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ON THE JACKKNIFE STATISTICS FOR EIGENVALUES AND EIGENVECTORS OF A CORRELATION MATRIX

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Abstract. This paper deals with some problems of eigenvalues and eigenvectors of a sample correlation matrix and derives the limiting distributions of their jackknife statistics with some numerical examples.

Key words and phrases: Jackknife statistics, eigenvalue, eigenvector, correlation matrix, nonnormal, limiting distribution.

1. Introduction

An important part in multivariate analysis is to reduce the dimension of multivariate data as small as possible without decreasing loss of information. Principal component analysis is a useful method for this problem and is fundamentally concerned with the eigenvalues and eigenvectors of a covariance matrix. Especially, eigenvalues of a covariance matrix play an important role in considering how much information is condensed into a small number of new variables. For this problem, we sometimes use a sample covariance matrix. Nagao (1985) obtained asymptotic distributions of jackknife statistics for the eigenvalues of a sample covariance matrix. Recently, the author finds that the jackknife estimator for eigenvector of a covariance matrix is not a robust one for a small sample size. Furthermore, a sample covariance matrix is not invariant under a change of scale. In practice, there are many situations in which variables are measured on different units. To avoid them, we shall consider a sample correlation matrix.

For the eigenvalue and eigenvector problems of a sample correlation matrix, a few authors (for example, Lawley (1963) and Konishi (1979)) have only dealt with them under a multivariate normal distribution. This may mainly be due to the fact that an exact expression for the distribution of a sample correlation matrix under the normal distribution has not been obtained yet. For a bivariate case, the sample correlation coefficient is well-

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known not to be a robust estimator as in Hinkley (1978). By applying a jackknife method to a correlation coefficient, he showed the validity of this method through the example though a little modification is needed. This paper treats some problems of eigenvalues and eigenvectors of a sample correlation matrix and gives the limiting distributions of their jackknife statistics with some numerical examples under normal and nonnormal situations. For the references on the jackknife statistics, see Miller (1974), Goto and Tazaki (1978), Parr and Schucany (1980) and Beran (1984). Also Beran and Srivastava (1985) have recently treated some problems of eigenvalues and eigenvectors of a covariance matrix without normality by using bootstrap method.

2. The limiting distribution of the eigenvalue

Let $p \times 1$ vectors $X_1, \ldots, X_N$ be a random sample from a p-variate distribution with mean μ, covariance matrix $\Sigma = (\sigma_{ij})$ and the finite fourth moments. Let $S = (s_{ij}) = \sum_{\alpha=1}^{N} (X_\alpha - \overline{X})(X_\alpha - \overline{X})'$ with $\overline{X} = N^{-1} \sum_{\alpha=1}^{N} X_\alpha$. Then a sample correlation matrix is defined by $R = D^{-1/2} SD^{-1/2}$, where $D = \text{diag}\,(s_{11}, \ldots, s_{pp})$. Letting $S \to \Sigma_0^{-1/2} S \Sigma_0^{-1/2}$, where $\Sigma_0 = \text{diag}\,(\sigma_{11}, \ldots, \sigma_{pp})$, since R is invariant, we can assume that the sample $X_1, \ldots, X_N$ has a covariance matrix P and the finite fourth moments. Here, P denotes a correlation matrix. Let l_j denote the j-th largest eigenvalue of R. Then the pseudo-values $\tilde{l}_j^\alpha$ and jackknife statistic $\bar{l}_j$ are, respectively, given by

$$\tilde{l}_j^\alpha = l_j + (N-1)(l_j - l_j^{-\alpha}) \quad (\alpha = 1, \ldots, N)\,, \tag{2.1}$$

and

$$\bar{l}_j = \frac{1}{N} \sum_{\alpha=1}^{N} \tilde{l}_j^\alpha\,, \tag{2.2}$$

where $l_j^{-\alpha}$ is the j-th largest eigenvalue of the sample correlation matrix $R_{-\alpha}$ obtained by deleting $X_\alpha = (x_{1\alpha}, \ldots, x_{p\alpha})'$ from a sample $X_1, \ldots, X_N$. Then the sample correlation matrix $R_{-\alpha} = (r_{ij,-\alpha})$ is given by $D_{-\alpha}^{-1/2} S_{-\alpha} D_{-\alpha}^{-1/2}$, where

