Ann. Inst. Statist. Math.
Vol. 40, No. 4, 693–714 (1988)

LOG-CONCAVITY OF STIRLING NUMBERS AND UNIMODALITY OF STIRLING DISTRIBUTIONS

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(Received August 7, 1987; revised December 17, 1987)

Abstract. A series of inequalities involving Stirling numbers of the first and second kinds with adjacent indices are obtained. Some of them show log-concavity of Stirling numbers in three different directions. The inequalities are used to prove unimodality or strong unimodality of all the subfamilies of Stirling probability functions. Some additional applications are also presented.

Key words and phrases: Inequalities, strong unimodality, total positivity 2, Stirling family of probability distributions.

1. Introduction

"Stirling family of distributions" is a collection of eight subfamilies of discrete probability distributions which involve Stirling numbers of the first and second kinds (Sibuya (1986)). The subfamilies are classified as shown in Table 2. Although the forms of their probability functions are different, all are unimodal if monotone increasing or decreasing probability functions in their distribution ranges are regarded also unimodal.

This fact can be proved by log-concavity of Stirling numbers or by inequalities involving Stirling numbers with adjacent indices. Each of the first and second kinds of Stirling numbers forms a triangular array, which is totally positive 2 (Karlin (1968)) as a function of two indices. The array includes finite and infinite sequences in three directions. All the sequences are log-concave (Keilson and Gerber (1971)) to the extent that some are just as the definition, and some are stronger and one is weaker than the definition (cf. Table 1).

Such type of inequalities was studied by Lieb (1968) and Neuman (1985). Results of this paper extend and improve some of their results. For some others alternative proofs are given. Lieb (1968) used Newton's inequality for symmetry functions and Neuman (1985) used moments of spline functions to obtain their inequalities. The methods used in this paper are elementary. Log-concavity conditions are given by Kurtz (1972) for triangular arrays defined by a recurrence formula.

In Section 2, preliminaries on Stirling numbers, Stirling distributions and log-concavity are given. In Section 3, a series of inequalities are given, and the main inequalities are summarized in Table 1. In Section 4, the inequalities are used to prove strong unimodality or unimodality of all the probability functions in Table 2 for any parameter value.

The inequalities are applied to other problems (Section 5). One is Poisson upper bounds of the distributions of **STR1F** and **STR2F** subfamilies of Table 2. Another problem is monotonicity of the convolutions of Stirling numbers. Finally, log-concavity of binomial coefficients are remarked (Section 6).

2. Preliminaries

'Unsigned' Stirling numbers of the first kind $\left[{n \atop m}\right]$ for nonnegative integers n and m are defined by the polynomial identity

$$t^{[n]} := t(t+1)\cdots(t+n-1) = \sum_m \left[{n \atop m}\right] t^m ,$$

in the domain $0 < m \leq n$, and are zero outside the domain except that $\left[{0 \atop 0}\right] = 1$ by convention. They satisfy the recurrence relation

$$\left[{n+1 \atop m}\right] = n\left[{n \atop m}\right] + \left[{n \atop m-1}\right], \quad n \geq 0, \quad m \geq 1 . \tag{2.1}$$

Stirling numbers of the second kind $\left\{{n \atop m}\right\}$ for $n,\ m \geq 0$ are defined similarly, in this case by

$$t^n = \sum_m \left\{{n \atop m}\right\} t^{(m)} ,$$

where $t^{(m)} := t(t-1)\cdots(t-m+1)$. They satisfy the recurrence relation

$$\left\{{n+1 \atop m}\right\} = m\left\{{n \atop m}\right\} + \left\{{n \atop m-1}\right\}, \quad n \geq 0, \quad m \geq 1 . \tag{2.2}$$

See Jordan (1947), Riordan (1958), Goldberg *et al.* (1964) and Knuth (1967–1981), for the introduction to Stirling numbers. The notation of the numbers differs by the literature; this paper follows the notation of Knuth.

Probability functions involving Stirling numbers as their main component are shown in Table 2, which appeared in Sibuya (1986). The table has two columns for each of two kinds of Stirling numbers. Good correspondence exists between them, and subfamilies have symbols **STR1*** in the first column, and **STR2*** in the second. The asterisks stand for **F**, **W**, **C** or **I**. The characters are initials of Finite range (or Fundamental), Waiting time, Complementary waiting time, and Infinite range. Some are very well known, some others are less known, and **STR1W**, **STR1C** and **STR2C** may be new.

A series of nonnegative numbers $(c_n)_{n=-\infty}^{\infty}$ is log-concave if

$$c_n^2 \geq c_{n-1}c_{n+1}\,, \qquad n = 0,\ \pm 1,\ \pm 2,\ldots,$$

(Hardy *et al.* (1952)). Log-concavity of a probability function on integers is a necessary and sufficient condition for its strong unimodality in Ibragimov's sense (Keilson and Gerber (1971)). Log-concavity of probability functions is preserved under the operations, convolution, shift, truncation and reverse. Strong unimodality implies unimodality, but not reversely.

Log-concavity of the subfamilies **STR1F**, **STR2W** and **STR2I** is obvious. Their probability functions are multiple convolutions of Bernoulli, geometric and 0-truncated Poisson probability functions, respectively, and these are log-concave. The probability function of **STR1I** is also a multiple convolution of logarithmic series probability functions which are, however, log-convex.

3. Inequalities

In Theorems 3.1–3.7, inequalities on Stirling numbers are shown. The main inequalities in Theorems 3.1–3.5 show log-concavity in both kinds of Stirling numbers in three directions. The schemes of indices of Stirling numbers in the inequalities are illustrated in Table 1. A duality exists between the first and second kinds of Stirling numbers, and the same type of inequalities often holds for both kinds. Occasionally, however, sharp differences arise.

The proofs for all the theorems, except for Theorem 3.7, are based on the recurrence relations (2.1) and (2.2) although some maneuvering is necessary.

THEOREM 3.1. (Lieb (1968)) *The following sequences are strictly decreasing for any* $n = 3, 4,\ldots;$

$$\text{(A1)} \qquad \frac{m-1}{n-m+1}\left[\begin{matrix} n \\ m \end{matrix}\right] \Big/ \left[\begin{matrix} n \\ m-1 \end{matrix}\right], \qquad m = 2, 3,\ldots, n\,,$$

Table 1. Log-concavity and related inequality relations among Stirling numbers.

