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LOCALLY MINIMAX TESTS IN SYMMETRICAL DISTRIBUTIONS*

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Abstract. In this paper we give an extension of the theory of local minimax property of Giri and Kiefer (1964, *Ann. Math. Statist.*, **35**, 21–35) to the family of elliptically symmetric distributions which contains the multivariate normal distribution as a member.

Key words and phrases: Hunt-Stein theorem, locally best invariant test, locally minimax test, uniformly most powerful invariant test, maximal invariant, nonnull robustness, null robustness, Wijsman's representation theorem.

1. Introduction

This paper represents an extension of local minimax results contained in Section 2 of Giri and Kiefer (1964), here after called G-K (1964), to the family of elliptically symmetric distributions which contains the multivariate normal distribution as a member. We shall call a test robust if certain property which the test enjoys for a given problem in the case of a multivariate normal distribution, can be extended to the class of elliptically symmetric distributions. Three types of robustness are commonly used for such problems: the null robustness, the nonnull robustness and the optimality robustness. The null robustness requires that the null distribution of the test statistic remains the same under any member of the family. The nonnull robustness requires the invariance of the nonnull distribution of the statistic. The optimality robustness requires that an optimality property which the test enjoys i.e. uniformly most powerful (UMP), uniformly most powerful invariant (UMPI), locally best invariant (LBI), etc. can be extended to every member of the family of distributions including the one for which it is known to be optimum. The null robustness has been considered by many authors including Dempster

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(1969), Kariya and Eaton (1977), Dawid (1977), Chielewski (1980), Jensen and Good (1981) and Eaton and Kariya (1981), among others. The optimality robustness for multivariate distribution concerning UMPI and LBI properties has been treated by Kariya and Eaton (1977), Kariya (1981) and Kariya and Sinha (1984). In this paper we develop simple techniques for proving the robustness of the local minimax property of some tests in some complex problems concerning the family of elliptically symmetric distributions. In our views such a theory serves two purposes. Firstly, there is the obvious point of demonstrating such properties which are valid for all members of the family. Secondly, by showing that a test (which is locally minimax for the multivariate normal setup) is robust for the entire family of distributions we add more to the credibility of the test.

As in the case of G-K (1964), our method of proof uses a slightly different version of the wellknown result that Bayes procedure with constant risk is minimax and the invariance approach through Hunt-Stein Theorem. This involves in finding the ratio of the probability density function of the maximal invariant under the alternative to that under the null hypothesis. We shall use Stein's theorem and Wijsman's representation theorem to find this ratio. In Section 2, we give sufficient conditions for a locally minimax test for the entire family of elliptically symmetric distributions. In Section 3, we treat the Gmanova problem to show that a generalized version of Pillai's test (see Kariya (1978)) is locally minimax for all elliptically symmetric probability density functions. Giri (1985*a*, 1985*b*) and Giri and Sinha (1984) have treated from this viewpoint the problem of independence and the problems of mean vector for this family.

2. Locally minimax test

In this section, we shall make clear the invariance structure under which the robustness of the locally minimax test holds and give sufficient conditions for a test to be locally minimax for all members of the family of elliptically symmetric distributions. Let χ be a space with associated σ-field which, along with other obvious measurability assumptions, we shall not mention in what follows. Let $x \in \chi$ have a probability density function

$$f_X(x|\theta) = B(\theta)q(\psi(x|\theta)) , \quad \theta \in \Omega , \tag{2.1}$$

where ψ is a known measurable function from χ onto y and q is a fixed function from y to $[0,\infty)$ and is independent of θ. Let G be a group of transformations which leaves the problem of testing H_0: $\theta \in \Omega_{H_0}$ against H_1: $\theta \in \Omega_{H_1}$ invariant. The function ψ satisfies

$$\bar{g}\psi(x|\theta) = \psi(gx|\theta) , \tag{2.2}$$

for all $x \in \chi$, $g \in G$, $\theta \in \Omega$ and $\bar{g} \in \bar{G}$, the induced group of G as a continuous homomorphic image. The ratio R of the pdf of the maximal invariant $T(x)$ on χ under $\theta_1 \in \Omega_{H_1}$ and under $\theta_0 \in \Omega_{H_0}$ (Stein (1956) and Wijsman (1967)) is given by

$$(2.3) \qquad R = \frac{dP^T_{\theta_1}}{dP^T_{\theta_0}} = \frac{\int_G f(gx|\theta_1)\delta(g)\mu(dg)}{\int_G f(gx|\theta_0)\delta(g)\mu(dg)} ,$$

where $\mu(dg)$ is an invariant measure on G and $\delta(g)$ is the inverse of the Jacobian of the transformation $X \to gX$.

ASSUMPTION 1. The group $\bar{G}$ acts transitively on the space $y \in \psi(x|\theta)$.

Remark 1. Under this assumption, it follows from Theorem 2.1 of Kariya and Sinha (1984) that R is independent of q.

