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RANK ORDER STATISTICS FOR TIME SERIES MODELS

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Abstract. Some invariance principles are obtained for the one-sample rank order statistics of a Φ-mixing or strong mixing type time series. The estimation of the center of symmetry of the time series and tests for serial dependence are considered as applications.

Key words and phrases: Invariance principles, mixing, rank order statistics.

1. Introduction

Rank order statistics have recently received much attention in time series analysis. For a bibliography, see Hallin *et al.* (1985). The purpose of this paper is to obtain invariance principles for the one-sample rank order statistics of a general class of time series. The estimation of the center of symmetry of the time series and tests for serial dependence are then studied as applications.

Let $X(t)$, $t=\dots,-1,0,1,\dots$ be a real valued strictly stationary time series defined on a probability space (Ω, F, P). Let M_1^k and M_{k+n} be, respectively, the σ-fields generated by $\{X_t: t\le k\}$ and $\{X_t: t\ge k+n\}$. Let α and Φ be functions of non-negative integers satisfying $\alpha(n)\downarrow 0$ and $\Phi(n)\downarrow 0$. Then X_t is said to be strong mixing if for all $A\in M_1^k$ and $B\in M_{k+n}$

$$|P(A\cap B)-P(A)P(B)|<\alpha(n),$$

where k and n are arbitrary positive integers; it is Φ-mixing if the above inequality holds with $\Phi(n)P(A)$ instead of $\alpha(n)$.

Let $X_1,\dots,X_n$ be n consecutive observations of the time series X_t. Define $u(x)=1$ or 0 according as $x\ge 0$ or <0. Let R_{ni} be the rank of $|X_i|$ among $|X_1|,\dots,|X_n|$ and

$$T_n=n^{-1}\sum_{i=1}^{n}u(X_i)J_n((n+1)^{-1}R_{ni}),\quad n\ge 1, \tag{1.1}$$

where $J_n(i/(n+1))=EJ(U_{ni})$ or $J(i/(n+1))$, $1\le i\le n$, $U_{n1}\le\cdots\le U_{nn}$ are the ordered random variables of a sample of size n from the uniform (0,1) distribution function and $J(u)=J^*((1+u)/2)$, $0\leq u<1$, is an absolutely continuous and twice differentiable score function. We assume that there are positive (finite) constants K, $0<\alpha\leq 1/2$, and $0<\mu\leq\alpha$ such that for $0<u<1$

$$(1.2)\qquad |J^{*(k)}(u)| = |d^kJ^*(u)/du^k| \leq K[u(1-u)]^{-\alpha-k+\mu},$$

for $k=0, 1, 2$; and X_t is Φ-mixing satisfying

$$(1.3)\qquad \sum_{n=1}^{\infty} n^2\Phi(n) < \infty,$$

or strong mixing satisfying

$$(1.4)\qquad \alpha(n) = O(e^{-\theta n}) \quad \text{for some} \quad \theta > 0.$$

Assume X_t has a continuous distribution function $F(x)$. Let $H(x) = P[|X_t|\leq x]$, $F_n(x)=n^{-1}\sum_{i=1}^{n} u(x-X_i)$, $-\infty<x<\infty$, and $H_n(x)=F_n(x)-F_n(-x-)$, $x\geq 0$.

Write $T_n=\int_0^\infty J_n(nH_n(x)/(n+1))\, dF_n(x)$. Then

$$(1.5)\qquad T_n = m + n^{-1}\sum_{i=1}^{n} B(X_i) + R_n,$$

where $B(X_i)=u(X_i)\,J(H(|X_i|)+\int_0^\infty[u(x-|X_i|)-H(x)]J'(H(x))dF(x)-m$, with $m=\int_0^\infty J(H(x))dF(x)$; and

$$\begin{aligned} R_n = &\int_0^\infty [J_n(nH_n(x)/(n+1)) - J(nH_n(x)/(n+1))]dF_n(x) \\ &+\int_0^\infty [J(nH_n(x)/(n+1)) - J(H(x))]dF_n(x) \\ &-\int_0^\infty [H_n(x) - H(x)]J'(H(x))dF(x). \end{aligned}$$

Let $\sigma^2=V[B(X_1)]+2\sum_{k=2}^{\infty}\operatorname{cov}[B(X_1), B(X_k)]$. Assume that

$$(1.6)\qquad \sigma^2 > 0.$$

THEOREM 1.1. *Let X_t be Φ-mixing satisfying* (1.3) *or strong mixing satisfying* (1.4). *If* (1.2) *and* (1.6) *hold, then*

$$L(n^{1/2}(T_n - m)/\sigma)\to N(0, 1) \quad \text{as} \quad n\to\infty.$$

THEOREM 1.2. *Assume the conditions of Theorem* 1.1 *hold. Then*

$$\limsup_{n\to\infty} n^{1/2}(T_n - m)/(2 \operatorname{loglog} n)^{1/2} = \sigma \quad a.s.$$

$$\liminf_{n\to\infty} n^{1/2}(T_n - m)/(2 \operatorname{loglog} n)^{1/2} = -\sigma \quad a.s.$$

