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THE PROGRESSIVELY TRUNCATED ESTIMATING FUNCTIONS AND ESTIMATORS

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Abstract. We consider the progressively truncated estimating functions and estimators as a generalization of the progressively truncated likelihood estimating functions and maximum likelihood estimators. We show the uniform consistency and weak convergence of the progressively truncated estimators.

Key words and phrases: Survival analysis, survivor function, progressive censoring, progressive truncation, estimating function and its estimator, martingale inequality, Gaussian process.

1. Introduction

In a survival analysis observations are lifetimes of individuals under study and are often censored either at a predetermined length of time or at a given proportion of the numbers of failures because of time, cost and the other consideration (see Lawless (1982) and Sen (1981)). Let observations $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with the distribution function $F_\theta(x)$ and the positive density function $f_\theta(x)$ on $R^+ = \{x > 0\}$ where $\theta \in \Theta$, a subset of $R^k = (-\infty, \infty)^k$. Sequentially in time we have at first the smallest observation $X_{n:1}$, next the second smallest $X_{n:2}$, and so on, and lastly the largest $X_{n:n}$, where $X_{n:1} \leq X_{n:2} \leq \cdots \leq X_{n:n}$ are the order statistics of $X_1, \ldots, X_n$. However, in fact, we must perform the statistical inference about parameter θ either with observations $\{X_i \leq t\}$ of type I censoring or with $\{X_{n:1} \leq \cdots \leq X_{n:r}\}$ of type II censoring.

Sen (1976) proposed the progressively censoring (**PC**) scheme "which allows us to monitor the experiment from beginning until a statistically valid decision with prescribed risks is made" and showed a Wiener process approximation for the PC likelihood ratio statistic. Sen and Tsong (1981) extended the result to the multiparameter case. Inagaki and Sen (1985) considered the progressively truncated (**PT**) maximum likelihood estimator and showed the uniform strong consistency and weak convergence of it to a Gaussian process.

In this paper, as a generalization of the PT maximum likelihood estimators, we consider the PT estimators which are defined as estimators based on the PT estimating functions. We discuss the relationship between the PT estimating functions and the PT estimators along the lines of Huber (1967) and Inagaki (1973). In Section 2 we define the PT score functions and the PT estimating functions. We state the assumptions used throughout this paper. It is verified that the PT estimating functions at the true parameter θ_0 converge to a Gaussian process. In Section 3 we prove the uniform consistency of the PT estimators. In Section 4 Huber's lemma about the asymptotic differentiability of estimating functions is extended to the PT situation. The weak convergence of the PT estimators to a Gaussian process is shown by using asymptotic relationships between the PT estimating functions and the PT estimators. An example is discussed in Section 5.

2. Assumptions and preliminaries

For the distribution function $F_\theta(x)$ and the empirical one $F_n(t)$, the survivor function and the empirical one are denoted by

$$\bar{F}_\theta(x) = 1 - F_\theta(x)\,,$$
$$\bar{F}_n(x) = 1 - F_n(x)\,,$$

respectively. Let $\psi(x,\theta)$ be a score function of an observation value x and a parameter θ and let us define the PT score function as

$$\psi_t(x,\theta) = \begin{cases} \psi(x,\theta) & \text{if} \quad x \le t\,, \\ \bar{\psi}(t,\theta) & \text{if} \quad x > t\,, \end{cases} \tag{2.1}$$

where

$$\bar{\psi}(t,\theta) = \int_t^\infty \psi(x,\theta) F_\theta(dx) / \bar{F}_\theta(t)\,. \tag{2.2}$$

Then, the PT estimating function is defined by

$$\Psi_{n:t}(\theta) = \sum_{i=1}^n \psi_t(x_i,\theta)\,. \tag{2.3}$$

When the observation for n subjects is truncated at time t (> 0), we obtain the observable random variables, $\{X_i \le t,\ i = 1,\dots,n\}$ and $\bar{F}_n(t)$, and then $\mathscr{B}_{n:t} = \mathscr{B}(X_{n:i} \le t,\ i = 1,\dots, nF_n(t))$ stands for the σ-field generated by them, letting $\mathscr{B}_{n:0} = \mathscr{B}(\{\phi\})$ and $\mathscr{B}_{n:\infty} = \mathscr{B}(X_1,\dots,X_n)$. Further, for an infinite sequence $\{X_i,\ i = 1,2,\dots\}$ of i.i.d. random variables, set $\mathscr{B}_t = \mathscr{B}(X_i \le t,\ i = 1,2,\dots)$, letting $\mathscr{B}_0 = \mathscr{B}(\{\phi\})$ and $\mathscr{B}_\infty = \mathscr{B}(X_i,\ i = 1,2,\dots)$.

We shall make the following assumptions throughout this paper.

(A1) The parameter space Θ is a compact subset of R^k and the true parameter θ_0 (say) is an inner point of Θ.

(A2) $\psi(x,\theta)$ is a k-vector valued measurable function for any fixed $\theta \in \Theta$ and is separable.

(A3) (i) For every $\theta \in \Theta$, the mean vector of $\psi(X_i,\theta)$,

$$\lambda(\theta) = E\psi(X_i,\theta) = \int_0^\infty \psi(x,\theta) F_{\theta_0}(dx)\,, \tag{2.4}$$

exists and vanishes only at θ_0: $\lambda(\theta_0) = 0$. (ii) For every $\theta \in \Theta$, the covariance matrix of $\psi(X_i,\theta)$,

$$\begin{aligned}\Gamma(\theta) &= \mathrm{Cov}_{\theta_0}[\psi(X_i,\theta)] \\ &= \int \{\psi(x,\theta) - \lambda(\theta)\}\{\psi(x,\theta) - \lambda(\theta)\}' F_{\theta_0}(dx)\,,\end{aligned} \tag{2.5}$$

exists and $\Gamma(\theta_0)$ is a positive definite.

(A4) (i) Let

$$u(x,\theta;d) = \sup\{|\psi(x,\tau) - \psi(x,\theta)| : |\tau - \theta| \le d\}\,.$$

There exist positive numbers d_0, b_1, b_2 and b_3 such that for every d with $0 < d \le d_0$,

$$E_{\theta_0}\{u(X_i,\theta;d)^p\} \le b_p d, \qquad p = 1,2\,. \tag{2.6}$$

(ii) For $\bar{\psi}(t,\theta)$ defined in (2.2) and for d with $0 < d \le d_0$,

$$\sup\{\bar{F}_{\theta_0}(t)|\bar{\psi}(t,\tau) - \bar{\psi}(t,\theta)| : t \in R^+,\ |\tau - \theta| \le d\} \le b_3 d\,. \tag{2.7}$$

(A5) (i) Let

$$\bar{\psi}^*(t,\theta) = \int_t^\infty \psi(x,\theta) F_{\theta_0}(dx)/\bar{F}_{\theta_0}(t)\,.$$

