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ON SEQUENTIAL PROCEDURES FOR THE POINT ESTIMATION OF THE MEAN OF A NORMAL POPULATION

AJIT CHATURVEDI

Department of Statistics, Lucknow University, Lucknow-226004, India

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Abstract. The sequential procedures developed by Starr (1966, *Ann. Math. Statist.*, **37**, 1173-1185) for estimating the mean of a normal population are further analyzed. Asymptotic properties of the "regret" and first two moments of the stopping rules are studied and second-order approximations are derived.

Key words and phrases: Normal mean, point estimation, loss, risk, stopping rule, regret, martingales, uniform integrability, second-order approximations.

1. Introduction

Let us consider a sequence $X_1, X_2, \ldots$ of independent random observations from a normal population having unknown mean $\mu \in (-\infty, \infty)$ and unknown variance $\sigma^2 \in (0, \infty)$. Given a random sample $X_1, X_2, \ldots, X_n$ of size n, let us define $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $\sigma_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$. Suppose the loss-occurred in estimating μ by $\overline{X}_n$ be

$$L_n(C) = A|\overline{X}_n - \mu|^s + Cn^t, \tag{1.1}$$

where A, s, C and t are known positive constants. Using the fact that $\overline{X}_n \sim N(\mu, \sigma^2/n)$, the risk corresponding to the loss (1.1) comes out to be

$$v_n(C) = \left(\frac{2}{s}\right) \frac{K\sigma^s}{n^{s/2}} + Cn^t, \tag{1.2}$$

where $K = (s/2)\, 2^{s/2}\, \Gamma((s+1)/2)/\Gamma(1/2)$. The fixed-sample size $n = n_0$, which minimizes $v_n(C)$, is given by

$$n_0 = \left(\frac{K\sigma^s}{Ct} \right)^{2/(s+2t)} . \tag{1.3}$$

But, as we have already assumed, σ is unknown, the fixed sample size procedure fails to minimize $v_n(C)$ simultaneously for all σ. We adopt a sequential procedure to obtain sample size "close" to the optimal but unknown n_0, and the following stopping rule N is defined in confirmity with (1.3).

$$N = \inf \left\{ n \geq m \colon n \geq \left(\frac{K\sigma_n^s}{Ct} \right)^{2/(s+2t)} \right\} , \tag{1.4}$$

where m (≥ 2) is the starting sample size. Using the fact that $(n-1) \cdot \sigma_n^2/\sigma^2 = \sum_{j=1}^{n-1} Z_j^2$, with $Z_j \sim N(0, 1)$, we can re-write the stopping rule N as follows:

$$N = \inf \left\{ n \geq m \colon \chi^2_{(n-1)} \leq (n-1) \left(\frac{n}{n_0} \right)^{(s+2t)/s} \right\} . \tag{1.5}$$

Following Starr (1966) and Starr and Woodroofe (1969), we define the "risk-efficiency" and "regret" of the above mentioned sequential procedure by

$$\eta(C) = \bar{v}(C)/v_{n_0}(C) , \tag{1.6}$$

and

$$\omega(C) = \bar{v}(C) - v_{n_0}(C) , \tag{1.7}$$

respectively, where $\bar{v}(C)$ is the risk associated with the sequential procedure, i.e.,

$$\bar{v}(C) = \left(\frac{2}{s} \right) K\sigma^s E(N^{-s/2}) + CE(N^t) , \tag{1.8}$$

and $v_{n_0}(C)$ is obtained on substituting $n = n_0$ in (1.2), i.e.,

$$v_{n_0}(C) = C\left(\frac{2t}{s} + 1 \right) n_0^t . \tag{1.9}$$

Starr (1966) determined a condition on the starting sample size m for which the above defined sequential procedure is asymptotically (as $C \to 0$)

risk-efficient. Later on, Starr and Woodroofe (1969) studied the asymptotic behaviour of the "regret" for $C = t = 1$, i.e., when the cost of sample is linear and is unity for each observation. For $s = 2$ and $t = 1$, Nagao and Takada (1980) further studied this sequential procedure. They obtained an upper bound for $E(N)$ and $E(N^l)$, for $l > 0$ and C fixed. They also proved that for all $m \geq 3$, $\lim_{C \to 0} \eta(C) = 1$ and $\lim_{C \to 0} \omega(C) = 0$. A stronger bound for $\omega(C)$ is available in Ghosh and Mukhopadhyay (1980).