Symbol in the text	the 1st kind	Pattern					the 2nd kind	Property
			$m-1$	m	$m+1$	$m+2$		
	$\left[{n \atop m}\right]$	$n-1$	.	.	.	.	$\left\{{n \atop m}\right\}$	
		n	.	.	.	.		
		$n+1$	.	.	.	.		
A	S		s	g^2	s	.	S	log-concave
B	—		g	s	.	.	—	totally positive
			s	g	.	.		
C	W		.	s	.	.	—	log-concave
			.	g^2	.	.		
			.	s	.	.		
G	—		s	.	.	.	—	log-concave
			.	g^2	.	.		
			.	.	s	.		
F	S		s	g	.	.	S	—
			.	.	g	s		
D	S		s	g	.	.	S	—
			.	g	s	.		
E	S		s	.	g	.	S	—
			.	g	.	s		

1. The central column, "pattern", indicates the positions of indices n and m appearing in each inequality. "g" represents the greater part and "s" the smaller part of the inequality. For example, in row A the pattern means

$$\left[{n \atop m}\right]^2 > \left[{n \atop m-1}\right]\left[{n \atop m+1}\right] \quad \text{and} \quad \left\{{n \atop m}\right\}^2 > \left\{{n \atop m-1}\right\}\left\{{n \atop m+1}\right\}.$$

2. "S" in the columns on both sides indicates that the inequality is actually stronger than what the pattern appears, and some factor larger than 1 can be multiplied to the smaller side of the inequality. "W" in the first kind column means the contrary; the inequality is weaker, and some factor larger than 1 must be multiplied to the greater part.

$$\text{(A1}'\text{)} \qquad \frac{m}{n-m}\left[{n \atop n-m}\right] \Big/ \left[{n \atop n-m+1}\right], \qquad m = 1, 2, \ldots, n-1\,,$$

$$\text{(A2)} \qquad \frac{(m-1)m}{n-m+1}\left\{{n \atop m}\right\} \Big/ \left\{{n \atop m-1}\right\}, \qquad m = 2, 3, \ldots, n\,,$$

and

$$\text{(A2}')\qquad \frac{m}{(n-m)(n-m+1)}\left\{ {n \atop n-m} \right\} \Big/ \left\{ {n \atop n-m+1} \right\}, \qquad m = 1, 2, \ldots, n-1 .$$

PROOF. The first half (A1) and (A1′) is equivalent to

$$(3.1)\qquad (m-1)(n-m)\left[{n \atop m} \right]^2 > m(n-m+1)\left[{n \atop m-1} \right]\left[{n \atop m+1} \right], \qquad m = 2, \ldots, n-1 .$$

Remark that both sides of (3.1) vanish for $m = 1$ and n.

To prove by induction on n, put $n = 3$ and $m = 2$ to see

$$\left[{3 \atop 2} \right]^2 = 9 > 2^2 \left[{3 \atop 1} \right]\left[{3 \atop 3} \right] = 8 .$$

To advance the induction step, calculate

$$\begin{aligned}
&(m-1)(n-m+1)\left[{n+1 \atop m} \right]^2 - m(n-m+2)\left[{n+1 \atop m-1} \right]\left[{n+1 \atop m+1} \right] \\
&= n^2 \frac{n-m+2}{n-m+1}\left((m-1)(n-m)\left[{n \atop m} \right]^2 \right. \\
&\qquad\qquad \left. - m(n-m+1)\left[{n \atop m-1} \right]\left[{n \atop m+1} \right] \right) \\
&\quad + n\left((m-2)(n-m)\left[{n \atop m-1} \right]\left[{n \atop m} \right] \right. \\
&\qquad\qquad \left. - m(n-m+2)\left[{n \atop m-2} \right]\left[{n \atop m+1} \right] \right) \\
&\quad + \frac{m}{m-1}\left((m-2)(n-m+1)\left[{n \atop m-1} \right]^2 \right. \\
&\qquad\qquad \left. - (m-1)(n-m+2)\left[{n \atop m-2} \right]\left[{n \atop m} \right] \right) \\
&\quad + \left(n\left(\frac{m-1}{n-m+1} \right)^{1/2}\left[{n \atop m} \right] - \left(\frac{n-m+1}{m-1} \right)^{1/2}\left[{n \atop m-1} \right] \right)^2 .
\end{aligned}$$

All the terms are nonnegative and some are positive for $m = 2, 3, \ldots, n$.

The second half (A2) and (A2′) is equivalent to

$$(3.2)\qquad (m-1)(n-m)\left\{ {n \atop m} \right\}^2 > (m+1)(n-m+1)\left\{ {n \atop m-1} \right\}\left\{ {n \atop m+1} \right\}, \qquad m = 2,\dots,n-1,$$

which will be proved in a similar way. For $n = 3$ and $m = 2$,

$$\left\{ {3 \atop 2} \right\}^2 = 9 > 6\left\{ {3 \atop 1} \right\}\left\{ {3 \atop 3} \right\} = 6.$$

To advance the induction step on n, calculate

$$\begin{aligned}
&(m-1)(n-m+1)\left\{ {n+1 \atop m} \right\}^2 - (m+1)(n-m+2)\left\{ {n+1 \atop m-1} \right\}\left\{ {n+1 \atop m+1} \right\} \\
&= (m^2-1)\frac{n-m+2}{n-m+1}\Bigg((m-1)(n-m)\left\{ {n \atop m} \right\}^2 \\
&\qquad\qquad - (m+1)(n-m+1)\left\{ {n \atop m-1} \right\}\left\{ {n \atop m+1} \right\}\Bigg) \\
&\quad + \frac{m-1}{n-m+1}\Big(m^2 + (n-m+2)(n-m)\Big)\left\{ {n \atop m} \right\}^2 \\
&\quad + \frac{m+1}{m}\Bigg((m-2)(m-1)(n-m)\left\{ {n \atop m-1} \right\}\left\{ {n \atop m} \right\} \\
&\qquad\qquad - m(m+1)(n-m+2)\left\{ {n \atop m-2} \right\}\left\{ {n \atop m+1} \right\}\Bigg) \\
&\quad + \frac{m-1}{m}\Big(2(n-2) + (m-2)^2 \Big)\left\{ {n \atop m-1} \right\}\left\{ {n \atop m} \right\} \\
&\quad + \frac{m+1}{m}\Bigg((m-2)(n-m+1)\left\{ {n \atop m-1} \right\}^2 \\
&\qquad\qquad - m(n-m+2)\left\{ {n \atop m-2} \right\}\left\{ {n \atop m} \right\}\Bigg) \\
&\quad + 2\,\frac{n-m+1}{m}\left\{ {n \atop m-1} \right\}^2 .
\end{aligned}$$