Then, there exist positive numbers b_4 and α with $0 < \alpha < 1$ such that

$$\sup\{\bar{F}_{\theta_0}(t)^{1-\alpha}|\bar{\psi}(t,\theta) - \bar{\psi}^*(t,\theta)| : t \in R^+, \theta \in \Theta\} < b_4\,. \tag{2.8}$$

(ii) For any fixed $t \in R^+$, and $\theta \in U_0 = \{\theta: |\theta - \theta_0| < d_0\}$ (say), there are positive numbers b_5 and α with $0 < \alpha < 1$ such that

$$\sup\{\bar{F}_{\theta_0}(t)^{1-\alpha}|\bar{\psi}(t,\theta) - \bar{\psi}^*(t,\theta)|/|\theta-\theta_0|: t \in R^+, \theta \in U_0\} \leq b_5 . \tag{2.9}$$

(A6) Set

$$\begin{aligned} \lambda_t(\theta) &= E_{\theta_0}\{\psi_t(X_i,\theta)\} \\ &= \int_0^t \psi(x,\theta)F_{\theta_0}(dx) + \bar{F}_{\theta_0}(t)\bar{\psi}(t,\theta) . \end{aligned} \tag{2.10}$$

For any fixed $t_0 > 0$, let $R_0 = \{t: t \geq t_0\}$. Then, $\lambda_t(\theta)$ vanishes only at $\theta = \theta_0$ for $t \in R_0$ and $\lambda_t(\theta)$ is differentiable at θ_0 where the Jacobian

$$\Lambda_t(\theta_0) = (\partial/\partial\theta^{(j)}\lambda_t^{(i)}(\theta_0)), \qquad i, j = 1,\ldots,k ,$$

is nonsingular and $\Lambda_0 = \sup\{|\Lambda_t(\theta_0)|: t \in R_0\}$ is finite. Furthermore,

$$\begin{aligned} \sup\{|n^{1/2}\lambda_t(\theta_0 + n^{-1/2}h) - \Lambda_t(\theta_0)h|: t \in R_0, |h| \leq M\} \\ \to 0 \quad \text{as} \quad n \to \infty . \end{aligned} \tag{2.11}$$

(A7) Set

$$\Gamma_t(\theta) = \mathrm{Cov}_{\theta_0}[\psi_t(X_i,\theta)] .$$

For every $t \in R_0$, $\Gamma_t(\theta_0)$ is a positive definite matrix.

Remarks. (a) Assumption (A3) implies that the mean vector and covariance matrix of $\psi_t(X_i,\theta)$, $\lambda_t(\theta)$ and $\Gamma_t(\theta)$, exist for every $\theta \in \Theta$. Furthermore, it follows from the definitions and Assumption (A4) that $\lambda_t(\theta)$ and $\Gamma_t(\theta)$ are continuous in $(t,\theta) \in R^+ \times \Theta$. (b) For any $\theta \in \Theta$,

$$\begin{aligned} \lambda_0(\theta) &= 0, \qquad \lambda_\infty(\theta) = \lambda(\theta) , \\ \Gamma_0(\theta) &= 0, \qquad \Gamma_\infty(\theta) = \Gamma(\theta) . \end{aligned}$$

For any $t > 0$

$$\begin{aligned} \lambda_t(\theta_0) &= 0 , \\ \bar{\psi}^*(t,\theta_0) &= \bar{\psi}(t,\theta_0) , \\ \Gamma_t(\theta_0) &= \Gamma(\theta_0) - \bar{\Gamma}_t(\theta_0) , \end{aligned} \tag{2.12}$$

where

$$\bar{\Gamma}_t(\theta_0) = \int_t^\infty \{\psi(x,\theta_0) - \bar{\psi}(x,\theta_0)\}\{\psi(x,\theta_0) - \bar{\psi}(x,\theta_0)\}' F_{\theta_0}(dx) . \tag{2.13}$$

Thus, $\Gamma_t(\theta_0)$ is nondecreasing in $t \in R^+$.

For $0 \leq t < \infty$, set

$$\psi_t^*(x,\theta) = \begin{cases} \psi(x,\theta) & \text{if} \quad x \leq t , \\ \bar{\psi}^*(t,\theta) & \text{if} \quad x > t , \end{cases} \tag{2.14}$$

and

$$\Psi_{n:t}^*(\theta) = \sum_{i=1}^n \psi_t^*(X_i,\theta) , \tag{2.15}$$

letting $\psi_0^*(x,\theta) = \lambda(\theta)$, $\psi_\infty^*(x,\theta) = \psi(x,\theta)$, $\Psi_{n:0}^*(\theta) = n\lambda(\theta)$ and

$$\Psi_{n:\infty}^*(\theta) = \sum_{i=1}^n \psi(x,\theta) = \Psi_n(\theta) \quad \text{(say)} . \tag{2.16}$$

LEMMA 2.1. *$\{\Psi_{n:t}^*(\theta), \mathscr{B}_{n:t}; t \in R^+\}$ is a martingale closed on the right by $\Psi_n(\theta)$.*

PROOF. It is easy to see that

$$\psi_t^*(x,\theta) = E_{\theta_0}[\psi(X_1,\theta) | \mathscr{B}_{1:t}]$$

and

$$E_{\theta_0}\{\psi_t^*(X_i,\theta)\} = E_{\theta_0}\{\psi(X_i,\theta)\} = \lambda(\theta) .$$

Immediately from the definitions, it follows that $\mathscr{B}_{n:t}$ is nondecreasing in $t \in R^+$, $\Psi_{n:\infty}^*(\theta) = \Psi_n(\theta)$ and

$$E_{\theta_0}[\Psi_n(\theta) | \mathscr{B}_{n:t}] = \Psi_{n:t}^*(\theta) .$$

The proof is complete.

LEMMA 2.2. *Sample function $\Psi_n^\circ = \{\Psi_{n:t}^\circ(\theta_0); t \in R^+\}$ with*

$$\Psi_{n:t}^\circ(\theta_0) = n^{-1/2} \Psi_{n:t}(\theta_0) \tag{2.17}$$

has the following moment structures:

$$E_{\theta_0}\{\Psi^{\circ}_{n:t}(\theta_0)\} = 0\ ,$$
$$E_{\theta_0}\{\Psi^{\circ}_{n:s}(\theta_0)\Psi^{\circ}_{n:t}(\theta_0)'\} = \Gamma_{s\wedge t}(\theta_0), \qquad s, t \in R^+\ ,$$

where $s \wedge t = \min(s, t)$.

PROOF. From (1.3), (2.3) and (2.5), we have

$$E_{\theta_0}\{\Psi^{\circ}_{n:t}(\theta_0)\} = 0\ .$$

The martingale property of Lemma 2.1 implies that for $0 < s \le t$,

$$\begin{aligned} E_{\theta_0}\{\Psi^{\circ}_{n:s}(\theta_0)\Psi^{\circ}_{n:t}(\theta_0)'\} &= E_{\theta_0}\{\Psi^{\circ}_{n:s}(\theta_0)\Psi^{\circ}_{n:s}(\theta_0)'\} \\ &= E_{\theta_0}\{\psi_s(X_i, \theta_0)\psi_s(X_i, \theta_0)'\} \\ &= \Gamma_s(\theta_0)\ . \end{aligned}$$

The proof is complete.