Let $\mathscr{F}$ be the space of values of $Z = T(X)$. For each (σ, η) in the parametric space of the distribution of Z suppose that $f(z; \sigma, \eta)$ is a pdf of Z with respect to some σ-finite measure u. Assume that the problem of testing H_0: $\theta \in \Omega_{H_0}$ against H_1: $\theta \in \Omega_{H_1}$ is reduced to that of testing H_0: $\sigma = 0$ against the alternative H_1: $\sigma = \lambda > 0$ in terms of Z. We are concerned here in a local theory in the sense that $f(z; \lambda, \eta)$ is close to $f(z; 0, \eta)$ when λ is small for all q in (2.1). Throughout this section notations like $o(1)$, $o(h(\lambda))$ are to be interpreted as $\lambda \to 0$ for any q in (2.1).

For each α, $0 < \alpha < 1$, consider a critical region of the form

$$(2.4) \qquad R = \{U(x) = U(t(x)) \geq C_\alpha\} ,$$

where U is bounded, positive and has a continuous distribution function for each (σ, η), equicontinuous in $(\sigma, \eta) < \sigma_0$ (fixed) for any q in (2.1) and

$$(2.5) \qquad P_{0,\eta}(R) = \alpha , \quad P_{\lambda,\eta}(R) = \alpha + h(\lambda) + r(\lambda, \eta) ,$$

for any q in (2.1), where, $r(\lambda, \eta) = o(h(\lambda))$ uniformly in η with $h(\lambda) > 0$ and $h(\lambda) = o(1)$. Without any loss of generality we shall take throughout this paper $h(\lambda) = b\lambda$ with $b > 0$.

Remark 2. $P_{0,\eta}(R) = \alpha$ for any q in (2.1) implies that the distribution of U and the test with critical region R is null robust.

Remark 3. $P_{\lambda,\eta}(R) = \alpha + h(\lambda) + r(\lambda, \eta)$ for any q in (2.1) implies that the distribution of U or the test with critical region R is locally nonnull robust as $\lambda \to 0$.

Let $\xi_{0\lambda}$, $\xi_{1\lambda}$ be the probability measures on the sets $\{\sigma=0\}$, $\{\sigma=\lambda\}$, respectively, such that

$$(2.6)\qquad \frac{\int f(z;\lambda,\eta)\xi_{1\lambda}(d\eta)}{\int f(z;0,\eta)\xi_{0\lambda}(d\eta)} = 1 + h(\lambda)[g(\lambda) + r(\lambda)u] + B(z,\lambda)\ ,$$

for any q in (2.1) where $0<c_1<r(\lambda)<c_2<\infty$ for λ sufficiently small, $g(\lambda)=O(1)$ and $B(z, \lambda)=O(\lambda)$ uniformly in z.

Remark 4. In many applications the set $\{\sigma=0\}$ is a single point and $\xi_{0\lambda}$ assigns measure one to that point. Here we get

$$(2.7)\qquad \frac{\int f(z;\lambda,\eta)\xi_{1\lambda}(d\eta)}{\int f(z;0,\eta)\xi_{0\lambda}(d\eta)} = \int \frac{f(z;\lambda,\eta)}{f(z;0,\eta)}\,\xi_{1\lambda}(d\eta)\ .$$

By Assumption 1 this ratio will not depend on q.

THEOREM 2.1. *If R satisfies* (2.5) *and for sufficiently small λ, there exist $\xi_{1\lambda}$ and $\xi_{0\lambda}$ satisfying* (2.7), *then R is locally minimax for testing H_0: $\sigma=0$ against the alternatives H_1: $\sigma=\lambda$ for any q in* (2.1) *as $\lambda\to 0$, i.e.*

$$(2.8)\qquad \lim_{\lambda\to 0} \frac{\inf_{\eta} P_{\lambda\eta}(R) - \alpha}{\sup_{\phi_\lambda\in Q_\alpha} \inf P_{\lambda\eta}\{\phi_\lambda \text{ rejects } H_0\} - \alpha} = 1\ ,$$

for any q in (2.1), *where Q_α is the class of tests of level α.*

PROOF. Let $\tau_\lambda=(2+h(\lambda)[g(\lambda)+C_\alpha r(\lambda)])^{-1}$. A Bayes critical region for (0,1) losses with respect to the a priori $\xi_\lambda=\tau_\lambda\,\xi_{1\lambda}+(1-\tau_\lambda)\,\xi_{0\lambda}$ is given by

$$(2.9)\qquad B_\lambda(z) = \left\{z\colon \int \frac{f(z;\lambda,\eta)}{f(z;0,\eta)}\,\xi_{1\lambda}(d\eta) \geq \frac{1-\tau_\lambda}{\tau_\lambda}\right\}$$

$$= \left\{z\colon u(z) + \frac{B(z,\lambda)}{r(\lambda)h(\lambda)} \geq C_\alpha\right\}.$$

By (2.5) and (2.7), $B_\lambda(z)$ holds for any q in (2.1). Write $V_\lambda=R-B_\lambda$, $W_\lambda=B_\lambda-R$. Since $\sup_z |B_\lambda(z)/h(\lambda)|=o(1)$ and the distribution of U is continuous, writing $P^*_{0\lambda}(A)=\int P_{0\eta}\xi_{0\lambda}(d\eta)$ and $P^*_{1\lambda}(A)=\int P_{\lambda\eta}(A)\xi_{1\lambda}(d\eta)$ we get $P^*_{0\lambda}(W_\lambda+V_\lambda)=o(1)$. Since with $A=V_\lambda$ or W_λ, $P^*_{1\lambda}(A)+P^*_{0\lambda}(A)(1+O(h(\lambda)))$, writing $r^*_\lambda(A)=(1-\tau_\lambda)\cdot P^*_{0\lambda}(A)+\tau_\lambda(1-P^*_{1\lambda}(A))$, we get the integrated Bayes risk with respect to ξ_λ as

It is interesting to note that the term of order $1/\sqrt{m_1}$ in the asymptotic expansion (2.6) reduces to zero, which will be discussed in the Appendix.