$$S_{-\alpha} = (s_{ij,-\alpha}) = S - \frac{N}{N-1}(X_\alpha - \overline{X})(X_\alpha - \overline{X})'\,, \tag{2.3}$$

and $D_{-\alpha} = \text{diag}\,(s_{11,-\alpha}, \ldots, s_{pp,-\alpha})$. At first, we shall derive the limiting distribution of $\bar{l}_j$. Let λ_j be the j-th largest eigenvalue of P and let $h_j = (h_{1j}, \ldots, h_{pj})'$ be the corresponding eigenvectors of P with $h_j'h_j = 1$ and $h_{jj} > 0$. Since we have

$$\sqrt{n}(\bar{l}_j - \lambda_j) = \sqrt{n}(l_j - \lambda_j) + \frac{\sqrt{n}(N-1)}{N} \sum_{\alpha=1}^{N} (l_j - l_j^{-\alpha}) , \tag{2.4}$$

we shall show that the last term in (2.4) converges to zero in probability, where $n = N - 1$. To prove it, we shall use an implicit function theorem. Consider the following equation on l;

$$F(S_{-\alpha}/(n-1), l) = |R_{-\alpha} - lI| = 0 , \tag{2.5}$$

where I is a $p \times p$ identity matrix. We note that we regard the equation (2.5) as function on the $S_{-\alpha}/(n-1)$. We shall show that l can be expanded around S/n. The partially derivative of F on l at $(S/n, l_j)$ is given by

$$F_{l_j}(S/n, l_j) = \begin{vmatrix} -1 & \\ 0 & \\ \vdots & R - l_j I \\ 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 \\ & \vdots \\ R - l_j I & 0 \\ & -1 \end{vmatrix} , \tag{2.6}$$

where the i-th determinant in (2.6) is of the matrix which is obtained by replacing the i-th column of $R - l_j I$ with the column vector $(0,\dots,0,-1,0,\dots,0)'$ having -1 at the i-th element. Then as $N \to \infty$, the equation (2.6) converges in probability to

$$\begin{vmatrix} -1 & \\ 0 & \\ \vdots & P - \lambda_j I \\ 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 \\ & \vdots \\ P - \lambda_j I & 0 \\ & -1 \end{vmatrix} . \tag{2.7}$$

The applications of an implicit function theorem need the following formulas; Then if λ_j is a simple root by them, the value (2.7) is shown not to be zero.

LEMMA 2.1. *Let $R = D^{-1/2}SD^{-1/2}$ be a sample correlation matrix and $h_j = (h_{1j},\dots,h_{pj})'$ be a normalized eigenvector corresponding to the j-th largest eigenvalue λ_j of a correlation matrix $P = (\rho_{ij})$. Then we have*

$$\frac{\partial}{\partial \lambda} |R - \lambda I| \Big|_{\lambda=\lambda_j} \xrightarrow{\text{in Prob.}} - \prod_{i \neq j}^{p} (\lambda_i - \lambda_j) , \tag{2.8}$$

$$\frac{\partial}{\partial (s_{kl}/n)} |R - \lambda I| \Big|_{\lambda=\lambda_j} \xrightarrow[(k \neq l)]{\text{in Prob.}} 2h_{kj}h_{lj} \prod_{i \neq j}^{p} (\lambda_i - \lambda_j) , \tag{2.9}$$

$$\frac{\partial}{\partial (s_{kk}/n)} |R - \lambda I| \Big|_{\lambda=\lambda_j} \xrightarrow{\text{in Prob.}} (1 - \lambda_j)h_{kj}^2 \prod_{i \neq j}^{p} (\lambda_i - \lambda_j) . \tag{2.10}$$

PROOF. Since the formula (2.8) is obvious, we shall show the formula (2.9). It can be verified that

$$\frac{\partial}{\partial(s_{kl}/n)}|R-\lambda I|\Big|_{\lambda=\lambda_j} \xrightarrow{\text{in Prob.}} -2|P-\lambda_j I-E_{kl}| \tag{2.11}$$

$$= -2\left| D_j - \begin{pmatrix} h_{l1} \\ \vdots \\ h_{lp} \end{pmatrix} (h_{k1},\dots,h_{kp}) \right| ,$$

where $D_j = \operatorname{diag}(\lambda_1,\dots,\lambda_p) - \lambda_j I$ and E_{kl} denotes a $p\times p$ matrix having 1 at (k, l)-element and zero otherwise. Then the right-hand side of (2.11) is

$$-2\begin{vmatrix} 1 & h_{k1}\cdots h_{kp} \\ h_{l1} & \\ \vdots & D_j \\ h_{lp} & \end{vmatrix} = -2(-1)^{j+2}(-1)^{j+1}\, h_{kl}h_{lj}\prod_{i\neq j}^{p}(\lambda_i-\lambda_j)\,. \tag{2.12}$$