All the terms are nonnegative and some are positive for $m = 2, 3,\dots, n$. □

Remarks. Lieb (1968) showed (3.1) from Newton's inequality and the fact that the generating function of $\left(\left[{n \atop m} \right]\right)_{m=1}^{n}$ has only real roots. He showed also the sequence

$$\frac{m-1}{n-m+1}\left\{ {n \atop m} \right\} \bigg/ \left\{ {n \atop m-1} \right\}, \qquad m = 2, 3, \ldots, n, \tag{3.3}$$

strictly decreasing. The inequality (A2) slightly improves this result.

Numerically, it is suggested that the sequence

$$\frac{(m-1)(2n+m)}{n-m+1}\left[{n \atop m} \right] \bigg/ \left[{n \atop m-1} \right], \qquad m = 2, 3, \ldots, n-1, \tag{3.4}$$

is strictly decreasing and the same for $m = n - 1$ and n, and that

$$\frac{(m-1)^2}{n-m+1}\left\{ {n \atop m} \right\} \bigg/ \left\{ {n \atop m-1} \right\}, \qquad m = 2, 3, \ldots, n, \tag{3.5}$$

is strictly decreasing.

COROLLARY 3.1. *Suppose a sequence* $(a_m)_{m=1}^{\infty}$ *satisfies* $0 = a_1 < a_2 \leq a_3 \leq \cdots$ *and* $2a_m \geq a_{m-1} + a_{m+1}$, $m = 2, 3, \ldots$. *The sequences*

$$a_m \left[{n \atop m} \right] \bigg/ \left[{n \atop m-1} \right], \qquad m = 2, 3, \ldots, n,$$

$$a_{m+1} \left[{n \atop n-m} \right] \bigg/ \left[{n \atop n-m+1} \right], \qquad m = 1, 2, \ldots, n-1,$$

$$a_m \left\{ {n \atop m} \right\} \bigg/ \left\{ {n \atop m-1} \right\}, \qquad m = 2, 3, \ldots, n,$$

and

$$a_{m+1} \left\{ {n \atop n-m} \right\} \bigg/ \left\{ {n \atop n-m+1} \right\}, \qquad m = 1, 2, \ldots, n-1,$$

are strictly decreasing for any $n = 3, 4, \ldots$.

PROOF. The sequence (a_m) satisfies $2 \geq a_3/a_2$, and if $m/(m-1) \geq a_{m+1}/a_m$, then

$$a_{m+2} \leq 2a_{m+1} - a_m \leq (m+1)a_{m+1}/m .$$

Since

$$\frac{m}{n-m}\bigg/\frac{m-1}{n-m+1} \geq \frac{m}{m-1} \geq a_{m+1}/a_m ,$$

the corollary follows from Theorem 3.1. □

COROLLARY 3.2. *Stirling numbers of the first and second kinds,* $\begin{bmatrix} n \\ m \end{bmatrix}$ *and* $\begin{Bmatrix} n \\ m \end{Bmatrix}$ *with n fixed, are log-concave sequences in m.*

Remark. These facts are well known. Hammersley (1951) and Erdös (1953) showed

$$\begin{bmatrix} n \\ m \end{bmatrix}\bigg/\begin{bmatrix} n \\ m-1 \end{bmatrix} \neq 1 ,$$

for any n and m.

COROLLARY 3.3. *Considering the ends of the sequences* (A1) *and* (A2),

$$\text{(3.6)} \qquad \frac{n-m+1}{(m-1)(n-1)} H_{n-1} \geq \begin{bmatrix} n \\ m \end{bmatrix}\bigg/\begin{bmatrix} n \\ m-1 \end{bmatrix} \geq \frac{2(n-m+1)}{(m-1)n} , \qquad 2 \leq m \leq n ,$$

where $H_k = \sum_{j=1}^{k} (1/j)$ *is the harmonic number, and*

$$\text{(3.7)} \qquad \frac{2^n-2}{n-1} \geq \begin{Bmatrix} n \\ m \end{Bmatrix}\bigg/\begin{Bmatrix} n \\ m-1 \end{Bmatrix} \geq \frac{2(n-m+1)}{(m-1)m} , \qquad 2 \leq m \leq n .$$

In both (3.6) *and* (3.7), *the left equalities hold for* $m = 2$ *and the right equalities for* $m = n$. *For* $2 < m < n$, *the inequalities are strict.*

Further (3.6) *and* (3.7) *are equivalent to*

$$\text{(3.6}') \qquad \frac{n(2n-m+1)}{2(n-m+1)} > \begin{bmatrix} n+1 \\ m \end{bmatrix}\bigg/\begin{bmatrix} n \\ m \end{bmatrix} > n + \frac{(m-1)(n-1)}{(n-m+1)H_{n-1}} ,$$

and

$$\text{(3.7}') \qquad \frac{(2n-m+1)m}{2(n-m+1)} > \begin{Bmatrix} n+1 \\ m \end{Bmatrix}\bigg/\begin{Bmatrix} n \\ m \end{Bmatrix} > m + (n-1)/(2^n-2) .$$

Remark. The inequalities (3.7) and (3.7′) improve the inequalities (4.17)–(4.20) in Neuman (1985).