In the case when $\mu_1=\mu_2=\cdots=\mu_k=0$ in $Q_k=Q_k(\lambda, \mu)$, the formula (2.6) yields an asymptotic expansion for the distribution of linear combination of independent chi-square variables. A number of papers have been published on the distribution of $Q_k(\lambda, 0)$. Among them, an approach based on linear differential equation by Davis (1977) appears to be useful for computation.

If $\lambda=(1, 1,\ldots, 1)=e$, say, in $Q_k(\lambda, \mu)$, then $Q_k(e, \mu)$ has the noncentral chi-square distribution with k degrees of freedom and noncentrality parameter $\omega^2=\sum_{j=1}^{k} \mu_j^2$. An asymptotic expansion for the distribution of $Q_k(e, \mu)$ is given by (2.6) with

$$
\begin{aligned}
&m_r = k + r\omega^2 , \quad w_j = (k + j\omega^2)/(k + \omega^2) , \\
&h = \frac{1}{3} + 2\omega^4/\{3(k + 2\omega^2)^2\} .
\end{aligned} \tag{2.7}
$$

In the special case when $\lambda=e$ and $\mu=0$, $Q_k(\lambda, \mu)$ has a chi-square distribution with k degrees of freedom, for which (2.7) further reduces to $m_r=k$, $w_j=1$ and $h=1/3$. In this case the formula (2.6) gives an asymptotic expansion for the distribution of the cube root transformation of the chi-square variate $Q_k(e, 0)$. A multivariate extension of the quadratic forms has been discussed by Khatri (1966) and Hayakawa (1966).

2.2 *Cornish-Fisher expansion*

The asymptotic expansion (2.6) can be used to calculate the probability $\Pr[Q_k<q_0]$ for an assigned value q_0. To obtain desired percentiles of the distribution of $Q_k(\lambda,\mu)$, the Cornish-Fisher inverse expansion is very convenient. The method suggested by Hill and Davis (1968) is useful for deriving the expansion of this type.

Suppose that an asymptotic expansion for the distribution of a certain variate X_n has the form

$$
\Pr[X_n < x] = \Phi(x) - \varphi(x) \sum_{j=1}^{\infty} A_j(x) n^{-j/2}
$$

We take u_α so that, for an assigned probability $(1-\alpha)$, $1-\alpha=\Pr[X_n<x_\alpha]=\Phi(u_\alpha)$. Then the Cornish-Fisher inverse expansion for x_α is given by

$$
x_\alpha = u_\alpha + \sum_{r=1}^{\infty} D_{(r)}\left\{ - \sum_{j=1}^{\infty} A_j(u)\right\}^r \Big/ r! \,\Big|_{u=u_\alpha} ,
$$

where $D_{(1)}$ denotes the identity operator and

$$D_{(r)} = (u - D_u)(2u - D_u) \cdots \{(r-1)u - D_u\} \quad \text{for} \quad r = 2, 3, \ldots ,$$

with $D_u = d/du$, the differential operator.

Applying this general formula to our problem, we have the following theorem.

THEOREM 2.2. *The Cornish-Fisher inverse expansion for the percentile q_α of the distribution of $Q_k(\lambda, \mu)$ defined by* (2.1) *is given by*

$$q_\alpha = m_1\{(2h^2 w_2/m_1)^{1/2} x_\alpha + 1 + h(h-1)w_2/m_1\}^{1/h}$$

and

$$x_\alpha = u_\alpha + \left(\sum_{j=1}^{6} m_1^{-j/2} b_j\right) + O(m_1^{-7/2}) , \tag{2.8}$$

where m_r and h are, respectively, defined in (2.2) *and* (2.4), *u_α is the percentile point of the standard normal distribution and the coefficients b_j, using the notation $w_j = m_j/m_1$ for $j = 2, 3, \ldots$, are given below.*

$$b_1 = 0 ,$$

$$b_2 = w_2^{-3}\left\{u_\alpha^3\left(\frac{1}{2} w_4 w_2 - \frac{20}{27} w_3^2 + \frac{2}{9} w_3 w_2^2\right) + u_\alpha\left(-\frac{3}{2} w_4 w_2 + \frac{14}{9} w_3^2\right)\right\} ,$$