In the next two sections, we shall derive second-order approximations for $\omega(C)$, $E(N)$ and $E(N^2)$ for all s and t. In the remaining part of this note, we shall denote by k any generic constant independent of C, $[y]$ will be used for the integral part of y, and $I(S)$ will stand for the indicator function defined on the set S.

2. Second-order approximation for $\omega(C)$

We first establish few basic results.

LEMMA 2.1. *$P(N = m) = O_e(C^{(m-1)/s})$, as $C \to 0$.*

PROOF. We have from (1.5) that

$$P(N = m) = P[\chi^2_{(m-1)} \leq kC^{2/s}] ,$$

or,

$$ke^{-kC^{2s^{-1}}} \cdot C^{(m-1)/s} \leq P(N = m) \leq kC^{(m-1)/s} ,$$

and the lemma follows.

LEMMA 2.2. *For any $0 < \theta < 1$,*

$$P(m + 1 \leq N \leq \theta n_0) = O(C^{(m-1)/s}), \quad as\ C \to 0 .$$

PROOF. We have

$$\begin{aligned} P(m + 1 \leq N \leq \theta n_0) &\leq \sum_{n=m+1}^{\theta n_0} P\left[\chi^2_{(n-1)} \leq (n-1)\left(\frac{n}{n_0}\right)^{(s+2t)/s} \right] \\ &\leq \sum_{n=m+1}^{\theta n_0} \inf_{h>0} \left[\exp\left\{ h(n-1)\left(\frac{n}{n_0}\right)^{(s+2t)/s} \right\} E(e^{-h\chi^2_{(n-1)}}) \right] \\ &= \sum_{n=m+1}^{\theta n_0} \inf_{h>0} \left[\exp\left\{ h(n-1)\left(\frac{n}{n_0}\right)^{(s+2t)/s} \right\} (1 + 2h)^{-(n-1)/2} \right]. \end{aligned}$$

This inequality is also valid for the value h_0 of h, which minimizes the function

$$f(h) = \exp\left\{ h(n-1)\left(\frac{n}{n_0}\right)^{(s+2t)/s}\right\}(1+2h)^{-(n-1)/2}$$

i.e., $h_0 = [(n_0/n)^{(s+2t)/s} - 1]/2$. Setting $h = h_0$, we obtain

$$\begin{aligned} &P(m+1 \le N \le \theta n_0) \\ &\quad \le \sum_{n=m+1}^{\theta n_0} \left[\left(\frac{n}{n_0}\right)^{(s+2t)/s} \cdot \exp\left\{1 - \left(\frac{n}{n_0}\right)^{(s+2t)/s}\right\}\right]^{(n-1)/2} \\ &\quad \le n_0^{-(s+2t)(m-1)/2s} \cdot \left[\exp\left\{1 - \left(\frac{m+1}{n_0}\right)^{(s+2t)/s}\right\}\right]^{(m-1)/2} \\ &\quad\quad \cdot \sum_{n=m+1}^{\theta n_0} n^{(m-1)(s+2t)/2s} \cdot (\xi e^{1-\xi})^{(n-m)/2}, \end{aligned}$$

where $\xi = (n/n_0)^{(s+2t)/s} < 1$ for all $n \le \theta n_0$, so that, $\xi e^{1-\xi} < 1$. Now, using ratio rule for series convergence, we obtain the lemma.

COROLLARY 2.1. *For any* $0 < \theta < 1$,

$$P(N \le \theta n_0) = O(C^{(m-1)/s}), \quad \text{as } C \to 0.$$

PROOF. We can write

$$P(N \le \theta n_0) = P(N = m) + P(m+1 \le N \le \theta n_0),$$

and the proof follows on applying Lemmas 2.1 and 2.2.

LEMMA 2.3. *As* $C \to 0$,

$$N_0 = \left(\frac{1}{n_0}\right)^{1/2} (N - n_0) \xrightarrow{\mathscr{L}} N(0, 1).$$

PROOF. The proof follows from Theorem 3 of Ghosh and Mukhopadhyay (1979).