Let us consider a k-variate Gaussian process $\Psi^{\circ} = \{\Psi^{\circ}_t(\theta_0); t \in R^+\}$ defined by

$$\Psi^{\circ}_t(\theta_0) = \int_0^\infty \psi_t(x, \theta_0) W^{\circ}(F_{\theta_0}(dx)), \qquad t \in R^+\ , \tag{2.18}$$

with $\Psi^{\circ}_0(\theta_0) = 0$ with probability one and

$$\Psi^{\circ}_\infty(\theta_0) = \Psi^{\circ}(\theta_0) = \int_0^\infty \psi(x, \theta_0) W^{\circ}(F_{\theta_0}(dx))\ ,$$

where $W^{\circ} = \{W^{\circ}(t), 0 \le t \le 1\}$ is the Brownian bridge on $[0, 1]$ so that W° is Gaussian with $E\{W^{\circ}(t)\} = 0$ and $E\{W^{\circ}(s)W^{\circ}(t)\} = s \wedge t - st$ for $s, t \in [0, 1]$. Then, the following lemma holds.

LEMMA 2.3. $\{\Psi^{\circ}_t(\theta_0), \mathscr{B}_t; t \in R^+\}$ *is a martingale closed on the right by* $\Psi^{\circ}(\theta_0)$ *and* Ψ° *has the same moment structure as* Ψ°_n *defined in* (2.17):

$$E\{\Psi^{\circ}_t(\theta_0)\} = 0\ ,$$
$$E\{\Psi^{\circ}_s(\theta_0)\Psi^{\circ}_t(\theta_0)'\} = \Gamma_{s\wedge t}(\theta_0), \qquad s, t \in R^+\ .$$

PROOF. From the definitions, we have, $\mathscr{B}_t$ is nondecreasing in $t \in R^+$ and $\Psi^{\circ}_\infty(\theta_0) = \Psi^{\circ}(\theta_0)$. Since

$$\Psi^\circ(\theta_0) = \int_0^\infty \psi(x,\theta_0) W^\circ(F_{\theta_0}(dx)) = \int_0^\infty \psi(x,\theta_0) W(F_{\theta_0}(dx)) ,$$

where $W(t)$, $0 \le t \le 1$ is a Wiener process, it follows that

$$E_{\theta_0}[\Psi^\circ(\theta_0) | \mathscr{B}_t] = \int_0^\infty \psi_t(x,\theta_0) W(F_{\theta_0}(dx))$$
$$= \int_0^\infty \psi_t(x,\theta_0) W^\circ(F_{\theta_0}(dx)) = \Psi_t^\circ(\theta_0) ,$$

and hence, $\{\Psi_t^\circ(\theta_0), \mathscr{B}_t; t \in R^+\}$ is a martingale. Thus the first part of the lemma is proved. The latter part is easy to understand.

Furthermore, we have the following theorem which could be proved similarly as in the proof of Theorem 1 of Sen and Tsong (1981).

THEOREM 2.1. *The process $\Psi_n^\circ = \{\Psi_{n:t}^\circ(\theta_0); t \in R^+\}$ converges weakly to the Gaussian process $\Psi^\circ = \{\Psi_t^\circ(\theta_0); t \in R^+\}$ in the extended Skorokhod's topology on $D^k[R^+]$.*

3. Uniform consistency of the PT estimators

We define the PT estimator $T_{n:t}$ by the value of θ which the PT estimating function $\Psi_{n:t}(\theta)$ vanishes. However, in order to prove the uniform consistency (3.2) below, it is only required that $T_{n:t}$ is $\mathscr{B}_{n:t}$-measurable and $\{T_{n:t}; t \in R^+\}$ belongs to $D^k[R^+]$ and satisfies the following condition:

$$\sup \{n^{-1} | \Psi_{n:t}(T_{n:t})| : t \in R^+\} \to 0 , \tag{3.1}$$

in probability as $n \to \infty$.

THEOREM 3.1. *Under the condition* (3.1), *it holds that for any fixed* $t_0 > 0$

$$\sup \{| T_{n:t} - \theta_0| : t \in R_0\} \to 0 , \tag{3.2}$$

in probability as $n \to \infty$, where $R_0 = \{t; t \ge t_0\}$.

For the proof of this theorem we provide the following lemma.

LEMMA 3.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\lim_{n\to\infty} P_{\theta_0}[\inf \{n^{-1} | \Psi_{n:t}(\theta)| : t \in R_0, \theta \in \Theta_0\} > \delta] = 1 , \tag{3.3}$$

where $\Theta_0 = \{\theta \in \Theta; |\theta - \theta_0| \geq \varepsilon\}$.

PROOF. From Assumptions (A3) and (A6) and Remarks (a) and (b), we have

$$\lambda_0 = \inf\{|\lambda_t(\theta)|: t \in R_0, \theta \in \Theta_0\} \quad \text{(say)} \tag{3.4}$$

is positive and

$$\Gamma_0 = \sup\{|\Gamma(\theta)|: \theta \in \Theta\} \quad \text{(say)} \tag{3.5}$$

is finite. It follows from (2.3), (2.10) and (2.15) that

$$\begin{aligned} n^{-1}\Psi_{n:t}(\theta) - \lambda_t(\theta) = &\{n^{-1}\Psi^*_{n:t}(\theta) - \lambda(\theta)\} \\ &+ \{\bar{F}_n(t) - \bar{F}_{\theta_0}(t)\}\{\bar{\psi}(t,\theta) - \bar{\psi}^*(t,\theta)\}, \end{aligned} \tag{3.6}$$

and hence, from (2.8) and (3.4)

$$\begin{aligned} &\inf\{n^{-1}|\Psi_{n:t}(\theta)|: t \in R_0, \theta \in \Theta_0\} \\ &\quad \geq \lambda_0 - \sup\{|n^{-1}\Psi^*_{n:t}(\theta) - \lambda(\theta)|: t \in R^+, \theta \in \Theta_0\} \\ &\quad\quad - b_4 \sup\{|\bar{F}_n(t)/\bar{F}_{\theta_0}(t) - 1|\,\bar{F}_{\theta_0}(t)^{\alpha}: t \in R^+\}. \end{aligned} \tag{3.7}$$