$$\begin{aligned} b_3 = \sqrt{2} w_2^{-9/2}\Bigg\{& u_\alpha^4\left(\frac{2}{5} w_5 w_2^2 - \frac{4}{3} w_4 w_3 w_2 + \frac{76}{81} w_3^3 + \frac{1}{9} w_3^2 w_2^2 - \frac{1}{9} w_3 w_2^4\right) \\ &+ u_\alpha^2\left(-\frac{12}{5} w_5 w_2^2 + 6 w_4 w_3 w_2 - \frac{272}{81} w_3^3 - \frac{2}{9} w_3^2 w_2^2\right) \\ &+ \frac{6}{5} w_5 w_2^2 - 2 w_4 w_3 w_2 + \frac{62}{81} w_3^3\Bigg\} , \end{aligned}$$

$$\begin{aligned} b_4 = w_2^{-6}\Bigg\{& u_\alpha^5\left(\frac{2}{3} w_6 w_2^3 - \frac{8}{3} w_5 w_3 w_2^2 - \frac{9}{8} w_4^2 w_2^2 + 6 w_4 w_3^2 w_2 + \frac{1}{3} w_4 w_3 w_2^3\right. \\ &\left. - \frac{1144}{405} w_3^4 - \frac{52}{135} w_3^3 w_2^2 - \frac{2}{15} w_3^2 w_2^4 + \frac{2}{15} w_3 w_2^6\right) \\ &+ u_\alpha^3\left(-\frac{20}{3} w_6 w_2^3 + \frac{64}{3} w_5 w_3 w_2^2 + 9 w_4^2 w_2^2 - \frac{367}{9} w_4 w_3^2 w_2 - w_4 w_3 w_2^3\right. \\ &\left. + \frac{4144}{243} w_3^4 + \frac{20}{27} w_3^3 w_2^2 + \frac{8}{27} w_3^2 w_2^4\right) \\ &+ u_\alpha\left(10 w_6 w_2^3 - 24 w_5 w_3 w_2^2 - \frac{87}{8} w_4^2 w_2^2 + \frac{113}{3} w_4 w_3^2 w_2\right. \\ &\left. - \frac{350}{27} w_3^4 + \frac{4}{27} w_3^3 w_2^2\right)\Bigg\} , \end{aligned}$$

From Gleser and Olkin (1970) and Kariya (1978), it follows that a maximal invariant in the space of X under the subgroup $H=G_l\times F$ of G is $(S_1(X, V), S_2(X, V))$

$$S_1(X,V) = (X_{(12)},X_{(13)})\begin{pmatrix} V_{(22)} & V_{(23)} \\ V_{(32)} & V_{(33)} \end{pmatrix}^{-1} (X_{(12)},X_{(13)})' ,$$

$$S_2(X,V) = \begin{pmatrix} X_{(13)} \\ X_{(23)} \end{pmatrix} V_{(33)}^{-1} \begin{pmatrix} X_{(13)} \\ X_{(23)} \end{pmatrix}' .$$

Thus, a maximal invariant under G depends on X only through $U=(U_1, U_2, U_3)$, where

$$U_1 = (X_{(12)},X_{(13)}) , \quad U_2 = X_{(23)} , \quad U_3 = (X_{(32)},X_{(33)}) . \tag{3.4}$$

Since $n_1+n_3>p$ by Assumption 2 we conclude that $U_1'U_1+U_2'U_2$ is nonsingular with probability one. Furthermore a corresponding maximal invariant in the parametric space of (θ, Σ) is given by

$$\xi\xi' \quad \text{where} \quad \xi = \theta_{(12)}\Sigma_{22\cdot 3}^{-1/2} . \tag{3.5}$$

Since the problem of testing H_0 against H_1 remains invariant under translations of X by F and since given $\Sigma>0$, there exists a $p\times p$ matrix $g_l \in G_l$ such that $\Sigma=g_lg_l'$, without any loss of generality we can assume that $\theta_{(11)}=0$, $\theta_{(21)}=0$, $\theta_{(22)}=0$, $\Sigma=I$ and take $G=O\times G_l$.

The marginal probability density function of U is given by

$$f(u|\xi) = \bar{q}(\mathrm{tr}[(u_1 - \xi^*)'(u_1 - \xi^*) + u_2'u_2 + u_3'u_3]) , \tag{3.6}$$

where $\theta_{(12)}=\xi$, $\xi^*=(\xi, 0)$: $n_1\times(p_2+p_3)$. The group G acting on the left on X reduces to the subgroup $H=O_{(11)}\times G_l^*$ acting on the left on U as

$$gU = (O_{(11)}U_1g_l^*,X_{(23)}g_{(23)}', U_2g_l^*) ,$$

where $g_l^* \in G_l^*$ with $g_l^*=\begin{pmatrix} g_{(22)} & g_{(23)} \\ 0 & g_{(33)} \end{pmatrix}$: $(p_2+p_3)\times(p_2+p_3)$ nonsingular matrices in the block form and $\bar{q}$ is the integral of q with respect to the remaining variables $X_{(11)}$, $X_{(22)}$, $X_{(21)}$. Note that the subgroup $O_{(22)}\times O_{(33)}$ can be ignored as it does not affect U. Kariya (1981) has shown that the distribution of the maximal invariant under the null hypothesis remains the same for any pdf of the form (3.1). Kariya and Sinha (1984) have shown that if $q \in Q$; the class of continuously thrice differentiable functions from $[0,\infty)$ to $[0,\infty)$ satisfying Assumption 2,

ASSUMPTION 2.