LEMMA 2.4. *For all* $m > 1 + 2s/(s+2t)$, N_0^2 *is uniformly integrable in* $C \le C_0$, *for some* $C_0 > 0$.

PROOF. Denoting by $F(x)$, the c.d.f. of $X = |N_0|$, we have, for some $a > 0$,

$$E[X^2 I(X > a)] = -\int_a^\infty x^2 d(1 - F(x)) \tag{2.1}$$
$$= a^2 P(X > a) + 2\int_a^\infty xP(X > x)\,dx$$
$$= \pi_1 + \pi_2 + \pi_3 + \pi_4 \,,$$

where

$$\pi_1 = a^2 P(N < n_0 - a(n_0)^{1/2}) \,,$$
$$\pi_2 = a^2 P(N > n_0 + a(n_0)^{1/2}) \,,$$
$$\pi_3 = 2\int_a^\infty xP(N < n_0 - x(n_0)^{1/2})\,dx \,,$$

and

$$\pi_4 = 2\int_a^\infty xP(N > n_0 + x(n_0)^{1/2})\,dx \,.$$

Let us choose $a > 2(n_0)^{-1/2}$ for $C \leq C_1$. Denoting by $L = [n_0 + x(n_0)^{-1/2}]$ one has for $x \geq a$ and $C \leq C_1$,

(i)
$$L - 1 \geq n_0 + x(n_0)^{1/2} - 2$$
$$\geq n_0 + a(n_0)^{1/2} - 2$$
$$> n_0 \,.$$

(ii)
$$L \geq n_0 + x(n_0)^{1/2} - 1$$
$$\geq n_0 + \frac{1}{2}x(n_0)^{1/2} + \frac{1}{2}a(n_0)^{1/2} - 1$$
$$\geq n_0 + \frac{1}{2}x(n_0)^{1/2}$$
$$\Rightarrow \left(\frac{L}{n_0}\right)^{(1+2t/s)} \geq 1 + kx(n_0)^{-1/2} \,.$$

From (i), (ii) and Markov's inequality, we obtain, for $q > 1$,

$$P(N > n_0 + x(n_0)^{1/2}) \tag{2.2}$$
$$\leq P(N \geq L + 1)$$
$$\leq P\left[\chi^2_{(L-1)} - (L-1) \geq (L-1)\left\{\left(\frac{L}{n_0}\right)^{(1+2t/s)} - 1\right\}\right]$$

$$\le P[\chi^2_{(L-1)} - (L-1) \ge kx(n_0)^{1/2}]$$
$$\le kx^{-2q}\, n_0^{-q}\, E\{\chi^2_{(L-1)} - (L-1)\}^{2q}$$
$$= kx^{-2q}\, n_0^{-q}\, (L-1)^q$$
$$= kx^{-2q} \le ka^{-2q}\,.$$

Thus,

$$\pi_2 \le ka^{2(1-q)} \tag{2.3}$$

and

$$\pi_4 \le k\int_a^\infty x^{1-2q}\, dx\,. \tag{2.4}$$

Now, choose C_2 such that $a > (n_0)^{1/2}/2$ for all $C \le C_2$. Hence, for $x \ge a$,

$$\begin{aligned}\pi_3 \le 2\Bigg[\int_a^{(n_0)^{1/2}/2} x\Bigg\{ P\left(N \le \frac{1}{2}n_0\right) \\ + P\left(\frac{1}{2}n_0 < N < n_0 - x(n_0)^{1/2}\right)\Bigg\} dx \\ + \int_{(n_0)^{1/2}/2}^{(n_0)^{1/2}} xP\left(N \le \frac{1}{2}n_0\right) dx\Bigg]\,.\end{aligned} \tag{2.5}$$

We have proved in Corollary 2.1 that, for $C \le C_3$,

$$P\left(N \le \frac{1}{2}n_0\right) \le kC^{(m-1)/s}\,,$$

so that, for $C \le C_4 = \min(C_2, C_3)$,

$$\int_{(n_0)^{1/2}/2}^{(n_0)^{1/2}} xP\left(N \le \frac{1}{2}n_0\right) dx \le ka^{2\{1-(m-1)(s+2t)/2s\}}\,. \tag{2.6}$$