Let us consider an open covering of the compact set Θ_0, $\{U_d(\theta); \theta \in \Theta_0\}$, with $U_d(\theta) = \{\tau; |\tau - \theta| < d\}$ where d is determined latter. Then, there is its finite open covering $\{U_d(\theta_j); j = 1,\ldots,m\}$ such that $\theta_j \in \Theta_0$, $j = 1,\ldots,m$ and $\bigcup_{i=1}^{n} U_d(\theta_j) \supset \Theta_0$. Thus, we see from (2.6) that

$$\begin{aligned} &\sup\{|n^{-1}\Psi^*_{n:t}(\theta) - \lambda(\theta)|: t \in R^+, \theta \in \Theta_0\} \\ &\quad \leq \sup\{|n^{-1}\Psi^*_{n:t}(\theta_j) - \lambda(\theta_j)|: t \in R^+, j = 1,\ldots,m\} \\ &\quad\quad + \sup\{|n^{-1}\Psi^*_{n:t}(\theta) - n^{-1}\Psi^*_{n:t}(\theta_j)|: \\ &\quad\quad\quad\quad t \in R^+, \theta \in U_d(\theta_j), j = 1,\ldots,m\} + b_1 d. \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} &n^{-1}\Psi^*_{n:t}(\theta) - n^{-1}\Psi^*_{n:t}(\theta_j) \\ &\quad = n^{-1}\sum_{X_i \leq t}\{\psi(X_i,\theta) - \psi(X_i,\theta_j)\} \\ &\quad\quad + \{\bar{F}_n(t)/\bar{F}_{\theta_0}(t)\}\int_t^{\infty}\{\psi(x,\theta) - \psi(x,\theta_j)\}F_{\theta_0}(dx), \end{aligned} \tag{3.9}$$

we have from (2.6)

$$(3.10)\quad \sup\{|n^{-1}\Psi^*_{n:t}(\theta) - n^{-1}\Psi^*_{n:t}(\theta_j)|: t \in R^+, \theta \in U_d(\theta_j), j = 1,\dots,m\}$$
$$\leq \sup\left\{\left| n^{-1}\sum_{i=1}^{n} u(X_i,\theta_j; d) - E_{\theta_0}u(X_i,\theta_j; d)\right| : j = 1,\dots,m\right\}$$
$$+ b_1 d[\sup\{\bar{F}_n(t)/\bar{F}_{\theta_0}(t): t \in R^+\} + 1].$$

Letting $\delta = \lambda_0/6$ and $L = \delta/(b_1 d) - 1$ for $0 < d < \delta/b_1$, we obtain the following inequality from (3.7), (3.8) and (3.10):

$$(3.11)\quad P_{\theta_0}[\inf\{n^{-1}|\Psi_{n:t}(\theta)|: t \in R_0, \theta \in \Theta_0\} > \delta]$$
$$\geq 1 - P_{\theta_0}[\sup\{|n^{-1}\Psi^*_{n:t}(\theta_j) - \lambda(\theta_j)|: j = 1,\dots,m, t \in R^+\} > \delta]$$
$$- P_{\theta_0}\left[\sup\left\{\left| n^{-1}\sum_{i=1}^{n} u(X_i,\theta_j; d) - E_{\theta_0}u(X_i,\theta_j; d)\right| : j = 1,\dots,m\right\} > \delta\right]$$
$$- P_{\theta_0}[\sup\{\bar{F}_n(t)/\bar{F}_{\theta_0}(t): t \in R^+\} \geq L]$$
$$- P_{\theta_0}[\sup\{|\bar{F}_n(t)/\bar{F}_{\theta_0}(t) - 1|\bar{F}_{\theta_0}(t)^{\alpha}: t \in R^+\} > \delta/b_4].$$

Lemma 2.1 and the martingale maximal inequality (see Karlin and Taylor (1981), p. 280) imply that

$$(3.12)\quad P_{\theta_0}[\sup\{|n^{-1}\Psi^*_{n:t}(\theta_j) - \lambda(\theta_j)|: j = 1,\dots,m, t \in R^+\} > \delta]$$
$$\leq \sum_{i=1}^{m} P_{\theta_0}[|n^{-1}\Psi_n(\theta_j) - \lambda(\theta_j)| > \delta]$$
$$\leq m\Gamma_0/(n\delta^2) \to 0 \quad \text{as} \quad n \to \infty,$$

recalling the definition of Γ_0 in (3.5). By the Chebyshev's inequality and (2.6), it follows that

$$(3.13)\quad P_{\theta_0}\left[\sup\left\{\left| n^{-1}\sum_{i=1}^{n} u(X_i,\theta_j; d) - E_{\theta_0}u(X_i,\theta_j; d)\right| : j = 1,\dots,m\right\} > \delta\right]$$
$$\leq m b_2 d/(n\delta^2) \to 0 \quad \text{as} \quad n \to \infty.$$

Chernoff and Rubin (1956) showed that $\sup\{\bar{F}_n(t)/\bar{F}_{\theta_0}(t): t \in R^+\}$ is bounded in probability. That is, suitably choosing $d > 0$ and equivalently $L > 0$ for any $\eta > 0$, it holds that

$$(3.14)\quad P_{\theta_0}[\sup\{\bar{F}_n(t)/\bar{F}_{\theta_0}(t): t \in R^+\} > L] < \eta \quad \text{for all} \quad n.$$

Similarly, for $T > 0$ such that $\bar{F}_{\theta_0}(T)^\alpha < \delta/\{b_4(L+1)\}$, it holds that

$$(3.15)\quad \begin{aligned} &P_{\theta_0}[\sup\{|\bar{F}_n(t)/\bar{F}_{\theta_0}(t) - 1|\bar{F}_{\theta_0}(t)^\alpha : t \geq T\} > \delta/b_4] \\ &\leq P_{\theta_0}[\sup\{\bar{F}_n(t)/\bar{F}_{\theta_0}(t) : t \in R^+\} > L] < \eta \quad \text{for all} \quad n . \end{aligned}$$

Furthermore, it follows by Kolmogorov's theorem that

$$(3.16)\quad \begin{aligned} &P_{\theta_0}[\sup\{|\bar{F}_n(t)/\bar{F}_{\theta_0}(t) - 1|\bar{F}_{\theta_0}(t)^\alpha : 0 \leq t \leq T\} > \delta/b_4] \\ &\leq P_{\theta_0}[\sup\{|\bar{F}_n(t) - \bar{F}_{\theta_0}(t)| : t \in R^+\} > (\delta/b_4)\bar{F}_{\theta_0}(T)^{1-\alpha}] \\ &\to 0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

Since $\eta > 0$ is arbitrary, (3.15) and (3.16) lead to

$$(3.17)\quad \lim_{n\to\infty} P_{\theta_0}[\sup\{|\bar{F}_n(t)/F_{\theta_0}(t) - 1|F_{\theta_0}(t)^\alpha : t \in R^+\} > \delta/b_4] = 0 .$$

Therefore, we conclude from (3.11)–(3.14) and (3.17) that

$$\lim_{n\to\infty} P_{\theta_0}[\inf\{n^{-1}|\Psi_{n:t}(\theta)| : t \in R_0, \theta \in \Theta_0\} > \delta] \geq 1 - \eta ,$$

for arbitrary $\eta > 0$. Thus, the proof of this lemma is complete.