(1) $\int_{R^{np}} q(\mathrm{tr}x'x)dx=1$,

(2) $\int_{G^*} (\mathrm{tr}g^*g^{*\prime})^{i/2}|\bar{q}^{(i)}(\mathrm{tr}g^*g^{*\prime})|\mu(dg^*)<\infty, \quad i=1,2,3$,

(3) $\bar{q}^{(3)}(x)\leq 0$ and $\bar{q}^{(3)}$ nondecreasing, where $\bar{q}^{(i)}(x)=d^i\bar{q}(x)/dx^i$.

Then the test which rejects H_0 whenever

$$R_1 = a\mathrm{tr}z(z'z + V_{22\cdot 3})z(I+T_2)^{-1} - \mathrm{tr}(I+T_2)^{-1} \geq c\,, \tag{3.7}$$

where c is a constant depending on the level of significance α, where $a=(\eta_1+\eta_3-p_3)/p_2$, $z=(I+T_2)^{-1/2}(X_{(12)}-X_{(13)}V_{(33)}^{-1}V_{(32)})$, $T_2=X_{(13)}V_{(33)}^{-1}X'_{(13)}$, is LBI for testing H_0 against H_1 for any pdf of the form (3.1) when $\delta=\mathrm{tr}\xi\xi'\rightarrow 0$. For the Manova problem R_1 reduces to

$$R_1 = \mathrm{tr}X_{(12)}(X'_{(12)}X_{(12)} + V_{(22)})^{-1}X'_{(12)} \geq c\,. \tag{3.8}$$

For the multivariate normal setup Schwartz (1967) has shown that the test given in (3.8) is LBI and locally minimax as $\sigma\rightarrow 0$.

Remark 5. The Assumption 2 is satisfied for a large class of distribution especially in the case of a normal mixture $q(x)=\int_0^\infty e^{-ax}dF(x)$ provided the condition on moments holds. Kariya (1981) has shown that when $\min(p_1,p_2)=1$, without any additional condition on q except its convexity the UMP property of the Manova test given in (3.8) holds.

In G-K (1964) the first step in verifying the local minimaxity of T^2 and R^2 tests, etc., is to reduce the problem using Hunt-Stein Theorem. Let $G_T(p_2+p_3)$ be the multiplicative group of $(p_2+p_3)\times(p_2+p_3)$ nonsingular upper triangular matrices. Both the groups $O_{(11)}$ and G_T satisfy the conditions of Hunt-Stein Theorem and so does their direct product. Hence, for each α, there exists a $g_T\in G_T$ and $O_{(11)}$ invariant level α test which is minimax. Using Stein (1956) and Wijsman (1967) the ratio R of the pdf of the maximal invariant with respect to G_T and $O_{(11)}$ under H_1 and under H_0 is given by

$$\begin{aligned} R = \Big\{ \int \bar{q}(\mathrm{tr}[(O_{(11)}U_1g_T - \xi^*)'(O_{(11)}U_1g_T - \xi^*) + g_{(33)}X'_{(23)}X_{(23)}g'_{(33)} \\ + g_TU_3U_3'g_T'])\mu(dg_T)\nu(dO_{(11)})\Big\} \Big/ \Big\{ \int \bar{q}(\mathrm{tr}[g_T'U_1U_1'g_T \\ + g_{(33)}X'_{(23)}X_{(23)}g'_{(33)} + g_TU_3U_3'g_T'])\mu(dg_T)\nu(dO_{(11)})\Big\}\,, \end{aligned} \tag{3.9}$$

where $g_T=\begin{pmatrix} g_{(22)} & g_{(23)} \\ 0 & g_{(33)} \end{pmatrix}=(g_{ij})$ as g above (but g_T is upper triangular, i.e. $g_{(22)}$,

$g_{(33)}$ are upper triangular)

$$(3.10)\qquad \mu(dg_T) = |g_{(22)}g'_{(22)}|^{(n_1+n_2-p_3)/2}|g_{(33)}g'_{(33)}|^{(n+p_2)/2}\prod_1^{p_2+p_3}(g_{ii}^2)^{-i/2}dgT\ ,$$

and $\nu(dO_{(11)})$ is the invariant measure on $O_{(11)}$. It may be noted that a left invariant Haar measure on G_T is

$$(3.11)\qquad \prod_1^{p_2+p_3}(g_{ii}^2)^{-(p_2+p_3+1-i)/2}dg_T\ .$$

It may be remarked that we are writing g_T also as

$$g_T = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ 0 & g_{22} & \cdots & g_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g_{kk} \end{pmatrix},\qquad k = p_2 + p_3\ ,$$

and the inverse of the Jacobian of the transformation $u \to g_T u$ is

$$|g_{(22)}g'_{(22)}|^{(\eta_1+\eta_3)/2}|g_{(33)}g'_{(33)}|^{\eta/2}.$$

Since $U_1'U_1+U_3'U_3$ is nonsingular, there exists $g_T^* \in G_T$ such that $g_T^*(U_1'U_1+U_3'U_3)g_T^{*\prime}=I$. Hence, the integrand in the numerator of (3.9) can be written as (writing $g_Tg_T^{*-1}=g_T=(g_{ij}) \in G_T$)