Let us write $L_1 = [n_0/2]$, $L_2 = [n_0 - x(n_0)^{1/2}]$. We note that

$$\begin{aligned}1 - \left(\frac{L_2}{n_0}\right)^{(1+2t/s)} &\ge 1 - (1 - x(n_0)^{-1/2})^{(1+2t/s)} \\ &\ge kx(n_0)^{-1/2}\,.\end{aligned}$$

Now, we have

$$P\left(\frac{1}{2}n_0 < N < n_0 - x(n_0)^{1/2}\right)$$
$$= P\left[\bigcup_{N=L_1+1}^{L_2}\left\{\chi^2_{(N-1)} - (N-1) \le -(N-1)\left(1-\left(\frac{N}{n_0}\right)^{(1+2t/s)}\right)\right\}\right]$$
$$\le P\left[\bigcup_{N=L_1+1}^{L_2}\left\{\chi^2_{(N-1)} - (N-1) \le -L_1\left(1-\left(\frac{L_2}{n_0}\right)^{(1+2t/s)}\right)\right\}\right]$$
$$\le P\left[\bigcup_{N=L_1+1}^{L_2}\{\chi^2_{(N-1)} - (N-1) \le -kx(n_0)^{1/2}\}\right].$$

Since $\{\chi^2_{(N-1)} - (N-1): N \ge 2\}$ is a stationary martingale sequence, using Kolmogorov's inequality for martingales, one gets, for $q > 1$,

$$(2.7)\qquad P\left(\frac{1}{2}n_0 < N < n_0 - x(n_0)^{1/2}\right)$$
$$\le kx^{-2q}(n_0)^{-q}\, E\{\chi^2_{(L_2-L_1-1)} - (L_2 - L_1 - 1)\}^{2q}$$
$$= kx^{-2q}.$$

Substituting from (2.6) and (2.7) in (2.5), we obtain

$$(2.8)\qquad \pi_3 \le k\left[a^{2\{1-(m-1)(1+2t/s)/2\}} + \int_a^{(n_0)^{1/2}/2} x^{1-2q}\,dx\right].$$

A similar inequality can also be obtained for π_1.

Utilizing the inequalities (2.3), (2.4) and (2.8), we obtain from (2.1), for $q > 1$ and $C \le C_0 = \min(C_1, C_4)$,

$$(2.9)\qquad E[X^2 I(X > a)] \le k\left[a^{2(1-q)} + \int_a^{\infty} x^{1-2q}\,dx + a^{2\{1-(m-1)(1+2t/s)/2\}} + \int_a^{(n_0)^{1/2}/2} x^{1-2q}\,dx\right].$$

The expression on the r.h.s. of (2.9) tends to zero as $a \to \infty$ for all $m > 1 + 2s/(s+2t)$, implying that N_0^2 is uniformly integrable in $C \le C_0$.

The main results of this section are stated in the next two theorems.

THEOREM 2.1. *For all $m > 1 + 2s/(s+2t)$, as $C \to 0$,*

$$\omega(C) = \frac{1}{4}Ct(s+2t)n_0^{t-1} + o(C^{(s+2)/(s+2t)}).$$

PROOF. From (1.8) and (1.9), substituting the values of $\bar{\nu}(C)$ and $\nu_{n_0}(C)$ in (1.7), and using Taylor series expansion, we obtain for $|W - n_0| \le$

$|N - n_0|$,

$$(2.10) \quad \omega(C) = \frac{2Ct}{s} n_0^{(s+2t)/2} \cdot E[N^{-s/2} - n_0^{-s/2}] + CE[N^t - n_0^t]$$

$$= \frac{2Ct}{s} n_0^{(s+2t)/2} \cdot E\left[-\frac{s}{2} n_0^{-(s/2+1)} \cdot (N - n_0) + \frac{s}{4}\left(\frac{s}{2}+1\right)(N-n_0)^2 W^{-(s/2+2)} \right]$$

$$+ CE\left[tn_0^{t-1} \cdot (N - n_0) + \frac{1}{2} t(t-1)(N-n_0)^2 W^{t-2} \right]$$

$$= I_1 + I_2 \quad \text{(say)},$$

where

$$I_1 = \frac{1}{4} Ct(s+2)n_0^{t-2} \cdot E\left[(N-n_0)^2 \left(\frac{n_0}{W}\right)^{(s/2+2)}\right],$$

and

$$I_2 = \frac{1}{2} Ct(t-1)n_0^{t-2} \cdot E\left[(N-n_0)^2 \left(\frac{W}{n_0}\right)^{t-2}\right].$$