By the similar calculation as in (2.9), we have the formula (2.10), though the calculation is more complicated.

Then by an implicit function theorem, we can get

$$l_j^{-\alpha} = l_j + \sum_{k\leq l} g_{kl}^j(S/n)t_{kl}^{\alpha} + \frac{1}{2}\langle t_{kl}^{\alpha}\rangle\, C_\alpha^j \langle t_{kl}^{\alpha}\rangle' , \tag{2.13}$$

where $g_{kl}^j(S/n) = -F_{s_{kl}/n}(S/n, l_j)/F_{l_j}(S/n, l_j)$, $(t_{kl}^{\alpha}) = S_{-\alpha}/(n-1) - S/n$, $\langle t_{kl}^{\alpha}\rangle = (t_{11}^{\alpha},\dots,t_{pp}^{\alpha}, t_{12}^{\alpha},\dots,t_{p-1,p}^{\alpha})$ and the element of C_α^j is derivative of $g_{kl}^j(S/n)$ at the value of element of some matrix between S/n and $S_{-\alpha}/(n-1)$. By the similar calculation as in Nagao (1985), we can show that the last term in (2.4) converges to zero in probability. Thus, the limiting distribution of $\sqrt{n}(\bar{l}_j-\lambda_j)$ is the same as that of $\sqrt{n}(l_j-\lambda_j)$, whose distribution was derived by Fang and Krishnaiah (1982). Since its derivation is based on the perturbation method (for example, Bellman (1960)), we have

$$\sqrt{n}(l_j-\lambda_j) = \operatorname{tr}(h_j h_j' - \lambda_j L_j)V + O_p(n^{-1/2}) , \tag{2.14}$$

where $V = (v_{ij}) = \sqrt{n}(S/n - P)$ and $L_j = \operatorname{diag}(h_{1j}^2,\dots,h_{pj}^2)$. Then we have the following theorem;

THEOREM 2.1. *If the j-th largest eigenvalue λ_j of P is a simple root, the limiting distribution of $\sqrt{n}(\bar{l}_j-\lambda_j)$ is a normal with mean zero and variance* $\tau_j^2 = \sum_{\alpha,\beta}\sum_{\gamma,\delta} a_{\alpha,\beta}^j a_{\gamma,\delta}^j \cdot \operatorname{cov}((X_{\alpha1}-\mu_\alpha)(X_{\beta1}-\mu_\beta),\ (X_{\gamma1}-\mu_\gamma)(X_{\delta1}-\mu_\delta)),$

where $(a^j_{\alpha,\beta}) = h_j h_j' - \lambda_j L_j$.

When the population eigenvalues have multiplicities, Fang and Krishnaiah (1982) have obtained the asymptotic distributions of eigenvalues. But it seems to be difficult to obtain the similar results in the case of the jackknife statistics for eigenvalues. Next we shall show that $\sum_{\alpha=1}^{N} (\tilde{l}_j^\alpha - \bar{l}_j)^2/(N-1)$ converges to τ_j^2 in probability. For some matrix ξ_α between $S_{-\alpha}/(n-1)$ and S/n, we have

$$(2.15)\quad \sum_{\alpha=1}^{N} (\tilde{l}_j^\alpha - \bar{l}_j)^2/(N-1) = (N-1) \sum_{\alpha=1}^{N} \left\{ \sum_{k\le l} g^j_{kl}(\xi_\alpha) t^\alpha_{kl} - \frac{1}{N} \sum_{\alpha=1}^{N} \sum_{k\le l} g^j_{kl}(\xi_\alpha) t^\alpha_{kl} \right\}^2 .$$

By Lemma 2.1, we have, after some tedious calculation, the following;

