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A THRESHOLD FOR THE SIZE OF RANDOM CAPS TO COVER A SPHERE

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Abstract. Consider a unit sphere on which are placed N random spherical caps of area $4\pi p(N)$. We prove that if $\overline{\lim}\,(p(N)\cdot N/\log N) < 1$, then the probability that the sphere is completely covered by N caps tends to 0 as $N \to \infty$, and if $\underline{\lim}\,(p(N)\cdot N/\log N) > 1$, then for any integer $n > 0$ the probability that each point of the sphere is covered more than n times tends to 1 as $N \to \infty$.

Key words and phrases: Coverage problem, random caps, threshold function.

1. Introduction

Consider N random spherical caps of area $4\pi p(N)$ placed on the surface of a sphere of unit radius. We suppose that the centers of these N caps are independently and uniformly distributed over the surface of the sphere. Then, what is the probability that the sphere is completely covered? This problem is called a coverage problem (see, e.g., Kendall and Moran (1963), Santaló (1976) and Solomon (1978)), and it seems that no exact solution is obtained yet. In this paper we consider the asymptotic behavior of the covering probability as N tends to infinity. We prove that the function $p_0(N) = (\log N)/N$ is a threshold function for the coverage in the following sense: if $\overline{\lim}\,(p(N)/p_0(N)) < 1$, then the covering probability tends to 0 as $N \to \infty$, and if $\underline{\lim}\,(p(N)/p_0(N)) > 1$, then for any integer $n > 0$, the probability that every point on the sphere is covered by more than n caps tends to 1 as $N \to \infty$.

The method used in the proof will be clearly applied to any dimensions. Hence the same function $p_0 = (\log N)/N$ is also a threshold function for the coverage on a sphere in any dimensions.

2. Random arcs on a circle

Here we recall how the similar result was obtained in the one dimensional case. Consider the circumference of a circle with unit perimeter on which N random arcs $I_1,\dots,I_N$, each of length p, are randomly placed. Let v_i be the clockwise endpoint of I_i, $i=1,\dots,N$. We assume that v_i, $i=1,\dots,N$, are independently and uniformly distributed over the circle. Let Y_i be a random variable such that $Y_i=1$ if I_i contains no other points v_j, where $j\neq i$, and $Y_i=0$ otherwise. Then the sum $Y=Y_1+\cdots+Y_N$ stands for the number of "gaps", that is, the number of connected components of the part of the circle that is not covered by any N arcs. Then it was proved by Maehara (1987) that if $p=(c\log N)/N$ with a constant $c>0$, then $P(Y=0)$ tends to 0 or 1 accordingly as $c<1$ or $c>1$, and that if $p=(t+\log N)/N$, then $P(Y=k)$ tends to $\mu^k e^{-\mu}/k!$ as N tends to infinity, where $\mu=e^{-t}$. Hence the covering probability $P(\text{cover})$ tends to 0 if $p\cdot N/\log N\to c<1$, and $P(\text{cover})$ tends to 1 if $p\cdot N/\log N\to c>1$. Furthermore, if $p=(t+\log N)/N$, then $P(\text{cover})$ tends to $e^{-\mu}$ as $N\to\infty$.

3. Random caps on a sphere

Let S denote the surface of a unit sphere in three dimensions. The area of a spherical cap of angular radius a is $2\pi(1-\cos a)=4\pi\sin^2(a/2)$. Thus any of random points uniformly distributed over S falls in a specified cap of angular radius a with probability

$$p=(1-\cos a)/2=\sin^2(a/2)\,.$$

Now consider N random caps of angular radius a on S, whose centers are independently and uniformly distributed over S. For each non-negative integer n, let $U_n(N,p)$ denote the set of points of S that are covered by at most n caps, that is, any point of $U_n(N,p)$ is not covered by more than n caps. Thus