Denoting by P, the c.d.f. of N, we can write

$$I_1 = I_{11} + I_{12},$$

where

$$I_{11} = \frac{1}{4} Ct(s+2)n_0^{t-2} \int_{N \le n_0/2} (N-n_0)^2 \left(\frac{n_0}{W}\right)^{(s/2+2)} dP$$

and

$$I_{12} = \frac{1}{4} Ct(s+2)n_0^{t-2} \int_{N > n_0/2} (N-n_0)^2 \left(\frac{n_0}{W}\right)^{(s/2+2)} dP.$$

Since, on the event "$N \le n_0/2$", $n_0/W \le 2$,

$$(2.11) \qquad I_{11} \le kCn_0^{t-2} \int_{N \le n_0/2} (N-n_0)^2 \, dP$$

$$\leq kCn_0^t P\left(N \leq \frac{1}{2} n_0 \right)$$
$$= kC^{(m-1)/s+s/(s+2t)}$$
$$= o(C^{(s+2)/(s+2t)}) ,$$

as $C \to 0$, for all $m > 1 + s/(s+2t)$. It is to be noted that the second last expression on the r.h.s. of (2.11) is obtained on using Corollary 2.1. On the event "$N > n_0/2$", $n_0/W \leq 2$. Moreover, since $W/n_0 \to 1$ w.p. 1 as $C \to 0$, we obtain on using Lemmas 2.3 and 2.4 that, for all $m > 1 + 2s/(s+2t)$, as $C \to 0$,

$$I_{12} \to \frac{1}{4} Ct(s+2)n_0^{t-1} . \tag{2.12}$$

In order to tackle the term I_2, we consider the following two cases.

Case 1. (When $t \leq 2$) Proceeding as for I_1, we can prove that for all $m > 1 + 2s/(s+2t)$, as $C \to 0$,

$$I_2 = \frac{1}{2} Ct(t-1)n_0^{t-1} + o(C^{(s+2)/(s+2t)}) . \tag{2.13}$$

Case 2. (When $t \geq 2$) We can write

$$I_2 = \frac{1}{2} Ct(t-1)n_0^{t-2} \left[\int_{N \leq n_0/2} \left(\frac{W}{n_0} \right)^{t-2} \cdot (N - n_0)^2 \, dP \right.$$
$$\left. + \int_{N > n_0/2} \left(\frac{W}{n_0} \right)^{t-2} \cdot (N - n_0)^2 \, dP \right] .$$

Since on the event "$N \leq n_0/2$", $W/n_0 \leq 1/2$, on the event "$N > n_0/2$", $W/n_0 \leq 3/2$, and $W/n_0 \to 1$ w.p. 1 as $C \to 0$, I_2 converges to the same limit as in Case 1.

The theorem now follows on making substitutions from (2.11), (2.12) and (2.13) in (2.10).

Remark 1. For $t = 1$, we conclude that $\omega(C) = (s+2)/4 + o(C)$ for all $m > 1 + 2s/(s+2)$. In this case, Starr and Woodroofe (1969) proved that $\omega(C) = O(1)$ iff $m \geq s + 1$. Thus, our bounds for $\omega(C)$ are sharper than that achieved by Starr and Woodroofe (1969).

Remark 2. For $s=2$, $t=1$, we obtain $\omega(C)=C+o(C)$ as $C\to 0$, for all $m\geq 3$, which is the result obtained by Ghosh and Mukhopadhyay (1980). Moreover, the result $\lim_{C\to 0}\omega(C)=0$, for all $m\geq 3$, obtained by Nagao and Takada (1980) also follows immediately.

The following theorem provides second-order approximations for the first two moments of N.

THEOREM 2.2. *For all* $m>1+2s/(s+2t)$, *as* $C\to 0$,

$$E(N)=n_0+(v-1)\left(1+\frac{2t}{s}\right)^{-1}+o(1)\,, \tag{2.14}$$

$$E(N^2)=n_0^2+2n_0\left(1+\frac{2t}{s}\right)^{-1}(v-1)+o(C^{-2/(s+2t)})\,, \tag{2.15}$$

where v *is specified.*

PROOF. Let us consider the difference

$$R_C=(N-1)\left(\frac{N}{n_0}\right)^{(s+2t)/s}-S_N\,, \tag{2.16}$$

where $S_N=\sum_{j=1}^{N-1}Z_j^2$, with $Z_j\sim N(0,1)$.