PROOF OF THEOREM 3.1. Since

$$\begin{aligned} &P_{\theta_0}[\sup\{|T_{n:t} - \theta_0| : t \in R_0\} \geq \varepsilon] \\ &\quad \leq P_{\theta_0}[\sup\{|n^{-1}\Psi_{n:t}(T_{n:t})| : t \in R_0\} > \delta] \\ &\qquad + P_{\theta_0}[\inf\{n^{-1}|\Psi_{n:t}(\theta)| : t \in R_0, \theta \in \Theta_0\} \leq \delta] , \end{aligned}$$

(3.1) and (3.3) lead to (3.2).

4. Weak convergence of the PT estimators to a Gaussian process

In this section we suppose that the PT estimator $\{T_{n:t}; t \in R^+\}$ satisfies the following condition stronger than (3.1): $\{T_{n:t}; t \in R^+\}$ belongs to $D^k[R^+]$ and

$$(4.1)\qquad \sup\{n^{-1/2}|\Psi_{n:t}(T_{n:t})| : t \in R^+\} \to 0 ,$$

in probability as $n \to \infty$. We shall, first, modify Lemma 3 of Huber (1967) and its proof in Lemmas 4.1, 4.2 and Theorem 4.1 (below) so that the uniform asymptotic differentiability holds for the PT estimating function $\Psi_{n:t}(\theta)$ defined in (2.3) at $\theta = \theta_0$. Next, we shall prove that the PT estimator satisfying (4.1) converges weakly to a Gaussian process.

Set

$$Z_{n:t}(\tau,\theta)=\frac{|\Psi_{n:t}(\tau)-\Psi_{n:t}(\theta)-n\lambda_t(\tau)+n\lambda_t(\theta)|}{n^{1/2}+n|\lambda_t(\tau)|} \tag{4.2}$$

and

$$Z^*_{n:t}(\tau,\theta)=\frac{|\Psi^*_{n:t}(\tau)-\Psi^*_{n:t}(\theta)-n\lambda(\tau)+n\lambda(\theta)|}{n^{1/2}+n|\lambda_t(\tau)|}\,. \tag{4.3}$$

It follows from (3.6) that

$$\begin{aligned}\Psi_{n:t}(\tau)-\Psi_{n:t}(\theta_0)-n\lambda_t(\tau)&=\{\Psi^*_{n:t}(\tau)-\Psi^*_{n:t}(\theta_0)-n\lambda(\tau)\}\\&\quad+n\{\bar F_n(t)-\bar F_{\theta_0}(t)\}\{\bar\psi(t,\tau)-\bar\psi^*(t,\tau)\}\,,\end{aligned}$$

and hence

$$Z_{n:t}(\tau,\theta_0)\le Z^*_{n:t}(\tau,\theta_0)+W_{n:t}(\tau,\theta_0)\,, \tag{4.4}$$

where

$$W_{n:t}(\tau,\theta_0)=\frac{|\bar F_n(t)-\bar F_{\theta_0}(t)|\,|\bar\psi(t,\tau)-\bar\psi^*(t,\tau)|}{n^{-1/2}+|\lambda_t(\tau)|}\,. \tag{4.5}$$

Let us choose $d_0>0$ as in Assumption (A4) and $U_0=\{\theta;|\theta-\theta_0|<d_0\}$ as in Assumptions (A5) and (A6). Let $R_0=\{t;t\ge t_0\}$ for any fixed $t_0>0$.

LEMMA 4.1.

$$\sup\{W_{n:t}(\tau,\theta_0):\,t\in R_0,\tau\in U_0\}\to 0\,, \tag{4.6}$$

in probability as $n\to\infty$.

PROOF. From Assumption (A6) we can choose $d_0>0$ in Assumption (A4) and $a>0$ such that

$$|\lambda_t(\tau)|\ge a|\tau-\theta_0|\,, \tag{4.7}$$

for $t\in R_0$ and $\theta\in U_0$. It follows from Assumption (A5) (ii) that

$$\sup\{W_{n:t}(\tau,\theta_0):\,t\in R_0,\tau\in U_0\}$$

$$\leq \sup \{ | \bar{F}_n(t) / \bar{F}_{\theta_0}(t) - 1 | \bar{F}_{\theta_0}(t)^{\alpha} b_5 / a \colon t \in R^+ \} .$$

This and (3.17) complete the proof of this lemma.

LEMMA 4.2.

$$\sup \{ Z^*_{n:t}(\tau, \theta_0) \colon t \in R_0, \tau \in U_0 \} \to 0 , \tag{4.8}$$

in probability as $n \to \infty$.

PROOF. Without loss of generality we take $\theta_0 = 0$ and $d_0 = 1$. P, E and V stand for those under $\theta = \theta_0$ and F stands for F_{θ_0}. As in the proof of Huber (1967), we define cubes

$$C_m = \{\theta;\ |\theta| \leq (1 - q)^m\}, \qquad m = 0, 1, \ldots, m_0 ,$$

and further, subcubes C_{m-1,ξ_j} of $C_{m-1} - C_m$ with center ξ_j and edges of length $2\delta_j$ such as

$$\begin{aligned} |\xi_j| &= (1 - q)^{m-1}(1 - q/2) , \\ 2\delta_j &= (1 - q)^{m-1} q , \end{aligned} \tag{4.9}$$

respectively, where $q = 1/M$ and M is an integer such that for any $\varepsilon > 0$ and L in (3.14),

$$M \geq (2Lb_1)/(\varepsilon a) , \tag{4.10}$$

then, the cube $C_0 = \{\theta; |\theta| \leq 1\}$ is divided into subcubes $C_{(j)}$, $j = 1, \ldots, N$ (of $C_0 - C_{m_0}$) with center and edges according to (4.9) and C with center 0 and edges of length

$$2\delta_0 = 2(1 - q)^{m_0} . \tag{4.11}$$

Here, defining $m_0 = m_0(n)$ by

$$m_0(n) - 1 < \gamma \log n / |\log (1 - q)| \leq m_0(n) , \tag{4.12}$$

for any γ with $1/2 < \gamma < 1$, we have the total number as follows:

$$N = O(\log n) . \tag{4.13}$$

Since, for $\tau \in C_{(j)}$ and $t \in R_0$,

$$|\lambda_t(\tau)| \geq a|\tau| \geq a(1-q)^m \qquad \text{(recalling (4.7))},$$

$$(4.14) \qquad |\lambda(\tau) - \lambda(\xi_j)| \leq Eu(X_i, \xi_j; \delta_j) \leq b_1\delta_j \leq b_1(1-q)^m q/2\,,$$

$$V\{u(X_i, \xi_j; \delta_j)\} \leq Eu(X_i, \xi_j; \delta_j)^2 \leq b_2\delta_j \leq b_2(1-q)^m q/2\,,$$