$$U_0(N,p)\subset U_1(N,p)\subset U_2(N,p)\subset\cdots,$$

and $U(N,p):=U_0(N,p)$ is the part of S that is not covered by any of the N caps. The proportion of the area covered by $U_n(N,p)$ is denoted by the lower case $u_n(N,p)$;

$$u_n(N,p)=\{\text{the area of } U_n(N,p)\}/(4\pi)\,.$$

THEOREM 3.1. *If $p=p(N)\le(c/N)\log N$ for a constant $c<1$, then*

$$P(U(N,p)\neq\varnothing)\to 1 \quad \textit{as} \quad N\to\infty\,.$$

PROOF. If $p' < p$, then clearly $P(U(N,p') \neq \emptyset) > P(U(N,p) \neq \emptyset)$. So, it will be sufficient to prove the theorem when $p = (c/N) \log N$. By Robbins' theorem (see Kendall and Moran (1963), p. 109), the expected value of $u(N,p)$ is

$$E(u(N,p)) = \int_S P(x \in U(N,p))dx/(4\pi)\,,$$

where dx is the area element of S at x. Since $x \in U(N,p)$ implies that all the centers of the N caps fall outside the cap of angular radius a centered at x (where $p = \sin^2(a/2)$), we have

$$P(x \in U(N,p)) = (1-p)^N\,,$$

and hence

$$E(u(N,p)) = (1-p)^N\,.$$

Similarly,

$$\begin{aligned} E(u(N,p)^2) &= \int_S \int_S P(x, y \in U(N,p))dxdy/(16\pi^2) \\ &= \int_S P(x_0, y \in U(N,p))dy/(4\pi)\,, \end{aligned}$$

where x_0 is a fixed point on S. Suppose that x_0 and y subtend an angle θ at the center of the sphere and let $q(\theta)$ denote the fraction of S inside both of the caps with angular radius a and centers x_0 and y. Since x_0, $y \in U(N,p)$ implies that all the centers of the N caps fall outside both of the caps with radius a and centers x_0 and y, we have

$$P(x_0, y \in U(N,p)) = (1 - (2p - q(\theta)))^N\,.$$

Two points x_0 and y subtend an angle between θ and $\theta + d\theta$ at the center of the sphere if and only if y falls in a ring of area $2\pi \sin\theta\, d\theta$. Hence

$$E(u(N,p)^2) = \int_0^\pi (1 - (2p - q(\theta)))^N (1/2) \sin\theta\, d\theta\,,$$

and noting that $q(\theta) = 0$ for $\theta > 2a$,

$$\begin{aligned} E(u(N,p)^2) &< \int_0^{2a} (1-p)^N (1/2) \sin\theta\, d\theta + \int_{2a}^\pi (1-2p)^N (1/2) \sin\theta\, d\theta \\ &< (1-p)^N [-(1/2)\cos\theta]_0^{2a} + (1-2p)^N \end{aligned}$$

$$= (1-p)^N(1-\cos 2a)/2 + (1-2p)^N .$$

Since $(1-2p)^N < (1-p)^{2N} = E(u(N,p))^2$, and $(1-\cos 2a)/2 = 4p(1-p)$, it follows that the variance of $u(N,p)$ is

$$V(u(N,p)) = E(u(N,p)^2) - E(u(N,p))^2 < 4p(1-p)^{N+1} .$$

Thus

$$V(u(N,p))/E(u(N,p))^2 < 4p/(1-p)^{N-1} \sim 4pe^{pN}$$
$$= (4c \log N)/N^{1-c} \to 0 \quad \text{as} \quad N \to \infty .$$

Therefore, using Chebyshev's inequality,

$$P(u(N,p)=0) \leq P(|u(N,p) - E(u(N,p))| \geq E(u(N,p)))$$
$$< V(u(N,p))/E(u(N,p))^2 \to 0 ,$$

that is,

$$P(U(N,p) \neq \varnothing) \to 1 \quad \text{as} \quad N \to \infty .$$

THEOREM 3.2. *If $p = p(N) \geq (c/N) \log N$ for a constant $c > 1$, then for any integer $n > 0$,*

$$P(U_n(N,p) \neq \varnothing) \to 1 \quad \textit{as} \quad N \to \infty .$$

First we prove the following lemma.