THEOREM 3.2. *As double sequences, Stirling numbers of the first and second kinds are strictly totally positive* 2 (*Karlin* (1968)) *in the sense that for any* $n_1 < n_2$ *and* $m_1 < m_2$

$$\text{(B1)}\qquad \left|\begin{matrix} \left[{n_1 \atop m_1}\right] & \left[{n_1 \atop m_2}\right] \\ \left[{n_1 \atop m_1}\right] & \left[{n_1 \atop m_2}\right] \end{matrix}\right| \begin{cases} > 0, & \text{if } \left[{n_1 \atop m_1}\right]\left[{n_2 \atop m_2}\right] \neq 0\,, \\ = 0, & \text{if } \left[{n_1 \atop m_1}\right]\left[{n_2 \atop m_2}\right] = 0\,. \end{cases}$$

This means the sequences

$$\left[{n_2 \atop m}\right] \Big/ \left[{n_1 \atop m}\right], \qquad m = 1, 2, \ldots, n_1, \qquad (n_1 < n_2)\,,$$

and

$$\left[{n \atop m_2}\right] \Big/ \left[{n \atop m_1}\right], \qquad n = m_2, m_2 + 1, \ldots, \qquad (m_1 < m_2)\,,$$

are strictly increasing.

A relation (B2) *which is completely the same as* (B1) *holds for* $\left\{{n \atop m}\right\}$.

PROOF. It is enough to prove

$$\left[{n \atop m}\right]\left[{n+1 \atop m+1}\right] - \left[{n \atop m+1}\right]\left[{n+1 \atop m}\right] = \left[{n \atop m}\right]^2 - \left[{n \atop m+1}\right]\left[{n \atop m-1}\right] \geq 0\,.$$

This is true due to Corollary 3.1, and the last equality holds only if all the terms vanish.

For Stirling numbers of the second kind,

$$\begin{aligned} &\left\{{n \atop m}\right\}\left\{{n+1 \atop m+1}\right\} - \left\{{n \atop m+1}\right\}\left\{{n+1 \atop m}\right\} \\ &\quad = \left\{{n \atop m}\right\}\left\{{n \atop m+1}\right\} + \left\{{n \atop m}\right\}^2 - \left\{{n \atop m+1}\right\}\left\{{n \atop m-1}\right\} \geq 0\,, \end{aligned}$$

by the above reason. □

Remark. Examining the above proof, we get a stronger result: The sequence

$$\frac{1}{m}\left\{ {n+1 \atop m} \right\} \Big/ \left\{ {n \atop m} \right\}, \qquad m = 1, 2, \ldots, n\,,$$

is strictly increasing.

THEOREM 3.3. *The sequences*

$$\text{(C1)} \qquad \frac{1}{n}\left[{n+1 \atop m} \right] \Big/ \left[{n \atop m} \right], \qquad n = m, m+1, \ldots\,,$$

and

$$\text{(C2)} \qquad \left\{ {n+1 \atop m} \right\} \Big/ \left\{ {n \atop m} \right\}, \qquad n = m, m+1, \ldots\,,$$

are strictly decreasing for any $m = 2, 3, \ldots$.

PROOF. (C1):

$$\begin{aligned} &\frac{1}{n}\left[{n+1 \atop m} \right]^2 - \frac{1}{n+1}\left[{n \atop m} \right]\left[{n+2 \atop m} \right] \\ &\quad = \frac{1}{n+1}\left(\left[{n \atop m-1} \right]\left[{n+1 \atop m} \right] - \left[{n \atop m} \right]\left[{n+1 \atop m-1} \right]\right) \\ &\quad\quad + \frac{1}{n(n+1)}\left[{n \atop m-1} \right]\left[{n+1 \atop m} \right] \geq 0\,, \end{aligned}$$

due to Theorem 3.2, and the last equality holds only if all the terms vanish

(C2):

$$\left\{ {n+1 \atop m} \right\}^2 - \left\{ {n \atop m} \right\}\left\{ {n+2 \atop m} \right\} = \left\{ {n+1 \atop m} \right\}\left\{ {n \atop m-1} \right\} - \left\{ {n \atop m} \right\}\left\{ {n+1 \atop m-1} \right\} \geq 0\,.$$

by the above reason. □

Remarks. The sequence

$$\left[{n+1 \atop m} \right] \Big/ \left[{n \atop m} \right], \qquad n = m, m+1, \ldots\,,$$

is equal to n if $m = 1$, increasing if $m = 2$, increasing first and decreasing afterward if $m \geq 3$. Thus the factor $1/n$ is necessary to obtain a decreasing sequence (see Theorem 3.7 for further discussion).

The inequality (C2), showing the log-concavity of $\left(\left\{ {n \atop m} \right\}\right)_{n=m}^{\infty}$, was proved by Neuman (1985) based on the log-concavity of the symmetric means.

THEOREM 3.4. *For* $n \geq m + 1 \geq 2$,

$$\text{(D1)} \qquad (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix} > n \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\begin{bmatrix} n \\ m+1 \end{bmatrix},$$

and for $n \geq m + 1 \geq 3$,

$$\text{(E1)} \qquad (n-1)^2\begin{bmatrix} n-1 \\ m \end{bmatrix}\begin{bmatrix} n \\ m-1 \end{bmatrix} > n^2 \begin{bmatrix} n-1 \\ m-2 \end{bmatrix}\begin{bmatrix} n \\ m+1 \end{bmatrix}.$$

PROOF. The inequalities are proved simultaneously by induction on both m and $k := n - m$. For $m = 1$ (D1) is true since the right-hand side vanishes. For $m = 2$ (E1) is similarly true. For $n - m = 1$ (D1) and (E1) are true, because

$$\text{(D1)} \Leftrightarrow (m+1)m^2/2 > (m+1)m(m-1)/2\,,$$

and

$$\begin{aligned}\text{(E1)} \Leftrightarrow\ & (m+1)m^3(m-1)(3m+2)/24 \\ & > (m+1)^2 m(m-1)(m-2)(3m-1)/24\,.\end{aligned}$$

Now assume that (D1) is satisfied for $k = n - m$ for all m, and in the induction step of $k + 1$ assume (D1) for $m - 1$. At the same time (E1) is assumed in step k and m.