The mean v of the asymptotic distribution of R_C can be obtained from Theorem 2.2 of Woodroofe (1977). By (2.16), Wald's lemma for cumulative sums, and Taylor series expansion, we obtain for $|W-n_0|\leq|N-n_0|$,

$$\begin{aligned}
E(S_N)&=E(N-1)\\
&=\frac{1}{n_0^{1+2t/s}}E[N^{(2+2t)/s}-N^{(1+2t)/s}]-v\\
&=E\Bigg[\Bigg\{n_0+\left(2+\frac{2t}{s}\right)(N-n_0)+\frac{1}{2}\left(2+\frac{2t}{s}\right)\left(1+\frac{2t}{s}\right)\\
&\qquad\cdot\left(\frac{W}{n_0}\right)^{2t/s}\cdot\frac{(N-n_0)^2}{n_0}\Bigg\}-\Bigg\{1+\left(1+\frac{2t}{s}\right)\cdot\frac{(N-n_0)}{n_0}\\
&\qquad+\frac{1}{2}\left(1+\frac{2t}{s}\right)\left(\frac{2t}{s}\right)\left(\frac{W}{n_0}\right)^{2t/s-1}\cdot\frac{(N-n_0)^2}{n_0}\Bigg\}\Bigg]-v\,.
\end{aligned}$$

Proceeding exactly along the lines of proof of Theorem 2.1, it can be shown that as $C \to 0$, for all $m > 1 + 2s/(s+2t)$

$$E(N-1) = E\left[\left\{ n_0 + \left(2 + \frac{2t}{s}\right)(N - n_0) + (1 + o(1)) \right\}\right.$$
$$\left. - \left\{ 1 + \left(1 + \frac{2t}{s}\right) \cdot \frac{(N - n_0)}{n_0} \right\}\right] - \nu ,$$

or,

$$\left(1 + \frac{1}{n_0}\right) E\{N - n_0\} = \nu \left(1 + \frac{2t}{s}\right)^{-1} - \left(1 + \frac{2t}{s}\right)^{-1} + o(1) ,$$

and (2.14) follows.

To obtain second-order approximations for $E(N^2)$, let us write

$$E(N^2) = n_0^2 + E\{N^2 - n_0^2\}$$
$$= n_0^2 + 2n_0 E\{N - n_0\} + n_0 E\left\{ \frac{(N - n_0)^2}{n_0} \right\} .$$

Utilizing (2.14), we obtain

$$E(N^2) = n_0^2 + 2n_0 \left\{\left(1 + \frac{2t}{s}\right)^{-1} (\nu - 1) + o(1) \right\} + n_0\{1 + o(1)\} ,$$

and (2.15) holds.

3. Estimation of μ under log-cost function

In this section, we consider a different loss function. Let us take

$$L_n(C) = A|\overline{X}_n - \mu|^s + C \log n . \tag{3.1}$$

This loss function was considered by Starr (1966) under Section 4, and it implies that the cost of sampling n observations is proportional to $\log n$, when C is the known cost per unit observation. The risk corresponding to the loss (3.1) is

$$v_n(C) = \frac{2K\sigma^s}{sn^{s/2}} + C \log n . \tag{3.2}$$

The value n^* of n, which minimizes (3.2) is

(3.3) $$n^* = \left(\frac{K}{C}\right)^{2/s} \cdot \sigma^2 ,$$

and setting $n = n^*$, the minimum risk is

(3.4) $$v_{n^*}(C) = \frac{2C}{s} + C \log n^* .$$

In the ignorance of σ, the following stopping rule N is suggested:

(3.5) $$N = \inf\left\{ n \geq m : n \geq \left(\frac{K}{C}\right)^{2/s} \sigma_n^2 \right\} .$$

The risk associated with the sequential procedure (3.5) is

(3.6) $$\begin{aligned} E[L_N(C)] &= v_N(C) \\ &= \frac{2C}{s} E\left(\frac{n^*}{N}\right)^{s/2} + CE(\log N) . \end{aligned}$$

As usual, we define the "risk-efficiency" and "regret" by

(3.7) $$\eta(C) = v_N(C)/v_{n^*}(C) ,$$

and

(3.8) $$\omega(C) = v_N(C) - v_{n^*}(C) ,$$

respectively.