it follows by the similar calculation as in (3.11) that

$$\begin{aligned} Z^*_{n:t}(\tau, 0) &\leq Z^*_{n:t}(\tau, \xi_j) + Z^*_{n:t}(\xi_j, 0) \\ &\leq \Bigg[\Bigg| n^{-1} \sum_{i=1}^{n} \{u(X_i, \xi_j; \delta_j) - Eu(X_i, \xi_j; \delta_j)\} \Bigg| + b_1(1-q)^m q \\ &\quad + \{\bar{F}_n(t)/\bar{F}(t)\} b_1(1-q)^m q \Bigg] \Bigg/ \{a(1-q)^m\}\,, \\ &\quad + |n^{-1}\{\Psi^*_{n:t}(\xi_j) - \Psi^*_{n:t}(0) - n\lambda(\xi_j)\}| / \{a(1-q)^m\}\,, \end{aligned}$$

and hence, from (4.10)

$$\begin{aligned} (4.15) \quad & \sup\{Z^*_{n:t}(\tau, 0) \colon t \in R^+, \tau \in C_{(j)}\} \\ & \leq U_n(\xi_j, \delta_j)/\{a(1-q)^m\} + b_1 q/a + \sup\{\bar{F}_n(t)/\bar{F}(t) \colon t \in R^+\}(\varepsilon/L) \\ & \quad + \sup\{V_{n:t}(\xi_j) \colon t \in R^+\}/\{a(1-q)^m\}\,, \end{aligned}$$

letting

$$(4.16) \qquad \begin{aligned} U_n(\xi, \delta) &= \Bigg| n^{-1} \sum_{i=1}^{n} \{u(X_i, \xi; \delta) - Eu(X_i, \xi; \delta)\} \Bigg|\,, \\ V_{n:t}(\xi) &= |n^{-1}\{\Psi^*_{n:t}(\xi) - \Psi^*_{n:t}(0) - n\lambda(\xi)\}|\,. \end{aligned}$$

By Chebyshev's inequality and from (4.10), (4.12) and (4.14) we have

$$\begin{aligned} (4.17) \qquad & P[U_n(\xi_j, \delta_j)/\{a(1-q)^m\} + b_1 q/a \geq \varepsilon] \\ & \leq P\{U_n(\xi_j, \delta_j) \geq L b_1 q (1-q)^m\} \\ & \leq V\{u(X_i, \xi_j; \delta_j)\}/[n\{L b_1 q(1-q)^m\}^2] \\ & \leq [b_2/\{2L^2 b_1 q(1-q)\}]\{n(1-q)^{m-1}\}^{-1} \\ & \leq K_1\{n(1-q)^{m_0-1}\}^{-1} \leq K_1 n^{\gamma-1}\,, \end{aligned}$$

where K_1 is a constant. Since $\{V_{n:t}(\xi_j), \mathscr{B}_{n:t}; t \in R^+\}$ is a submartingale closed on the right by

$$V_n(\xi_j) = |n^{-1}\Psi_n(\xi_j) - \Psi_n(0) - n\lambda(\xi_j)|\,,$$

we have, by the martingale maximal inequality,

$$
\begin{aligned}
(4.18)\qquad & P[\sup\{V_{n:t}(\xi_j)/(a(1-q)^m): t \in R^+\} \geq \varepsilon] \\
& \leq V(V_n(\xi_j))/\{\varepsilon a(1-q)^m\}^2 \\
& \leq Eu(X_i, 0; |\xi_j|)^2/[n\{2Lb_1 q(1-q)^m\}^2] \\
& \leq [b_2/\{2Lb_1 q(1-q)\}^2]\{n(1-q)^{m-1}\}^{-1} \\
& \leq K_2\{n(1-q)^{m_0-1}\}^{-1} \leq K_2 n^{\gamma-1},
\end{aligned}
$$

where K_2 is a constant. Therefore, (4.15), (4.17) and (4.18) lead to

$$
\begin{aligned}
(4.19)\qquad & P[\sup\{Z^*_{n:t}(\tau, 0): t \in R^+, \tau \in C_{(j)}, j = 1,\dots, N\} \geq 3\varepsilon] \\
& \leq N(K_1 + K_2)n^{\gamma-1} + P[\sup\{\bar{F}_n(t)/\bar{F}(t): t \in R^+\} \geq L].
\end{aligned}
$$

Similarly, we see from (4.11) and (4.12) that for $\tau \in C_{m_0}$

$$
\begin{aligned}
Z^*_{n:t}(\tau, 0) \leq{} & n^{-1/2} \sum_{i=1}^{n} \{u(X_i, 0; \delta_0) - Eu(X_i, 0; \delta_0)\} \\
& + 2n^{1/2} Eu(X_i, 0; \delta_0) \\
& + \{\bar{F}_n(t)/\bar{F}(t)\} n^{1/2} Eu(X_i, 0; \delta_0) \\
\leq{} & n^{1/2} U_n(0, \delta_0) + 2n^{1/2} b_1 + \{\bar{F}_n(t)/\bar{F}(t)\} n^{-\gamma+1/2} b_1,
\end{aligned}
$$

and

$$
P\{n^{1/2} U_n(0, \delta_0) \geq \varepsilon\} \leq b_2(1-q)^{m_0}/\varepsilon^2 \leq K_3 n^{-\gamma},
$$

where K_3 is a constant, and hence, that for n with $2n^{-\gamma+1/2} b_1 L \leq \varepsilon$

$$
\begin{aligned}
(4.20)\qquad & P[\sup\{Z^*_{n:t}(\tau, 0): t \in R^+, \tau \in C_{m_0}\} \geq 3\varepsilon] \\
& \leq K_3 n^{-\gamma} + P[\sup\{\bar{F}_n(t)/\bar{F}(t): t \in R^+\} \geq L].
\end{aligned}
$$

Consequently, (4.19) and (4.20) imply that

$$
\begin{aligned}
& P[\sup\{Z^*_{n:t}(\tau, 0): t \in R^+, \tau \in C_0\} \geq 3\varepsilon] \\
& \quad \leq P[\sup\{Z^*_{n:t}(\tau, 0): t \in R^+, \tau \in C_{(j)}, j = 1,\dots, N\} \geq 3\varepsilon] \\
& \qquad + P[\sup\{Z^*_{n:t}(\tau, 0): t \in R^+, \tau \in C_{m_0}\} \geq 3\varepsilon] \\
& \quad \leq N(K_1 + K_2)n^{\gamma-1} + K_3 n^{-\gamma} + 2P[\sup\{\bar{F}_n(t)/F(t): t \in R^+\} \geq L] \\
& \quad \leq O(n^{\gamma-1} \log n) + O(n^{-\gamma}) + o(1) + 2\eta,
\end{aligned}
$$

for arbitrary $\eta > 0$, according to (3.14) and (4.13). The proof of this lemma is complete.

Immediately from (4.4), (4.6) and (4.8) we have the uniform asymptotic differentiability of the PT estimating function $\Psi_{n:t}(\theta)$ at $\theta = \theta_0$:

THEOREM 4.1.

$$\sup \{Z_{n:t}(\tau, \theta_0) \colon t \in R_0, \tau \in U_0\} \to 0 , \tag{4.21}$$

in probability as $n \to \infty$.

