\} \\
& + P\left\{ X(T_j^-) \leq T_j < X(T_j^-) + \frac{1}{n}, j = 1,\ldots, J \right\}.
\end{aligned}$$

Since $X(T_j) = K_1 + \cdots + K_j$ where the K_j's are independently distributed as H and $T_j = \xi_1 + \cdots + \xi_j$ where the ξ_j's are independent exponential random variables with mean $1/\lambda$, and the K_j's and ξ_j's are independent, it can be shown from (2.9) that

$$P\{M \neq M_n\} \leq 2(1 - e^{-\lambda/n})E(J) = 2\lambda c_n(1 - e^{-\lambda/n})\,,$$

which $\to 0$ as $n \to \infty$ since $c_n = o(n)$. Therefore, $\theta_n \equiv P\{M_n \geq 1\} \to P\{M \geq 1\} \equiv \theta$ as $n \to \infty$.

Finally, let $E\{\exp(-sX_n(t))\} = \exp(-t\Phi_n(s))$. From the proof for the lattice case, $\theta_n = \Phi_n'(\omega_n)$ where ω_n is the root of the equation $\Phi_n(s) = s$. The result now follows from the fact that $\Phi_n(s) \to \Phi(s)$ and $\Phi_n'(s) \to \Phi'(s)$ as $n \to \infty$ since $\sup_{0<t<\infty} |X_n(t) - X(t)| \to 0$ everywhere as $n \to \infty$.

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