Starr (1966) proved that $\lim_{C \to 0} \eta(C) = 1$ for all $m \geq s + 1$. Here, we shall study the asymptotic behaviours of "regret" and first two moments of the stopping time N. We shall repeatedly use the notation Result A[B] to indicate that the proof of result A is similar to that of B, where the result may be in the form of a lemma, corollary or theorem.

LEMMA 3.1[2.1]. *As* $C \to 0$,

$$P(N = m) = O_e(C^{(m-1)/s}) .$$

LEMMA 3.2[2.2]. *For any* $0 < \theta < 1$, *as* $C \to 0$,

$$P(m + 1 \leq N \leq \theta n^*) = O(C^{(m-1)/s}) .$$

COROLLARY 3.1[2.1]. *For any* $0 < \theta < 1$, *as* $C \to 0$,

$$P(N \leq \theta n^*) = O(C^{(m-1)/s}) .$$

LEMMA 3.3[2.3]. *As* $C \to 0$,

$$N^* = \left(\frac{1}{n^*}\right)^{1/2} (N - n^*) \xrightarrow{\mathscr{L}} N(0, 1) .$$

LEMMA 3.4[2.4]. *For all* $m > 1 + 2/s$, N^{*2} *is uniformly integrable in* $C \leq C_0$ *for some* $C_0 > 0$.

The following theorem provides second-order approximation for the regret $\omega(C)$.

THEOREM 3.1. *As* $C \to 0$,

$$\omega(C) = \frac{Cs}{4n^*} + o(C^{1+2/s}) ,$$

for all $m > 1 + 2/s$.

PROOF. From (3.4) and (3.6), substituting the values of $v_{n^*}(C)$ and $v_N(C)$ in (3.8) and using Taylor series expansion, we get after some algebraic manipulations, for $|W - n^*| \leq |N - n^*|$,

$$\begin{aligned} \omega(C) &= \frac{2C}{s} n^{*s/2} E[N^{-s/2} - n^{*-s/2}] + CE[\log N - \log n^*] \\ &= I_1 - I_2 \quad \text{(say)} , \end{aligned} \tag{3.9}$$

where

$$I_1 = \frac{C}{2n^{*2}}\left(1 + \frac{s}{2}\right) E\left[\left(\frac{n^*}{W}\right)^{(s/2+2)} \cdot (N - n^*)^2\right] ,$$

and

$$I_2 = \frac{C}{2n^{*2}} E\left[\left(\frac{n^*}{W}\right)^2 \cdot (N - n^*)^2\right] .$$

Proceeding along the lines of proofs of various steps in Theorem 2.1, we can show that, for all $m > 1 + 2/s$, as $C \to 0$,

$$I_1 = \frac{C}{2n^*}\left(\frac{s}{2}+1\right) + o(C^{1+2/s}),$$

and

$$I_2 = \frac{C}{2n^*} + o(C^{1+2/s}).$$

The proof now follows on substituting the values of I_1 and I_2 in (3.9).

In the next theorem, we shall establish second-order approximations for $E(N)$ and $E(N^2)$.

THEOREM 3.2. *As* $C \to 0$,

$$E(N) = n^* + \nu_1 - 1 + o(1), \tag{3.10}$$

$$E(N^2) = n^{*2} + 2n^*\nu_1 - n^* + o(C^{-2/s}), \tag{3.11}$$

for all $m > 1 + 2/s$, *where* ν_1 *is specified.*

PROOF. It follows from the definition of N that

$$N = \inf\left\{ n \geq m : S_N \leq (N-1)\left(\frac{n}{n^*}\right)\right\},$$

where S_N is the same as defined in Section 2. Let ν_1 be the mean of the asymptotic distribution of

$$R_C^* = (N-1)\left(\frac{N}{n^*}\right) - S_N .$$

The proofs of (3.10) and (3.11) are now similar to that of (2.14) and (2.15), respectively, with necessary modifications at various places.

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