LEMMA 4.3. *For every $M > 0$ and $\Lambda_t(\theta_0)$ in Assumption* (A6),

$$\sup \{|n^{-1/2}\Psi_{n:t}(\theta_0 + n^{-1/2}h) - n^{-1/2}\Psi_{n:t}(\theta_0) - \Lambda_t(\theta_0)h| \colon t \in R_0, |h| \le M\} \to 0 , \tag{4.22}$$

in probability as $n \to \infty$.

PROOF. Since $|n^{-1/2}h| \le d_0$ for $|h| \le M$ and sufficiently large n, (4.21) holds:

$$\sup \{Z_{n:t}(\theta_0 + n^{-1/2}h, \theta_0) \colon t \in R_0, |h| \le M\} \to 0 , \tag{4.23}$$

in probability as $n \to \infty$. It follows from (4.2), (4.16) and (4.24) that for $\varepsilon > 0$ and all sufficiently large n

$$Z_{n:t}(\theta_0 + n^{-1/2}h, \theta_0) \ge n^{1/2} V_{n:t}(\theta_0 + n^{-1/2}h)/(1 + \varepsilon + \Lambda_0 M) ,$$

recalling $\Lambda_0 = \sup \{\Lambda_t(\theta_0) \colon t \in R_0\}$, and thus, from (4.23) that

$$\sup \{n^{1/2} V_{n:t}(\theta_0 + n^{-1/2}h) \colon t \in R_0, |h| \le M\} \to 0 ,$$

in probability as $n \to \infty$. Therefore, we conclude from (2.11)

$$\begin{aligned} &\sup \{|n^{-1/2}\Psi_{n:t}(\theta_0 + n^{-1/2}h) - n^{-1/2}\Psi_{n:t}(\theta_0) - \Lambda_t(\theta_0)h| \colon t \in R_0, |h| \le M\} \\ &\quad \le \sup \{n^{1/2} V_{n:t}(\theta_0 + n^{-1/2}h) \colon t \in R_0, |h| \le M\} \\ &\qquad + \sup \{|n^{1/2}\lambda(\theta_0 + n^{-1/2}h) - \Lambda_t(\theta_0)h| \colon t \in R_0, |h| \le M\} \\ &\quad \to 0 \end{aligned}$$

in probability as $n \to \infty$.

THEOREM 4.2. *Under the condition* (4.1), *for any* $t_0 > 0$, *the PT estimator* $\{n^{1/2}(T_{n:t} - \theta_0);\ t_0 \le t < \infty\}$ *converges weakly in* $D^k([t_0, \infty))$ *to a Gaussian vector process*

$$(4.24) \qquad [-\Lambda_t(\theta_0)]^{-1}\Psi_t^\circ(\theta_0) = [-\Lambda_t(\theta_0)]^{-1}\int_0^\infty \psi_t(x, \theta_0) W^\circ(F_{\theta_0}(dx)) ,$$

$$t_0 \le t < \infty \text{ (recalling (2.8))}.$$

PROOF. The condition (4.1) means the condition (3.1) and hence, by Theorem 3.1, it holds that for any fixed $t_0 > 0$ and $d_0 > 0$

$$(4.25) \qquad \lim_{n\to\infty} P_{\theta_0}[\sup\{|T_{n:t} - \theta_0|: t \in R_0\} \ge d_0] = 0 ,$$

where $R_0 = \{t; t \ge t_0\}$. Therefore, it follows from Theorem 4.1 and (4.25) that

$$\sup\{Z_{n:t}(T_{n:t}, \theta_0): t \in R_0\} \to 0 ,$$

in probability as $n \to \infty$. Since, from (4.2) and (4.7),

$$(1 - \varepsilon)^{-1}\{\varepsilon + n^{-1/2}|\Psi_{n:t}(T_{n:t})| + n^{-1/2}|\Psi_{n:t}(\theta_0)|\}$$
$$> n^{1/2}|\lambda_t(T_{n:t})| \ge a n^{1/2}|T_{n:t} - \theta_0| ,$$

if $Z_{n:t}(T_{n:t}, \theta_0) < \varepsilon$, we have from Theorem 2.1 and (4.1) that $\sup\{n^{1/2} \cdot |T_{n:t} - \theta_0|: t \in R_0\}$ is stochastically bounded. Thus, we conclude from Lemma 4.3 that

$$\sup\{|n^{-1/2}\Psi_{n:t}(T_{n:t}) - n^{-1/2}\Psi_{n:t}(\theta_0) - \Lambda_t(\theta_0)n^{-1/2}(T_{n:t} - \theta_0)|: t \in R_0\} \to 0 ,$$

in probability as $n \to \infty$, and hence, from (4.1) and Assumption (A6)

$$\sup\{|n^{1/2}(T_{n:t} - \theta_0) - [-\Lambda_t(\theta_0)]^{-1}n^{-1/2}\Psi_{n:t}(\theta_0)|: t \in R_0\} \to 0 ,$$

in probability as $n \to \infty$. This and Theorem 2.1 complete the proof of this theorem.

5. An example

Let $F(x)$ be a distribution function with the density function $f(x) > 0$ on R^+. Let us consider a Lehmann alternative defined by

$$(5.1) \qquad F_\theta(x) = 1 - (1 - F(x))^{\gamma(\theta)} ,$$

and equivalently,

$$\bar{F}_\theta(x) = \bar{F}(x)^{\gamma(\theta)}, \tag{5.1'}$$

where the parameter θ is the median of $F_\theta(x)$ and belongs to a finite closed interval $\Theta = [0, K]$ for any fixed $K > 0$. Then, the hazard function is proportional:

$$\begin{aligned} h_\theta(x) &= f_\theta(x)/\bar{F}_\theta(x) \\ &= \gamma(\theta)f(x)/\bar{F}(x) = \gamma(\theta)h(x). \end{aligned} \tag{5.2}$$

The fact that $F_\theta(\theta) = 1/2$ leads to

$$\gamma(\theta) = \left(\log \frac{1}{2}\right) \Big/ \log \bar{F}(\theta). \tag{5.3}$$

Let the score function be

$$\psi(x, \theta) = \operatorname{sgn}(x - \theta), \quad \text{the signature function.} \tag{5.4}$$

Then,

$$\begin{aligned} \lambda(\theta) &= E_{\theta_0}[\psi(X_i, \theta)] = 1 - 2F_{\theta_0}(\theta), \\ \Gamma(\theta) &= \mathrm{Cov}_{\theta_0}[\psi(X_i, \theta)] = 4F_{\theta_0}(\theta)(1 - F_{\theta_0}(\theta)). \end{aligned}$$

The PT score function is

$$\psi_t(x, \theta) = \begin{cases} \operatorname{sgn}(x - \theta) & \text{if} \quad x \le t, \\ \bar{\psi}(t, \theta) & \text{if} \quad x > t, \end{cases} \tag{5.5}$$

where

$$\begin{aligned} \bar{\psi}(t, \theta) &= \int_t^\infty \psi(x, \theta) F_\theta(dx)/\bar{F}_\theta(t) \\ &= \begin{cases} 1 & \text{if} \quad t \ge \theta, \\ F_\theta(t)/\bar{F}_\theta(t) & \text{if} \quad t < \theta. \end{cases} \end{aligned} \tag{5.6}$$