THEOREM 2.2. *If the j-th largest eigenvalue λ_j of a correlation matrix P is a simple root,* $\sum_{\alpha=1}^{N} (\tilde{l}_j^\alpha - \bar{l}_j)^2/(N-1)$ *converges to τ_j^2 in probability.*

Hence from the above two theorems, we have

THEOREM 2.3. *If the j-th largest eigenvalue λ_j of a correlation matrix P is a simple root, we have*

$$(2.16)\quad \frac{n(\bar{l}_j - \lambda_j)}{\sqrt{\sum_{\alpha=1}^{N} (\tilde{l}_j^\alpha - \bar{l}_j)^2}} \xrightarrow{\text{in law}} N(0,1) ,$$

where $n = N - 1$.

3. The jackknife statistic for a function of eigenvalues

In this section, we shall generalize the above results for the function of eigenvalues of R. For example, in principal component analysis, the fraction of the total variance accounted for by the first q principal components is measured by $d = \sum_{\alpha=1}^{q} \lambda_\alpha / p$ $(q < p)$, since $\sum_{i=1}^{p} \lambda_i = p$, which was proposed by Rao (1964). Thus, applying the jackknife method to an estimator $\hat{d} = \sum_{\alpha=1}^{q} l_\alpha / p$ and so on, we can obtain the confidence interval of d, etc. Let $f(\cdot)$ be a real-valued function with the second continuous derivatives on some neighbourhood of $(\lambda_1, \ldots, \lambda_{p-1})$. By the same notations as sections mentioned before, the pseudo-values and the jackknife statistic of $f(l_1,\dots,l_{p-1})$ are, respectively, given by

$$f^{-\alpha}=f(l_1,\dots,l_{p-1})+(N-1)\{f(l_1,\dots,l_{p-1})-f(l_1^{-\alpha},\dots,l_{p-1}^{-\alpha})\}\,, \tag{3.1}$$

and

$$\bar{f}=\frac{1}{N}\sum_{\alpha=1}^{N}f^{-\alpha}\,. \tag{3.2}$$

Since the method of the argument is similar as before, we only mention the result.

THEOREM 3.1. *If the eigenvalues $\lambda_1,\dots,\lambda_p$ of P are all simple, then for any function $f(\cdot)$ with continuous second derivatives about $(\lambda_1,\dots,\lambda_{p-1})$, we have*

$$\frac{n(\bar{f}-f)}{\sqrt{\sum_{\alpha=1}^{N}(f^{-\alpha}-\bar{f})^2}}\xrightarrow{\text{in law}}N(0,1)\,, \tag{3.3}$$

where $f=f(\lambda_1,\dots,\lambda_{p-1})$ and $n=N-1$.

4. The jackknife statistic of eigenvector

Let $c_j=(c_{1j},\dots,c_{pj})'$ be an eigenvector with $c_j'c_j=1$ and $c_{jj}>0$ corresponding to the j-th largest eigenvalue l_j. Then the pseudo-values and the jackknife statistic of the i-th component c_{ij} are given by

$$c_{ij}^{\alpha}=c_{ij}+(N-1)(c_{ij}-c_{ij}^{-\alpha})\qquad(\alpha=1,\dots,N)\,, \tag{4.1}$$

and

$$\bar{c}_{ij}=\frac{1}{N}\sum_{\alpha=1}^{N}c_{ij}^{\alpha}\,, \tag{4.2}$$

where $c_{ij}^{-\alpha}$ is the i-th component of an eigenvector $c_j^{-\alpha}$ corresponding to $l_j^{-\alpha}$. To obtain the limiting distribution of $\bar{c}_{ij}$, first of all, we shall consider the matrix $H'RH$, where $H=[h_1,\dots,h_p]=[\tilde{h}_1,\dots,\tilde{h}_p]'$. Since H is an orthogonal matrix such that $H'PH=\Lambda=\mathrm{diag}\,(\lambda_1,\dots,\lambda_p)$, its eigenvalues of $H'RH$ are the same as ones of R. Denoting $d_j=(d_{1j},\dots,d_{pj})'$ as an eigenvector corresponding to the eigenvalue l_j of $H'RH$, we have $c_{ij}=\tilde{h}_i'd_j$. Thus after giving the limiting distribution of the jackknife statistic for d_j, we shall derive the limiting distribution of $\bar{c}_{ij}$. We shall define the pseudo-values and the jackknife statistic of d_j corresponding to (4.1) and (4.2), that is,