$$\begin{aligned}(3.12)\qquad &\bar{q}(\mathrm{tr}[g_Tg_T' - 2\xi'O_{(11)}(W_2g'_{(22)} + W_3g'_{(23)}) + g_{(33)}x_{(23)}x'_{(23)}g'_{(33)}] + \sigma) \\ &= \bar{q}(\mathrm{tr}[g_{(22)}g'_{(22)} + g_{(23)}g'_{(23)} + g_{(33)}(I + x_{(23)}x'_{(23)})g'_{(33)} \\ &\quad - 2\xi'O_{(11)}(W_2g'_{(22)} + W_3g'_{(23)})] + \sigma)\ ,\end{aligned}$$

where $\sigma=\mathrm{tr}\xi\xi'$ and $(W_2,\ W_3)=U_1g_T'=(X_{(12)}g'_{(22)}+X_{(13)}g'_{(23)},\ X_{(13)}g'_{(33)})$. It may be verified that

$$(3.13)\qquad \begin{aligned}\mathrm{tr}\,W_2'W_2 &= \mathrm{tr}z(z'z + V_{22\cdot3})^{-1}z'(I + T_2)^{-1}\ , \\ \mathrm{tr}\,W_3'W_3 &= -\ \mathrm{tr}(I + T_2)^{-1} + \eta_1\ .\end{aligned}$$

Now transforming $g_{(33)} \to g_{(33)}\,(I+X_{(23)}X'_{(23)})$ and writing $g_{(33)}\,(I+X_{(23)}X'_{(23)})^{-1/2}$ $=g_{(33)}$, the ratio R in (3.9) can be written as

$$\begin{aligned}(3.14)\qquad R &= \frac{dP_\sigma^T}{dP_0^T} \\ &= \frac{\int\bar{q}(\mathrm{tr}[g_Tg_T' - 2\xi'O_{(11)}(W_2g'_{(22)} + W_3g'_{(23)}) + \sigma])\mu(dg_T)\nu(dO_{(11)})}{\int\bar{q}(\mathrm{tr}g_Tg_T')\mu(dg_T)\nu(dO_{(11)})}\ .\end{aligned}$$

To evaluate the numerator of R, we expand the integrand q in the numerator of (3.14) when $\sigma \to 0$ as

$$\text{(3.15)} \quad \bar{q}(\mathrm{tr} g_T g_T') + \bar{q}^{(1)}(\mathrm{tr} g_T g_T')(-2\eta + \delta) + \frac{\bar{q}^{(2)}}{2}(\mathrm{tr} g_T g_T')(-2\eta + \delta)^2 + \frac{\bar{q}^{(3)}(V)}{6}(-2\eta + \delta)^3 ,$$

where

$$\text{(3.16)} \quad \begin{aligned} \eta &= \mathrm{tr}\xi' O_{(11)}(W_2 g_{(22)}' + W_3 g_{(23)}') , \\ V &= \mathrm{tr} g_T g_T' + (1 - \alpha)(-2\eta + \delta) , \quad 0 \le \alpha \le 1 . \end{aligned}$$

We shall now evaluate the integral of each term of (3.15) by using the following wellknown results

$$\text{(3.17)} \quad \int_{O_{(11)}} (\mathrm{tr} OQ)^k \nu(dO) = \begin{cases} \dfrac{\mathrm{tr} QQ'}{n_1} & \text{if} \quad k = 2 , \\ 0 & \text{if} \quad k \text{ is odd} . \end{cases}$$

This is due to James (1964).

$$\text{(3.18)} \quad \int_G g_{(22)} g_{(23)} \bar{q}^{(i)}(\mathrm{tr} g_T g_T')\mu(dg_T) = 0 , \quad i = 1, 2, 3 .$$

This follows from the fact that the measure $\bar{q}^{(i)}(\mathrm{tr} g_T g_T')\mu(dg_T)$ is invariant under the change of sign of $g_{(22)}$ to $-g_{(22)}$.

$$\text{(3.19)} \quad \int_{G_T} g_{ij} g_{lk} \bar{q}^{(i)}(\mathrm{tr} g_T g_T')\mu(dg_T) = 0 \quad \text{if} \quad i \ne l, j \ne k .$$

$$\text{(3.20)} \quad \int_{G_T} g_T \bar{q}^{(i)}(\mathrm{tr} g_T g_T')\mu(dg_T) = 0 .$$

The integration of the second term of (3.15) (using (3.20)) gives

$$\text{(3.21)} \quad \begin{aligned} \sigma \int \bar{q}^{(1)}(\mathrm{tr} g_T g_T')\mu(dg_T)\nu(dO_{(11)}) &= \sigma \int_{G_T} \bar{q}^{(1)}(\mathrm{tr} g_T g_T')(dg_T) \\ &= \sigma\beta_1 \text{ (say)} . \end{aligned}$$