$$\begin{aligned}&(m+k)\begin{bmatrix} m+k \\ m \end{bmatrix}\begin{bmatrix} m+k+1 \\ m \end{bmatrix} - (m+k+1)\begin{bmatrix} m+k \\ m-1 \end{bmatrix}\begin{bmatrix} m+k+1 \\ m+1 \end{bmatrix} \\ &\quad = (m+k)^2(m+k-1)\begin{bmatrix} m+k-1 \\ m \end{bmatrix}\begin{bmatrix} m+k \\ m \end{bmatrix} \\ &\qquad - ((m+k)^2-1)(m+k)\begin{bmatrix} m+k-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m+k \\ m+1 \end{bmatrix}\end{aligned}$$

$$+\frac{m+k}{m+k-1}(m+k-1)^2\begin{bmatrix} m+k-1 \\ m \end{bmatrix}\begin{bmatrix} m+k \\ m-1 \end{bmatrix}$$

$$-\frac{m+k+1}{m+k}(m+k)^2\begin{bmatrix} m+k-1 \\ m-2 \end{bmatrix}\begin{bmatrix} m+k \\ m+1 \end{bmatrix}$$

$$+(m+k)^2\begin{bmatrix} m+k-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m+k \\ m \end{bmatrix}$$

$$-((m+k)^2-1)\begin{bmatrix} m+k-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m+k \\ m \end{bmatrix}$$

$$+\frac{m+k}{m+k-1}(m+k-1)\begin{bmatrix} m+k-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m+k \\ m-1 \end{bmatrix}$$

$$-\frac{m+k+1}{m+k}(m+k)\begin{bmatrix} m+k-1 \\ m-2 \end{bmatrix}\begin{bmatrix} m+k \\ m \end{bmatrix},$$

which is positive since all the lines are positive; the first line is so by (D1) with m and $n = m + k$, so is the second by (E1) with m and $n = m + k$, the third trivially, and the fourth by (D1) with $m - 1$ and $n = m - 1 + k + 1$. Thus, (D1) is true for m and $n = m + k + 1$.

This new inequality is multiplied by (D1) with $m + 1$ and $n = m + 1 + k$ side by side. The result is (E1) with $m + 1$ and $n = m + 1 + k$. Now the induction on m is advanced one step for both (D1) and (E1). □

COROLLARY 3.4.

$$\text{(F1)} \qquad (n-1)\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix} > n\begin{bmatrix} n-1 \\ m-2 \end{bmatrix}\begin{bmatrix} n \\ m+1 \end{bmatrix}.$$

$$\text{(G1)} \qquad \begin{bmatrix} n \\ m \end{bmatrix}^2 \geq \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\begin{bmatrix} n+1 \\ m+1 \end{bmatrix},$$

where the equality holds only if both sides are 0 *or* 1. *The latter is the case* $n = m$.

PROOF. Multiply (D1) with

$$\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}^2 > \begin{bmatrix} n-1 \\ m-2 \end{bmatrix}\begin{bmatrix} n-1 \\ m \end{bmatrix},$$

(Corollary 3.1) to obtain (F1). (G1) is a direct result of (D1). □

THEOREM 3.5. *For* $n \geq m + 1 \geq 2$, (D2) $$m\left\{ {n-1 \atop m} \right\}\left\{ {n \atop m} \right\} > (m+1)\left\{ {n-1 \atop m-1} \right\}\left\{ {n \atop m+1} \right\},$$

and for $n \geq m+1 \geq 3$,

(E2) $$(m-1)\left\{ {n-1 \atop m} \right\}\left\{ {n \atop m-1} \right\} > (m+1)\left\{ {n-1 \atop m-2} \right\}\left\{ {n \atop m+1} \right\}.$$

PROOF. The way to prove is the same as Theorem 3.4. For $m = 1$ and $m = 2$, (D2) and (E2) are true, respectively. For $n = m + 1$,

$$\text{(D2)} \Leftrightarrow (m+1)m^2/2 > (m+1)m(m-1)/2\,,$$

and

$$\begin{aligned}\text{(E2)} \Leftrightarrow\ & (m+1)m(m-1)^2(3m-2)/24 \\ & > (m+1)m(m-1)(m-2)(3m-5)/24\,,\end{aligned}$$

and these inequalities are true.

Assume the induction step $k := n - m$ for all m, and assume in step $k + 1$ (D2) holds for $m - 1$. Further assume (E2) holds for k and m.

$$\begin{aligned}
& m\left\{ {m+k \atop m} \right\}\left\{ {m+k+1 \atop m} \right\} - (m+1)\left\{ {m+k \atop m-1} \right\}\left\{ {m+k+1 \atop m+1} \right\} \\
&= m^2 m\left\{ {m+k-1 \atop m} \right\}\left\{ {m+k \atop m} \right\} \\
&\qquad - (m^2-1)(m+1)\left\{ {m+k-1 \atop m-1} \right\}\left\{ {m+k \atop m+1} \right\} \\
&\quad + \frac{m^2}{m-1}(m-1)\left\{ {m+k-1 \atop m} \right\}\left\{ {m+k \atop m-1} \right\} \\
&\qquad - (m+1)^2\left\{ {m+k-1 \atop m-2} \right\}\left\{ {m+k \atop m+1} \right\} \\
&\quad + m^2\left\{ {m+k-1 \atop m-1} \right\}\left\{ {m+k \atop m} \right\} \\
&\qquad - (m^2-1)\left\{ {m+k-1 \atop m-1} \right\}\left\{ {m+k \atop m} \right\} \\
&\quad + \frac{m}{m-1}(m-1)\left\{ {m+k-1 \atop m-1} \right\}\left\{ {m+k \atop m-1} \right\}
\end{aligned}$$

$$-\frac{m+1}{m}m\left\{\begin{matrix} m+k-1 \\ m-2 \end{matrix}\right\}\left\{\begin{matrix} m+k \\ m \end{matrix}\right\},$$

which is positive since the first line is so by (D2) with m and k, so is the second by (E2) with m and k, the third trivially and the fourth by (D2) with $m-1$ and $k+1$.

This new inequality is multiplied by (D2) with k and $m+1$ side by side to give (E2) with k and $m+1$. □

COROLLARY 3.5.