$$d_j^\alpha = d_j + (N-1)(d_j - d_j^{-\alpha}) \qquad (\alpha = 1, \ldots, N)\,, \tag{4.3}$$

and

$$\bar{d}_j = \frac{1}{N}\sum_{\alpha=1}^{N} d_j^\alpha\,, \tag{4.4}$$

where $d_j^{-\alpha}$ is an eigenvector corresponding to an eigenvalue $l_j^{-\alpha}$ of $H'R_{-\alpha}H$. To obtain the limiting distribution of $\bar{d}_j$, we need the following lemma.

LEMMA 4.1. *Let $A = (a_{ij})$ be a $p \times p$ real symmetric matrix and we assume $|(A)_{jj} - \lambda I| = 0$ for some j, where $(A)_{jj}$ denotes a $(p-1) \times (p-1)$ matrix deleting the j-th row and the j-th column of A. The necessary and sufficient condition for $x = (x_1', x_j, x_2')'$ to be an eigenvector corresponding to an eigenvalue λ is*

$$((A)_{jj} - \lambda I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + a_j x_j = 0 \quad \text{and} \quad x_j \neq 0\,, \tag{4.5}$$

where x_1 and x_2 are $(j-1) \times 1$ and $(p-j) \times 1$ vector, respectively, and x_j is some fixed constant. Also a_j denotes the j-th $(p-1) \times 1$ column vector of A omitted an element a_{jj}.

Since λ_j is a simple root, a matrix $((H'RH)_{jj} - \lambda_j I)$ is nonsingular for large N. We shall show that $(\sqrt{n}(N-1)/N) \sum_{\alpha=1}^{N} (d_j - d_j^{-\alpha})$ converges to zero in probability. By Lemma 4.1, we consider the equation

$$G(S_{-\alpha}/(n-1), (x_1, x_2)) = ((H'R_{-\alpha}H)_{jj} - l_j^{-\alpha} I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tilde{r}_{j,-\alpha} x_j = 0\,, \tag{4.6}$$

where $\tilde{r}_{j,-\alpha}$ is given by a $(p-1) \times 1$ column vector $(h_1' R_{-\alpha} h_j, \ldots, h_p' R_{-\alpha} h_j)'$ omitted $h_j' R_{-\alpha} h_j$ and $x_j \neq 0$. Since we can choose that the j-th component of d_j^α is the same as that of d_j by Lemma 4.1, we shall show that the vector $d_j^\alpha(j)$ deleted the j-th component of d_j^α can be expanded around $(S/n, d_j(j))$, where $d_j(j)$ is the subvector of d_j corresponding to $d_j^\alpha(j)$. Then the partially derivative of G with respect to (x_1', x_2') at $(S/n, d_j(j))$ is given by $((H'RH)_{jj} - l_j I)$. Since this matrix is nonsingular, by an implicit function theorem for a multivariate case, we have

$$d_j^a(j) = d_j(j) + A_j\langle t_{kl}^a\rangle + \frac{1}{2}\begin{pmatrix} \langle t_{kl}^a\rangle\, C_1^a\, \langle t_{kl}^a\rangle \\ \vdots \\ \langle t_{kl}^a\rangle\, C_p^a\, \langle t_{kl}^a\rangle \end{pmatrix}, \tag{4.7}$$

where $A_j = -[G_{d_j(j)}(S/n, d_j(j))]^{-1}(G_{s_{kl}/n}(S/n, d_j(j))$ and C_k^a $(k = j)$ is the derivative of the k-th row vector of A_j at some matrix between $S_{-a}/(n-1)$ and S/n. Then by a tedious calculation, we find that the limiting distribution of $\sqrt{n}\,\bar{d}_j$ is the same as one of $\sqrt{n}\,d_j$. Let $U = (u_{ij}) = (n/2)((1/n)\Lambda^{-1/2} \cdot H'SH\Lambda^{-1/2} - I)$, then if λ_j is a simple root, by a perturbation method,