The integral of the third term of (3.15) becomes

$$(3.22)\quad I = \frac{2}{\eta_1}\int_{G_T}\mathrm{tr}(w_2 g'_{(22)} + w_3 g'_{(23)})\xi'\xi(w_2 g'_{(22)} + w_3 g'_{(23)})'\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)$$

$$+ \frac{\sigma^2}{2}\int_{G_T}\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)$$

$$= \frac{2}{\eta_1}\int_{G_T}\mathrm{tr}(w_2 g'_{(22)}\xi'\xi g_{(22)} w'_2)\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)$$

$$+ \frac{2}{\eta_1}\int_{G_T}\mathrm{tr}(w_3 g'_{(23)}\xi'\xi g_{(23)} w'_3)\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)$$

$$+ \frac{\sigma^2}{2}\int_{G_T}\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)\ .$$

Write $\Gamma=(\gamma_{ij})=\xi'\xi$, $B=(b_{ij})=W'_2W_2$, $E=(e_{ij})=W'_3W_3$, $\delta_i=\gamma_{ii}$,

$$(3.23)\quad \begin{aligned} d_{ij} &= \int_{G_T} g_{ij}^2\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T), \qquad i \neq j\ , \\ c_j &= \int_{G} g_{jj}^2\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T)\ . \end{aligned}$$

Hence

$$(3.24)\quad \begin{aligned} &\int_{G_T}\mathrm{tr}(W_2 W_2 g'_{(22)}\xi'\xi g_{(22)})\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T) \\ &\quad = \int \mathrm{tr}(\Gamma g'_{(22)} B g_{(22)})\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T) \\ &\quad = \Sigma\left(\sum_{i>j}\sigma_i d_{ij} + \sigma_j c_j\right) b_{jj}\ , \end{aligned}$$

and

$$(3.25)\quad \begin{aligned} &\int_{G_T}\mathrm{tr}(W'_3 W_3 g_{(23)}\xi'\xi g_{(23)})\bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\mu(dg_T) \\ &\quad = \sum_k\left(\sum_i \sigma_i d_{iK}\right) e_{kk}\ . \end{aligned}$$

To evaluate d_{ij} and c_j we proceed as follows. Let

$$\begin{aligned} &L = \mathrm{tr} g_T g'_T\ , && h(g_T) = \bar{q}^{(2)}(\mathrm{tr} g_T g'_T)\ , \\ &k = \int_{G_T} h(g_T) dg_T\ , && m = p_2 + p_3\ , \\ &e_i = g_{ii}^2/L\ , && i = 1,\dots, m\ , \\ &e_{m+i} = g_{i\,i+1}^2/L\ , && i = 1,\dots, m-1\ , \\ &e_{2m+i-1} = g_{i\,i+2}^2/L\ , && i = 1,\dots, m-2\ , \end{aligned}$$

$$e_{m(m+1)/2} = g_{m1}^2 / L \ ,$$
$$D = \int_{G_T} q(\mathrm{tr} g_T g_T') \prod_{1}^{p_2} (g_{ii}^2)^{(n_1+n_3-p_3-i)/2} \prod_{j=p_2+1}^{m} (g_{jj}^2)^{(n+p_3-i)/2} dg_T \ ,$$
$$N = \int_{G_T} L^{\sum_1^{p_2}(n_1+n_3-p_3-i)/2 + \sum_{p_2+1}^{m}(n+p_3-i)/2} \overline{q}^{(2)}(L) dg_T \ .$$

Since $h(g_T)/k$ is a spherical density of g_{ij}'s $i \le j$, L and $e=(e_1,\ldots, e_{m(m+1)/2})$ are independent and e obeys Dirichlet distribution $D(1/2,\ldots, 1/2)$ (Kariya and Eaton (1977)). Hence, with $N_1=n_1+n_3-p_3$, $N_2=n+p_3$,

$$\begin{aligned} (3.26) \qquad d_{ij} &= \int_{G_T} g_{ij}^2 \overline{q}^{(2)}(\mathrm{tr} g_T g_T') \prod_{1}^{p_2} (g_{ii}^2)^{(N_1-i)/2} \prod_{p_2+1}^{m} (g_{jj}^2)^{(N_2-i)/2} dg_T \\ &= N\ E\left(e_j \prod_{l=1}^{p_2} e^{(N_1-i)/2} \prod_{p_2+1}^{m} e^{(N_2-i)/2} \right), \quad j > m \\ &= N\ M \text{ (say)} \ , \end{aligned}$$

and for $j \le p_2$

$$\begin{aligned} c_j &= \int_{G_T} g_{ij}^2 \overline{q}^{(2)}(\mathrm{tr} g_T g_T') \prod_{1}^{p_2} (g_{ii}^2)^{(N_1-i)/2} \prod_{p_2+1}^{m} (g_{jj}^2)^{(N_2-i)/2} dg_T \\ &= N\ E\left(e_j \prod_{l=1}^{p_2} e^{(N_1-i)/2} \prod_{p_2+1}^{m} e_k^{(N_2-i)/2} \right) \\ &= N\ M(N_1 - j + 1) \ . \end{aligned}$$