(F2) $$m\left\{\begin{matrix} n-1 \\ m-1 \end{matrix}\right\}\left\{\begin{matrix} n \\ m \end{matrix}\right\} > (m+1)\left\{\begin{matrix} n-1 \\ m-2 \end{matrix}\right\}\left\{\begin{matrix} n \\ m+1 \end{matrix}\right\},$$

(G2) $$\left\{\begin{matrix} n \\ m \end{matrix}\right\}^2 \geq \left\{\begin{matrix} n-1 \\ m-1 \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ m+1 \end{matrix}\right\},$$

where the equality holds only if both sides are 0 *or* 1. *The latter is the case* $n = m$.

PROOF. The proof is similar to that of Corollary 3.4. □

LEMMA 3.1. *For* $c > 0$,

(3.8) $$1 > \frac{1}{n+c}\begin{bmatrix} n+1 \\ m \end{bmatrix} \Big/ \begin{bmatrix} n \\ m \end{bmatrix},$$

if and only if

$$\begin{bmatrix} n \\ m-1 \end{bmatrix} < c\begin{bmatrix} n \\ m \end{bmatrix}.$$

PROOF. Just expand $\begin{bmatrix} n+1 \\ m \end{bmatrix}$. □

Remark. The sequence $\left(\begin{bmatrix} n \\ m \end{bmatrix}\right)_{m=1}^{n}$ is unimodal, and the mode is approximately equal to log n (Jordan (1947)). When m is fixed and if n is larger than some value close to e^m the inequality (3.8) with $c = 1$ is satisfied.

THEOREM 3.6. *For any* $c > 0$ *and for any* $n = m, m+1, \ldots$

(3.9) $$\frac{1}{n+c}\begin{bmatrix} n \\ m \end{bmatrix} \Big/ \begin{bmatrix} n-1 \\ m \end{bmatrix} < 1,$$

implies (3.9) *itself with n replaced by* $n+1$.

PROOF. For $m=1$ and for any $n>1$,

$$\frac{1}{n+c}\begin{bmatrix} n \\ 1 \end{bmatrix} \Big/ \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = \frac{n-1}{n+c} < 1 .$$

For $m \geq 2$, (3.9) is equivalent to

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} \Big/ \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} > \frac{1}{c+1},$$

because of Lemma 3.1. Since the left-hand side is increasing in n (Theorem 3.2),

$$\begin{bmatrix} n \\ m \end{bmatrix} \Big/ \begin{bmatrix} n \\ m-1 \end{bmatrix} > \frac{1}{c+1},$$

which is equivalent to (3.9) with n replaced by $n+1$. □

THEOREM 3.7. *Put*

$$r(n; m, \theta) := \frac{1}{n+1+\theta}\begin{bmatrix} n+1 \\ m \end{bmatrix} \Big/ \begin{bmatrix} n \\ m \end{bmatrix}.$$

For any $m = 1, 2, \ldots$ *and for any* $\theta > 0$, $(r(n; m, \theta))_{n=m}^{\infty}$ *is not a non-increasing sequence.*

Remark. Compare Theorem 3.7 with Theorem 3.3.

PROOF. The existence of n such that

$$r(n; m, \theta) < r(n+1; m, \theta),$$

should be proved, and this is equivalent to

$$\frac{n+1}{n+1+\theta} r(n; m, 0) < \frac{n+2}{n+2+\theta} r(n+1; m, 0).$$

Since $(n+1)/(n+1+\theta) < (n+2)/(n+2+\theta)$ it is enough to prove the above-mentioned fact for $\theta = 0$. For $m=1$,

$$r(n; 1, 0) = n/(n+1),$$

which is increasing. For $m = 2$,

$$r(n; 2, 0) = \frac{1}{n+1}(n + H_{n-1}^{-1}),$$

where $H_n = \sum_{j=1}^{n}(1/j)$. Therefore,

$$\begin{aligned} r(n; 2, 0) &< r(n+1; 2, 0) \\ &\Leftrightarrow (n+2)/H_{n-1} < 1 + (n+1)/H_n \\ &\Leftrightarrow 1 + 1/n < H_n(H_{n-1} - 1), \end{aligned}$$

and the last expression is valid for $n = 3, 4, \ldots$. Now for $m > 2$, $r(n; m, 0) < 1$ if n is larger than some value close to e^m (Remark of Lemma 3.1). On the other hand it is known (Jordan (1947)) that

$$\lim_{n\to\infty} r(n; m, 0) = 1,$$

so that $r(n; m, 0)$ cannot be nonincreasing for all $n > e^m$. □

4. Unimodality of Stirling distributions

The theorems in Section 3 make it possible to check unimodality of the probability distributions in Table 2. The following theorem summarizes the results.

THEOREM 4.1. *Among the eight subfamilies of Stirling probability distributions in Table 2, the following six are strongly unimodal in Ibragimov's sense for any parameter value:*

STR1F, **STR2F**, **STR2W**, **STR1C**, **STR2C** *and* **STR2I**.

The other two subfamilies are not strongly unimodal but are unimodal for any parameter value:

STR1W *and* **STR1I**.

PROOF. For each of the distributions in Table 2, the ratio of consecutive probabilities $r(x) = f(x+1)/f(x)$ is as follows, and each is decreasing due to the corresponding inequality shown by its symbol.

STR1F: $$r(x) = \theta \begin{bmatrix} n \\ x+1 \end{bmatrix} \Big/ \begin{bmatrix} n \\ x \end{bmatrix}, \tag{A1}$$

Table 2. Stirling family of probability distributions.