$$d_{aj} = -(\lambda_a - \lambda_j)^{-1}\left\{\frac{1}{\sqrt{n}}\left[(2\lambda_a\lambda_j)^{1/2}u_{aj} - \frac{1}{\sqrt{2}}(\lambda_a + \lambda_j)\sum_k\sum_l(\lambda_k\lambda_l)^{1/2}b_{klaj}u_{kl}\right]\right\} + O_p(n^{-1}), \tag{4.8}$$

and

$$d_{jj} = 1 + O_p(n^{-1}), \tag{4.9}$$

where $b_{klaj} = \sum_{i=1}^{p} h_{ik}h_{il}h_{ia}h_{ij}$. Hence the limiting distribution of c_{ij} is given by

$$\sqrt{n}(c_{ij} - h_{ij}) \xrightarrow{\text{in law}} N(0, \omega_{ij}^2), \tag{4.10}$$

where $\omega_{ij}^2 = \sum_{u,v=j}^{p} \tau_{u,v}^j h_{i,u}h_{j,v}$,

$$\begin{aligned}\tau_{u,v}^j = (\lambda_u - \lambda_j)^{-1}(\lambda_v - \lambda_j)^{-1}\Bigg\{ &\sum_a\sum_b\sum_c\sum_d h_{au}h_{bj}h_{cv}h_{dj}\kappa_{ab,cd} \\ &- \frac{1}{2}(\lambda_v + \lambda_j)\sum_a\sum_b\sum_c h_{au}h_{bj}h_{cv}h_{cj}\kappa_{ab,cc} \\ &- \frac{1}{2}(\lambda_u + \lambda_j)\sum_a\sum_b\sum_c h_{au}h_{aj}h_{bj}h_{cj}\kappa_{aa,bc} \\ &+ \frac{1}{4}(\lambda_u + \lambda_j)(\lambda_v + \lambda_j)\sum_a\sum_b h_{au}h_{aj}h_{bv}h_{bj} \\ &\cdot \kappa_{aa,bb}\Bigg\},\end{aligned} \tag{4.11}$$

and $\kappa_{ab,cd} = \operatorname{cov}((X_{a\alpha} - \mu_a)(X_{b\alpha} - \mu_b), (X_{c\alpha} - \mu_c)(X_{d\alpha} - \mu_d))$.

Next we shall give

$$\frac{1}{N-1}\sum_{\alpha=1}^{N}(c_{ij}^{\alpha}-\bar{c}_{ij})^2 \xrightarrow{\text{in Prob.}} \omega_{ij}^2 . \tag{4.12}$$

To prove the above, we use the formula (4.7) up to the second term. Also the following lemma is helpful in a multivariate implicit function theorem.

LEMMA 4.2. *Let R be a sample correlation matrix and $h_j = (h_{1j},\ldots,h_{pj})'$ be an eigenvector corresponding to an eigenvalue λ_j of P $(j = 1,\ldots,p)$. Then we have*

$$\frac{\partial h_\alpha' R h_j}{\partial(s_{kl}/n)} \xrightarrow{\text{in Prob.}} 2h_{k\alpha}h_{l\alpha} \qquad (k \neq l) , \tag{4.13}$$

$$\frac{\partial h_\alpha' R h_j}{\partial(s_{kk}/n)} \xrightarrow{\text{in Prob.}} -\frac{1}{2}\{(\lambda_\alpha+\lambda_j)-2\}\, h_{kj}h_{k\alpha} . \tag{4.14}$$

PROOF. The formula (4.13) is obvious. For (4.14), using the relationship between an eigenvalue and an eigenvector, we can show it.

Then by the similar calculation as (2.15), we can prove (4.12). Thus we have

THEOREM 4.1. *Let $c_j = (c_{1j},\ldots,c_{pj})'$ $(c_{jj} > 0)$ be the eigenvector with the length corresponding to an eigenvalue l_j of R. If an eigenvalue λ_j of P is a simple root,*

$$\frac{n(\bar{c}_{ij}-h_{ij})}{\sqrt{\sum_{\alpha=1}^{N}(c_{ij}^{\alpha}-\bar{c}_{ij})^2}} \xrightarrow{\text{in law}} N(0,1) , \tag{4.15}$$

where h_{ij} is the (i,j) element of an orthogonal matrix $H = (h_{ij})$ with $h_{jj} > 0$ $(j = 1,\ldots,p)$ such that $H'PH = \operatorname{diag}(\lambda_1,\ldots,\lambda_p)$.