Thus, we get from (3.22),

$$(3.27) \quad I = 2\frac{NM}{\eta_1}\left[\sum_j \left(\sum_{i>j} \sigma_i + (N_1 - j + 1)\sigma_j \right) b_{ij} + \sum_k \left(\sum_i \sigma_i \right) e_{kk} \right] + o(\sigma) \ .$$

Now consider the integral of the fourth term in (3.15).

$$\begin{aligned} |\eta| &= |\mathrm{tr}(\xi' O_{(11)}(W_2 g_{(22)}' + W_3 g_{(23)}'))| \\ &\le (\mathrm{tr}(g_T g_T'))^{1/2} (\mathrm{tr}(\xi'(O_{(11)}W_2, O_{(11)}W_3)(O_{(11)}W_2, O_{(11)}W_3)'\xi))^{1/2} \\ &\le (\mathrm{tr} g_T g_T')^{1/2} (\mathrm{tr}\xi'\xi)^{1/2} \ , \end{aligned}$$

as $(W_2, W_3)(W_2, W_3)' = U_1(U_1'U_1 + U_2'U_2)^{-1}$, $U_1' \le I$. Since $V \ge \mathrm{tr} g_T g_T' - 2\sigma^{1/2} \cdot (\mathrm{tr} g_T g_T')^{1/2} + \sigma$ and

$$(-2\eta + \sigma)^3 \le 8(\mathrm{tr} g_T g_T')^{3/2}\sigma^{3/2} + 12(\mathrm{tr} g_T g_T')\sigma^2 + 6(\mathrm{tr} g_T g_T')^{1/2}\sigma^{5/2} + \sigma^3 \ ,$$

by Assumption 2 we get

$$\left|\int_{G_T} \bar{q}^{(3)}(V)(-2\eta + \sigma)^3 \nu(dO_{(11)})\mu(dg_T)\right|$$
$$\leq \int_{G_T} \bar{q}^{(3)}(V)|-2\eta + \sigma|^3 \nu(dO_{(11)})\mu(dg_T)$$
$$\leq \int_{G_T} \bar{q}^{(3)}(\mathrm{tr} g_T g_T' - 2(\mathrm{tr}(g_T g_T'))^{1/2}\sigma^{1/2})|-2\eta + \sigma|^3 \nu(dO_{(11)})\mu(dg_T)$$
$$\leq \sum_1^4 c_i \int_{G_T} \bar{q}^{(3)}(\mathrm{tr} g_T g_T' - 2(\mathrm{tr} g_T g_T')^{1/2}\sigma^{1/2})(\mathrm{tr} g_T g_T')^{(4-i)/2}\sigma^{(1+i)/2}\mu(dg_T)$$
$$= o(\sigma) \ ,$$

where $c_1=8$, $c_2=12$, $c_3=6$ and $c_4=1$. Hence, we can write with $\xi_i=\sigma_i/\sigma$, $i=1,\ldots,p_2$

$$(3.28)\quad R = 1 + \frac{\sigma}{D}\left(\beta_1 + 2\frac{MN}{\eta_1 K}\left(\sum_{j=1}^{p_2}\left(\sum_{i>j}\xi_i + (N_1 - j + 1)\xi_j\right)b_{jj} + \mathrm{tr} W_3' W_3\right)\right)$$
$$+ B(x,\xi,\sigma) \ ,$$

where $\xi=(\xi_1,\ldots,\xi_{p_2})$ and $B(x,\sigma,\xi)=o(\sigma)$ uniformly in x and ξ. Now letting ξ_0 assign measure 1 to the single point $\xi=0$ while $\xi_{1\lambda}$ gives measure 1 to the single point $\xi^*=(\xi_1^*,\ldots,\xi_{p_2}^*)$ (say) whose j-th coordinate $\xi_j^*=(N_1-j)(N_1-j+1)^{-1}\cdot p_2^{-1}N_1(N_1-p_2)$ so that

$$\sum_{i>j}^{p_2}\xi_i^* + (N_1 - j + 1)\xi_j^* = \frac{N_1}{p_2} \ ,$$

for all j, we see that for testing H_0: $\sigma=0$ against H_1: $\sigma=\lambda$ the condition (2.6) is satisfied for any q in (3.1). The condition (2.5) is now obvious. Hence, we have the following:

THEOREM 3.1. *The test given by the critical region $R_1\geq c$ is locally minimax with respect to the contour*

$$\{(\theta_{(12)},\theta_{22\cdot3})|\mathrm{tr}\theta_{(12)}\Sigma_{22\cdot3}^{-1}\theta_{(12)}' = \lambda\} \ ,$$

as $\lambda\to0$.

Note. This also establishes the local minimaxity of the test $R_1\geq C$ for general Manova problem in the normal multivariate setup. The most general result known so far in this context is due to Kariya (1978) where he took

$$\theta = \begin{pmatrix} \theta_{(11)} & \theta_{(12)} & \theta_{(13)} \\ \theta_{(21)} & \theta_{(22)} & \theta_{(23)} \end{pmatrix} \quad \text{with} \quad \theta_{(ij)}: n_i \times p_j \ ,$$

with $\eta_1+\eta_2=\eta$, $p_1+p_3+p_3=n$.

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