	the 1st kind	the 2nd kind
Finite Interval	**STR1F**(n,θ) $\begin{bmatrix} n \\ x \end{bmatrix} \frac{\theta^x}{\theta^{[n]}}$ $1 \le x \le n,\ 0 < \theta < \infty$ no. breaking records; binary search tree	**STR2F**(n,m) $\begin{Bmatrix} n \\ x \end{Bmatrix} \frac{m^{(x)}}{m^n}$ $1 \le x \le \min(n,m)$ m: pos. int. or $n-1<m$ real occupancy; poker test, test of empty boxes, collision test.
Waiting Time	**STR1W**(k,θ) $\begin{bmatrix} x-1 \\ k-1 \end{bmatrix} \frac{\theta^k}{\theta^{[x]}}$ $2 \le k \le x < \infty,\ 0 < \theta < \infty$ waiting new records	**STR2W**(k,m) $\begin{Bmatrix} x-1 \\ k-1 \end{Bmatrix} \frac{m^{(k)}}{m^x}$ $2 \le k \le x < \infty$, $k-1 < m < \infty$, m: real coupon collector's test
Complementary Waiting Time	**STR1C**(k,θ) $\begin{bmatrix} x-1 \\ x-k \end{bmatrix} \frac{(x-1)\theta^{x-k}}{\theta^{[x]}}$ $k+1 \le x \le \infty,\ 0 < \theta < \infty$ waiting non-records	**STR2C**(k,m) $\begin{Bmatrix} x-1 \\ x-k \end{Bmatrix} \frac{(x-k)m^{(x-k)}}{m^x}$ $k+1 \le x \le k+m$, m: pos. int. waiting collisions
Infinite Interval	**STR1I**(k,θ) $\frac{k!}{(-\log(1-\theta))^k} \begin{bmatrix} x \\ k \end{bmatrix} \frac{\theta^x}{x!}$ $0 < k \le x < \infty,\ 0 < \theta < 1$ logarithmic series	**STR2I**(k,θ) $\frac{k!}{(e^\theta - 1)^k} \begin{Bmatrix} x \\ k \end{Bmatrix} \frac{\theta^x}{x!}$ $0 < k \le x < \infty,\ 0 < \theta < \infty$ 0-truncated Poisson

θ: real; k, n, x: positive integer; m: positive integer or real

$$a^{[n]} = a(a+1)\cdots(a+n-1), \qquad a^{(n)} = a(a-1)\cdots(a-n+1)\,.$$

$$\textbf{STR2F:} \quad r(x) = (m-x)\begin{Bmatrix} n \\ x+1 \end{Bmatrix} \Big/ \begin{Bmatrix} n \\ x \end{Bmatrix}, \tag{A2}$$

$$\textbf{STR2W:} \quad r(x) = \frac{1}{m}\begin{Bmatrix} x \\ k-1 \end{Bmatrix} \Big/ \begin{Bmatrix} x-1 \\ k-1 \end{Bmatrix}, \tag{C2}$$

$$\textbf{STR1C:} \quad r(x) = \frac{\theta}{\theta+x}\,\frac{x}{x-1}\begin{bmatrix} x \\ x+1-k \end{bmatrix} \Big/ \begin{bmatrix} x-1 \\ x-k \end{bmatrix}, \tag{G1}$$

$$\textbf{STR2C:} \quad r(x) = \frac{(x+1-k)(m-x+k)}{m(x-k)}\begin{Bmatrix} x \\ x+1-k \end{Bmatrix} \Big/ \begin{Bmatrix} x-1 \\ x-k \end{Bmatrix}, \tag{G2}$$

$$\textbf{STR2I:} \qquad r(x) = \frac{\theta}{x+1} \left\{ \begin{matrix} x+1 \\ k \end{matrix} \right\} \bigg/ \left\{ \begin{matrix} x \\ k \end{matrix} \right\} . \tag{C2}$$

Notice that the coefficients $\theta x/(\theta + x)(x-1)$ in **STR1C** and $(x+1-k) \cdot (m-x+k)/m(x-k)$ in **STR2C** are decreasing in x.

Regarding the other two subfamilies

$$\textbf{STR1W:} \quad r(x) = (x+\theta)^{-1} \left[\begin{matrix} x \\ k-1 \end{matrix} \right] \bigg/ \left[\begin{matrix} x-1 \\ k-1 \end{matrix} \right],$$

and

$$\textbf{STR1I:} \qquad r(x) = \theta(x+1)^{-1} \left[\begin{matrix} x+1 \\ k \end{matrix} \right] \bigg/ \left[\begin{matrix} x \\ k \end{matrix} \right],$$

Theorem 3.7 is applied to show that they are not decreasing for any k or θ. However, Theorem 3.6 shows that $1 > r(x)$ implies

$$f(x) > f(x+1) > f(x+2) > \cdots ,$$

and this means $f(x)$ is unimodal (including the case of increasing or decreasing).

Unimodality of **STR1I** was also stated in Patil and Wani (1965). □

5. Other applications

Distributions of the subfamilies **STR1F** and **STR2F** can be approximated by Poisson distributions. Based on Theorem 3.1, Poisson distributions, which are stochastically larger than these distributions, can be obtained.

Let f and g be probability functions on the integers. A partial order $f \prec g$ is defined by

$$f(x+1)g(x) \le f(x)g(x+1) \qquad \text{for any} \qquad x .$$

This relation implies $\Sigma_x^\infty f(t) \le \Sigma_x^\infty g(t)$, namely g is statistically larger than f, and moreover on any interval the conditional g is statistically larger than the conditional f (Yanagimoto and Sibuya (1972)).

THEOREM 5.1. (i) *Let f_1 and f_2 be the 'reversed'* **STR1F** *and* **STR2F** *probability functions, respectively*:

$$f_1(x) = \left[\begin{matrix} n \\ n-x \end{matrix} \right] \frac{\theta^{n-x}}{\theta^{[n]}} , \qquad 0 \le x \le n-1 ,$$

and

$$f_2(x) = \left\{ \begin{matrix} n \\ n-x \end{matrix} \right\} \frac{m^{(n-x)}}{m^n}, \qquad 0 \leq x \leq n-1 < m .$$

Let g_0 be the Poisson probability function with mean λ. Then

$$f_1 \prec g_0 \quad \textit{if} \quad \lambda \geq f_1(1)/f_1(0) = n(n-1)/2\theta ,$$

and

$$f_2 \prec g_0 \quad \textit{if} \quad \lambda \geq f_2(1)/f_2(0) = n(n-1)/2(m-n+1) .$$

(ii) *Let f_3 and f_4 be the probability functions of* **STR1F** *and* **STR2F**, *respectively. Let g_1 and g_2 be the probability functions of 'shifted' and '0-truncated' Poisson distributions*:

$$g_1(x) = e^{-\lambda}\lambda^{x-1}/(x-1)!, \qquad x = 1, 2, \dots ,$$
$$g_2(x) = (e^{\lambda}-1)^{-1}\lambda^x/x!, \qquad x = 1, 2, \dots .$$