Also the similar calculation yields the following theorem;

THEOREM 4.2. *If the λ_j is a simple root of P, we have*

$$n(\tilde{c}_j-\tilde{h}_j)'\,\hat{\Omega}^{-1}(\tilde{c}_j-\tilde{h}_j) \xrightarrow{\text{in law}} \chi^2_{[p-1]} , \tag{4.16}$$

where $\tilde{c}_j$, $\tilde{h}_j$ and $\tilde{c}_j^{\alpha}(j)$ denote $(p-1)\times 1$ vector deleting the j-th component of $\bar{c}_j = (\bar{c}_{1j},\ldots,\bar{c}_{pj})'$, h_j and $c_j^{\alpha} = (c_{1j}^{\alpha},\ldots,c_{pj}^{\alpha})'$, respectively, and $\hat{\Omega} = (1/(N-1))$ $\cdot \sum_{\alpha=1}^{N}(\tilde{c}_j^{\alpha}(j)-\tilde{c}_j)(\tilde{c}_j^{\alpha}(j)-\tilde{c}_j)'$. $\chi^2_{[p-1]}$ stands for a chi-square distribution with $(p-1)$ degrees of freedom.

5. The jackknife statistic in Brillinger case

In this section, we shall deal with the case of the sample size $N = gh$, where the group g is fixed and $h \to \infty$. The problem mentioned before is the case of group $g = N \to \infty$ and $h = 1$. Let the $p \times 1$ vectors $X_1, \ldots, X_h, \ldots, X_{(g-1)h+1}, \ldots, X_{gh}$ be a random sample from a p-variate continuous distribution with mean μ, correlation matrix P and the finite fourth moments. By using the same notations before, we define the pseudo-values and jackknife statistic for an eigenvalues l_j and the k-th component c_{kj} of an eigenvector c_j, respectively, as follows;

$$l_j^i = gl_j - (g-1)l_j^{-i} \quad (i = 1, \ldots, g), \qquad \bar{l}_j = \frac{1}{g}\sum_{i=1}^{g} l_j^i, \tag{5.1}$$

and

$$c_{kj}^i = gc_{kj} - (g-1)c_{kj}^{-i} \quad (i = 1, \ldots, g), \qquad \bar{c}_{kj} = \frac{1}{g}\sum_{i=1}^{g} c_{kj}^i, \tag{5.2}$$

where c_{kj}^{-i} is the k-th component of the eigenvector corresponding to an eigenvalue l_j^{-i} of $D_i^{-1/2} S_{-i} D_i^{-1/2}$, which represents a sample correlation matrix of a sub-sample obtained by deleting $X_{(i-1)h+1}, \ldots, X_{ih}$ from a sample $X_1, \ldots, X_{gh}$. Then S_{-i} is given by

$$S_{-i} = \sum_{\alpha \notin \mathscr{A}_i} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' - \frac{h^2}{N-h}(\bar{X}^i - \bar{X})(\bar{X}^i - \bar{X})', \tag{5.3}$$

where $\bar{X}^i = h^{-1} \sum_{\alpha \in \mathscr{A}_i} X_\alpha$ and $\mathscr{A}_i$ denotes the set $\{(i-1)h + 1, \ldots, ih\}$. Applying (2.14) and (4.8) to (5.1) and (5.2), respectively, we have the following two theorems after some tedious calculation;

THEOREM 5.1. *If the λ_j is a simple root, then for the statistics l_j^i and $\bar{l}_j$ defined in* (5.1), *we have, as $h \to \infty$,*

$$\frac{\sqrt{g}(\bar{l}_j - \lambda_j)}{\sqrt{\sum_{i=1}^{g}(l_j^i - \bar{l}_j)^2/(g-1)}} \xrightarrow{\text{in law}} t_{[g-1]}, \tag{5.4}$$

where $t_{[g-1]}$ is a t-distribution with $(g-1)$ *degrees of freedom.*

THEOREM 5.2. *If the λ_j is a simple root, then for the statistics c_{kj}^i and $\bar{c}_{kj}$ defined in* (5.2), *we have, as $h \to \infty$,*

$$\frac{\sqrt{g}(\bar{c}_{kj} - h_{kj})}{\sqrt{\sum_{i=1}^{g}(c_{kj}^{i} - \bar{c}_{kj})^2/(g-1)}} \xrightarrow{\text{in law}} t_{[g-1]}, \tag{5.5}$$