LEMMA 3.1. *If $p = (c/N) \log N$ $(c > 1)$, then for any $n > 0$ and $\varepsilon > 0$,*

$$P(U_n(N,p) > \varepsilon p) \to 0 \quad \textit{as} \quad N \to \infty .$$

PROOF. By Robbins' theorem,

$$E(u_n(N,p)) = \int_S P(x \in U_n(N,p))dx/(4\pi) = P(x_0 \in U_n(N,p))$$
$$= \sum_{j=0}^{n} \binom{N}{j} p^j(1-p)^{N-j} \sim \sum_{j=0}^{n} (N^j/j!)p^j(1-p)^N$$
$$\sim (c \log N)^n/(n!N^c) .$$

Since $u_n(N,p)$ is a non-negative random variable, Markov's inequality asserts that

$$P(u_n(N,p) \geq tE(u_n(N,p))) < 1/t .$$

Hence, letting $t = \varepsilon p / E(u_n(N,p))$, we have

$$P(U_n(N,p) > \varepsilon p) < E(u_n(N,p))/(\varepsilon p) .$$

Since

$$\frac{E(u_n(N,p))}{\varepsilon p} \sim \frac{N(c \log N)^n}{(\varepsilon c \log N) n! N^c} = \frac{(c \log N)^{n-1}}{\varepsilon n! N^{c-1}} \to 0 ,$$

we have $P(U_n(N,p) > \varepsilon p) \to 0$ as $N \to \infty$.

PROOF OF THEOREM 3.2. It will be enough to prove the theorem when $p = (c/N) \log N (c > 1)$. Let $c' = (c+1)/2$ and $p' = (c'/c)p = (c' \log N)/N$. Then, since $c' > 1$, it follows from the above lemma that for any $\varepsilon > 0$,

$$P(U_n(N,p') > \varepsilon p') \to 0 \quad \text{as} \quad N \to \infty .$$

Let a' be the angular radius of a spherical cap of area $4\pi p'$. Then $a - a' > 0$. Choose a constant $\varepsilon > 0$ so that $\varepsilon(4\pi p')$ is less than the area of a cap of angular radius $a - a'$. For example, we may let $\varepsilon = (\sqrt{c/c'} - 1)^2/2$. To see this, note that since a is small, $\pi a^2 \sim 4\pi p$ and $a \sim 2\sqrt{p}$. Similarly, $a' \sim 2\sqrt{p'}$. Hence

$$\pi(a - a')^2 \sim 4\pi(\sqrt{p} - \sqrt{p'})^2 = 4\pi p'(\sqrt{c/c'} - 1)^2 ,$$

and

$$\varepsilon(4\pi p') = \frac{1}{2}(\sqrt{c/c'} - 1)^2 4\pi p' < \pi(a - a')^2 .$$

Now, let $x_1, \ldots, x_N$ be N random points on S and suppose that $u_n(N,p') \leq \varepsilon p'$ for the N caps of angular radius a' centered at x_i, $i = 1, \ldots, N$. Suppose that $U_n(N,p') \neq \emptyset$ and let y be a point of $U_n(N,p')$, i.e., y is covered at most n times by those N caps. Then there must be a point $z \notin U_n(N,p')$ within angular distance $a - a'$ from y. For otherwise, we have $u_n(N,p') > \varepsilon p'$, a contradiction. If a cap C of angular radius a' covers z, then the cap of angular radius a concentric with C covers y. Therefore, if we extend the size of each cap from angular radius a' to angular radius a, then those caps that cover z extend to caps that cover y, and hence y will be covered by more than n extended caps. This implies that

$$P(u_n(N,p') < \varepsilon p') < P(U_n(N,p) = \varnothing) .$$

Hence, $P(U_n(N,p) = \varnothing) \to 1$ as $N \to \infty$.

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