Then,

$$f_3 \prec g_1, \quad \textit{if} \quad \lambda \geq f_3(2)/f_3(1) = \theta H_{n-1} ,$$
$$f_3 \prec g_2, \quad \textit{if} \quad \lambda \geq 2 f_3(2)/f_3(1) ,$$
$$f_4 \prec g_1, \quad \textit{if} \quad \lambda \geq f_4(2)/f_4(1) = (m-1)(2^{n-1}-1) ,$$

and

$$f_4 \prec g_2, \quad \textit{if} \quad \lambda \geq 2 f_4(2)/f_4(1) .$$

PROOF. Notice that for the Poisson and the 0-truncated Poisson distributions $(x+1)g_i(x+1)/g_i(x) = \lambda$ $(i = 0, 2)$, and for the shifted Poisson distribution $xg_1(x+1)/g_1(x) = \lambda$. Since

$$(x+1)\frac{f_1(x+1)}{f_1(x)} = \frac{x+1}{\theta}\left[\begin{matrix} n \\ n-x-1 \end{matrix} \right] \Big/ \left[\begin{matrix} n \\ n-x \end{matrix} \right],$$

is decreasing due to Theorem 3.1,

$$\lambda \geq \frac{f_1(1)}{f_1(0)} > (x+1)\frac{f_1(x+1)}{f_1(x)}, \qquad x = 1, 2, \dots ,$$

and this means $g_0(x+1)/g_0(x) \geq f_1(x+1)/f_1(x)$ or $f_1 \prec g_0$.

The other five cases are similarly proved. □

The last application is related to the convolution of Stirling numbers.

THEOREM 5.2. *Put*

$$c_1(n_1, n_2, m) := \sum_t \begin{bmatrix} n_1 \\ t \end{bmatrix} \begin{bmatrix} n_2 \\ m-t \end{bmatrix},$$

and

$$c_2(n_1, n_2, m) := \sum_t \begin{Bmatrix} n_1 \\ t \end{Bmatrix} \begin{Bmatrix} n_2 \\ m-t \end{Bmatrix}.$$

When $n_1 + n_2$ is fixed, $c_i(n_1, n_2, m)$ decreases when $|n_1 - n_2|$ decreases for $i = 1, 2$ and for any m. Except that when $n_1 + n_2 = 2s$,

$$c_1(s-1, s+1, m) = c_1(s, s, m) = c_1(s+1, s-1, m).$$

PROOF. It is enough to prove

$$c_i(k-1, n-k+1, m) > c_i(k, n-k, m) \quad \text{if} \quad k \leq n-k, \tag{5.1}$$

(replaced by equality if $i = 1$ and $n = 2k$). For $i = 1$, (5.1) can be rewritten as

$$\sum_t (n-k) \begin{bmatrix} k-1 \\ t \end{bmatrix} \begin{bmatrix} n-k \\ m-t \end{bmatrix} > \sum_t k \begin{bmatrix} k-1 \\ t \end{bmatrix} \begin{bmatrix} n-k \\ m-t \end{bmatrix}.$$

and this is true unless $n - k = k$.

For $i = 2$, (5.1) is rewritten as

$$\sum_t (m-t) \begin{Bmatrix} k-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-k \\ m-t \end{Bmatrix} > \sum_t t \begin{Bmatrix} k-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-k \\ m-t \end{Bmatrix}.$$

Take the two terms $t = s$ and $t = m - s$ $(s < m - s)$ from both sides, to obtain

$$\begin{aligned} &(m-s) \begin{Bmatrix} k-1 \\ s \end{Bmatrix} \begin{Bmatrix} n-k \\ m-s \end{Bmatrix} + s \begin{Bmatrix} k-1 \\ m-s \end{Bmatrix} \begin{Bmatrix} n-k \\ s \end{Bmatrix} \\ &\quad > s \begin{Bmatrix} k-1 \\ s \end{Bmatrix} \begin{Bmatrix} n-k \\ m-s \end{Bmatrix} + (m-s) \begin{Bmatrix} k-1 \\ m-s \end{Bmatrix} \begin{Bmatrix} n-k \\ s \end{Bmatrix}. \end{aligned}$$

This is true because

$$\begin{vmatrix} \left\{ {k-1 \atop s} \right\} & \left\{ {n-k \atop s} \right\} \\ \left\{ {k-1 \atop m-s} \right\} & \left\{ {n-k \atop m-s} \right\} \end{vmatrix} > 0\,,$$

if $k-1 < n-k$ and $s < m-s$. □

Part of both Theorems 5.1 and 5.2 has been used in the discussion of the occupancy problem with two types of balls (Nishimura and Sibuya (1988)).

6. Log-concavity of binomial coefficients

Finally, we remark that binomial coefficients, to which Stirling numbers are similar somewhat, are log-concave.

THEOREM 6.1. *Let a be a positive real number, and let n and m be positive integers. The sequences*

$$(x+1)\binom{a}{x+1}\Big/\binom{a}{x}, \qquad x = 0, 1, \ldots, n; \quad n < a\,,$$

$$\binom{a}{n-x-1}\Big/\binom{a}{n-x}, \qquad x = 0, 1, \ldots, n-1; \quad n-1 < a\,,$$

$$\binom{a+x+1}{x+1}\Big/\binom{a+x}{x}, \qquad x = 0, 1, 2, \ldots; \quad 0 < a\,,$$

and

$$\binom{a+x+1}{m}\Big/\binom{a+x}{m}, \qquad x = 0, 1, 2, \ldots; \quad m-1 < a\,,$$

are strictly decreasing.

The proof is straightforward and omitted.

Keilson and Gerber (1971) showed strong unimodality of binomial, negative binomial and Poisson distributions for any parameter value. The facts are due to part of the above theorem. Strong unimodality of positive and negative hypergeometric distributions

$$\binom{a}{x}\binom{b}{n-x}\Big/\binom{a+b}{n}, \qquad x = 0, 1, \ldots, n,$$

$$a, b > n - 1 \quad \text{or} \quad a, b < -1,$$

is also due to Theorem 6.1.

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