6. Numerical examples

Finally, we shall give some numerical examples under normal and nonnormal distributions. We consider a correlation matrix

$$P = \begin{pmatrix} 1 & 0.80 & -0.40 \\ 0.80 & 1 & -0.56 \\ -0.40 & -0.56 & 1 \end{pmatrix}. \tag{6.1}$$

Then the eigenvalues are given by $\lambda_1 = 2.1895$, $\lambda_2 = 0.6342$ and $\lambda_3 = 0.1763$ and the eigenvectors corresponding to them are $t_1 = (0.59307, 0.63308, -0.49748)'$, $t_2 = (0.50245, 0.19179, 0.84306)'$ and $t_3 = (-0.62914, 0.74995, 0.20434)'$, respectively. In this section, we shall treat the following cases: Let $X = (X_1, X_2, X_3)' = P^{1/2}(Y_1, Y_2, Y_3)'$, where $P^{1/2}P^{1/2} = P$. (i) Y_1, Y_2 and Y_3 are independent and normally distributed with mean 0 and variance 1. For another cases (ii) and (iii), let $Y_1 = a(Z_1 + Z_2Z_3)$, $Y_2 = a(Z_2 + Z_3Z_1)$ and $Y_3 = a(Z_3 + Z_1Z_2)$, where Z_1, Z_2 and Z_3 are independent and identically distributed random variables. (ii) Z_i $(i = 1, 2, 3)$ are uniformly distributed on $(-1, 1)$ with $a = 3/2$ and (iii) Z_i $(i = 1, 2, 3)$ are normally distributed with mean 0 and variance 1 with $a = 1/\sqrt{2}$. Then all correlation matrices of X are P. Under these assumptions, we shall give the accuracy of coverage when we shall use 95% point under the standard normal distribution. The repeated number is 1,000 times. The computations were carried out on the FACOM, M-380 of the University of Tsukuba.

Example 6.1. Applying (2.16) and (3.3), we shall give the coverage ratio of confidence interval for each eigenvalue and $(\lambda_1 + \lambda_2)/3$ for some sample sizes.

From Table 1, we give some comments for the three distributions (i), (ii) and (iii). For a normal distribution (i), the coverage values are very nice especially when a sample size is large. In case of a short distribution (ii), these values do not depend on a sample size so much. For a long tail distribution (iii), these values improve when a sample size increaces, but the speed of convergence is slow.

Finally, we shall consider the problem of the eigenvectors. In the principal component analysis, we are interested in the components of each eigenvector.

Table 1.

		λ_1	λ_2	λ_3	$(\lambda_1+\lambda_2)/3$
$N=50$	(i)	0.930	0.919	0.919	0.920
	(ii)	0.932	0.928	0.938	0.938
	(iii)	0.918	0.905	0.901	0.901
$N=100$	(i)	0.946	0.943	0.937	0.938
	(ii)	0.950	0.938	0.952	0.953
	(iii)	0.932	0.923	0.918	0.919
$N=150$	(i)	0.953	0.956	0.946	0.946
	(ii)	0.937	0.930	0.939	0.939
	(iii)	0.936	0.941	0.936	0.936

Example 6.2. We shall give the coverage ratio for the simultaneous confidence bounds of components of t_i ($i=1, 2, 3$), applying the formula (4.10). Then by Bonferroni inequality, we approximately determine the percentage point as 2.37 for 95% simultaneous confidence interval.

Table 2.

		t_1	t_2	t_3
$N=50$	(i)	0.909	0.948	0.897
	(ii)	0.911	0.856	0.796
	(iii)	0.875	0.834	0.775
$N=100$	(i)	0.962	0.944	0.930
	(ii)	0.917	0.917	0.878
	(iii)	0.909	0.902	0.851
$N=150$	(i)	0.962	0.948	0.951
	(ii)	0.937	0.938	0.911
	(iii)	0.936	0.917	0.886

Comparing the above values with the case of eigenvectors in a covariance matrix, for (i), the values improve in proportion as a sample size is large. For (ii) and (iii), the values for the eigenvectors corresponding to smaller eigenvalues are poor, but terms corresponding to larger eigenvalues are good, which are useful in a principal component analysis.

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