Assumptions (A1)–(A4) are easily seen to hold. The PT estimating function is

$$\Psi_{n:t}(\theta) = \sum_{i=1}^{n} \psi_t(X_i, \theta). \tag{5.7}$$

If the sample median X_{med} (say) $< t$,

$$\Psi_{n:t}(X_{\text{med}}) = 0 .$$

Therefore, we can take, for the PT estimator $T_{n:t}$,

$$T_{n:t} = X_{\text{med}} \quad \text{if} \quad X_{\text{med}} < t . \tag{5.8}$$

If $X_{n:r} < t < X_{n:r+1}$ for $r < n/2$, we can take $T_{n:t} =$ the solution of the following equation:

$$\Psi_n(\theta) = - r + (n - r)F_\theta(t)/\bar{F}_\theta(t) = 0 , \tag{5.9}$$

that is,

$$\log \frac{1}{2} \Big/ \log \bar{F}(\theta) = \log \bar{F}_n(t)/\log \bar{F}(t) . \tag{5.9$'$}$$

It is apparent that the PT estimator $T_{n:t}$ has the uniform consistency for $t \in R_0$ and satisfies the condition (4.1).

Now, we have

$$\begin{aligned} \lambda_t(\theta) &= E_{\theta_0}[\psi_t(X_i, \theta)] \\ &= \int_0^t \operatorname{sgn}(x - \theta) F_{\theta_0}(dx) + \bar{F}_{\theta_0}(t)\bar{\psi}(t, \theta) \\ &= \begin{cases} 1 - 2F_{\theta_0}(\theta) & \text{if} \quad t \geq \theta , \\ \bar{F}_{\theta_0}(t)/\bar{F}_\theta(t) - 1 & \text{if} \quad t < \theta , \end{cases} \end{aligned}$$

and hence

$$\Lambda_t(\theta) = \begin{cases} -2f_{\theta_0}(\theta) & \text{if} \quad t \geq \theta , \\ -\{\gamma(\theta)^2/\gamma(t)\}h(\theta)\{\bar{F}_{\theta_0}(t)/\bar{F}_\theta(t)\} & \text{if} \quad t < \theta . \end{cases}$$

$\lambda_t(\theta)$ is continuous in $(\theta, t) \in \Theta \times R_0$ and $\lambda_t(\theta_0) = 0$. Immediately,

$$\Lambda_t(\theta_0) = \begin{cases} -2f_{\theta_0}(\theta_0) & \text{if} \quad t \geq \theta_0 , \\ -2\{\gamma(\theta_0)/\gamma(t)\} f_{\theta_0}(\theta_0) & \text{if} \quad t < \theta_0 . \end{cases} \tag{5.10}$$

Further, we have

$$\Gamma_t(\theta) = V_{\theta_0}[\psi_t(X_i, \theta)]$$

$$= \begin{cases} 4F_{\theta_0}(\theta)\bar{F}_{\theta_0}(\theta) & \text{if} \quad t \geq \theta\,, \\ F_{\theta_0}(t)\bar{F}_{\theta_0}(t)/\bar{F}_{\theta}(t)^2 & \text{if} \quad t < \theta\,, \end{cases}$$

and hence

$$\Gamma_t(\theta_0) = \begin{cases} 1 & \text{if} \quad t \geq \theta_0\,, \\ F_{\theta_0}(t)/\bar{F}_{\theta_0}(t) & \text{if} \quad t < \theta_0\,. \end{cases} \tag{5.11}$$

Thus, it is easy to see that Assumptions (A6) and (A7) hold.

On the other hand, we have

$$\begin{aligned} \bar{\psi}^*(t,\theta) &= \int_t^\infty \psi(x,\theta)F_{\theta_0}(dx)/\bar{F}_{\theta_0}(t) \\ &= \begin{cases} 1 & \text{if} \quad t \geq \theta\,, \\ \{1 - 2F_{\theta_0}(\theta) + F_{\theta_0}(t)\}/\bar{F}_{\theta_0}(t) & \text{if} \quad t < \theta\,, \end{cases} \end{aligned}$$

and so,

$$\partial/\partial\theta\,\bar{\psi}^*(t,\theta) = \begin{cases} 0 & \text{if} \quad t > \theta\,, \\ -\,2f_{\theta_0}(\theta)/\bar{F}_{\theta_0}(t) & \text{if} \quad t < \theta\,. \end{cases}$$

From (5.6),

$$\partial/\partial\theta\,\bar{\psi}(t,\theta) = \begin{cases} 0 & \text{if} \quad t > \theta\,, \\ -\,d/d\theta\,\gamma(\theta)\{\log \bar{F}(t)/\bar{F}_{\theta}(t)\} & \text{if} \quad t < \theta\,. \end{cases}$$

Therefore, we can see Assumption (A5) holds. From the definition (2.18) and (5.5), we obtain that, if $t \geq \theta_0$,

$$\begin{aligned} \Psi_t^\circ(\theta_0) &= -\int_0^{\theta_0} W(F_{\theta_0}(dx)) + \int_{\theta_0}^\infty W(F_{\theta_0}(dx)) \\ &= -\,W\left(\frac{1}{2}\right) + \left\{W(1) - W\left(\frac{1}{2}\right)\right\} \sim N(0,1)\,, \end{aligned} \tag{5.12}$$

and if $t < \theta_0$,

$$\begin{aligned} \Psi_t^\circ(\theta_0) &= -\int_0^t W(F_{\theta_0}(dx)) + \{F_{\theta_0}(t)/\bar{F}_{\theta_0}(t)\}\int_t^\infty W(F_{\theta_0}(dx)) \\ &= -\,W(F_{\theta_0}(t)) + \{F_{\theta_0}(t)/\bar{F}_{\theta_0}(t)\}\{W(1) - W(F_{\theta_0}(t))\} \\ &\sim N(0, \{F_{\theta_0}(t)/\bar{F}_{\theta_0}(t)\})\,. \end{aligned} \tag{5.13}$$

Thus, the asymptotic distribution of the PT estimator, $n^{1/2}(T_{n:t} - \theta_0)$, comes from (4.24) as follows:

$$(5.14)\quad [-A_t(\theta_0)]^{-1}\Psi_t^{\circ}(\theta_0)$$
$$= \begin{cases} [2f_{\theta_0}(\theta_0)]^{-1}N(0,1) & \text{if} \quad t \geq \theta_0\,, \\ [2f_{\theta_0}(\theta_0)\gamma(\theta_0)/\gamma(t)]^{-1}N(0,\{F_{\theta_0}(t)/\bar{F}_{\theta_0}(t)\}) & \text{if} \quad t < \theta_0\,. \end{cases}$$

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