In the Φ-mixing case, Sen and Ghosh (1973) have obtained some invariance principles for T_n in a spirit similar to those considered here. They have shown that if $\sum_{n=1}^{\infty} n^k \Phi^{1/2}(n)<\infty$ and (1.2) is replaced by $|J^{*(k)}(u)| < K[u(1-u)]^{-\alpha+k+\mu}$ with $\alpha=(2k-1)/2(2k+1)$ for some $k>1$, then the conclusion of Theorem 1.1 holds if $\sigma^2>0$. They also showed that if $\sum_{n=1}^{\infty} n^k \Phi^{1/2}(n)<\infty$ and (1.2) is replaced by $|J^{*(k)}(u)|\leq K[u(1-u)]^{-\alpha+k+\mu}$ with $\alpha=(k-2)/2k$ for some $k\geq 3$, then the conclusion of Theorem 1.2 is valid of $\sigma^2>0$. Thus Theorems 1.1 and 1.2 give significant improvement of Sen and Ghosh's results for the Φ-mixing case.

Our main effort is devoted to the strong mixing case which is more interesting than the Φ-mixing case since the Φ-mixing condition is much more restrictive than the strong mixing condition. In fact, if X_t is Gaussian and Φ-mixing, then X_t is m-dependent. From the definition of the Φ-mixing condition, it is easily seen that if X_t is a sequence of independent random variables or if X_t is m-dependent, then X_t is Φ-mixing. Thus moving average time series models are Φ-mixing. Recently, Pham and Tran (1985) have shown that a large class of ARMA models are absolutely regular and hence strong mixing. Condition (1.4) is satisfied by a general class of ARMA models. For an account of this information, see Theorem 3.1 of Pham and Tran (1985).

In Section 2, some preliminaries and auxiliary lemmas are presented. The proofs of the theorems are given in Section 3. Section 4 considers some applications.

Throughout the paper, c will be used to denote constants whose values are unimportant and may be different from line to line.

2. Preliminaries and auxiliary lemmas

Let ξ be M_1^k measurable and η be M_{k+n} measurable, then if X_t is Φ-mixing

$$|\operatorname{Cov}(\xi, \eta)| \leq 2\|\xi\|_a \|\eta\|_b [\Phi(n)]^{1/a}, \tag{2.1}$$

for all $1\leq a, b\leq\infty$ with $a^{-1}+b^{-1}=1$, and if X_t is strong mixing

$$|\operatorname{Cov}(\xi, \eta)| \leq 10\|\xi\|_a \|\eta\|_b [\alpha(n)]^{1/c}, \tag{2.2}$$

for all $1\leq a,b,c\leq\infty$ with $a^{-1}+b^{-1}+c^{-1}=1$. See Puri and Tran (1980) for more information. Let $Y_i=F(X_i)$.

LEMMA 2.1. *If X_t is Φ-mixing satisfying* (1.3), *then*

$$E\left[\sum_{i=1}^{n}(u(t-Y_i)-t)\right]^4 \le c[n^2\tau^2+n\tau] \quad \textit{where} \quad \tau=t(1-t)\,.$$

Lemma 2.1 can be obtained by a slight variation of the proof of Lemma 2.6 (ii) of Mehra and Rao (1975).

LEMMA 2.2. *Let $\rho>0$ and $0\le t\le 1$. Let $\tau=t(1-t)$. If X_t is Φ-mixing satisfying* (1.3), *then*

$$E\left|\sum_{i=1}^{n}(u(t-Y_i)-t)\right|^l \le cn^{\rho}((n\tau)^{l/2}+n\tau) \quad \textit{where} \quad l\ge 4\,.$$

The proof of Lemma 2.2 is given in Puri and Tran ((1980), p. 409).

LEMMA 2.3. *Let X_t be Φ-mixing satisfying* (1.3). *Then*

$$\sup_{-\infty<x<\infty} n^{1/2}|F_n(x)-F(x)|\{F(x)(1-F(x))\}^{-1/2}=o(n^{\eta}) \quad a.s.\,,$$

for every $\eta>0$, as $n\to\infty$.

Remark 2.1. Sen and Ghosh (1973) have obtained a somewhat weaker version of Lemma 2.3.

For examples of Φ-mixing sequences satisfying (1.3) without satisfying Sen and Ghosh's condition, see Bradley (1980).

PROOF. Let $G_n(t)=n^{-1}\sum_{i=1}^{n}u(t-Y_i), 0\le t\le 1, n\ge 1$, and $h_n(t)=n^{1/2}|G_n(t)-t|\cdot\{t(1-t)\}^{-1/2}$. The conclusion of Lemma 2.3 is equivalent to $\sup_{0\le t\le 1} h_n(t)=o(n^{\eta})$ a.s. for some $\eta>0$, as $n\to\infty$. For reasons of symmetry, it is sufficient to prove $\sup_{0\le t\le 1/2} h_n(t)=o(n^{\eta})$ a.s., as $n\to\infty$. Let $r>1$. Then

$$\begin{aligned}
&P[G_n(n^{-r})\ge n^{-1} \text{ for some } n\ge k]\\
&\quad\le\sum_{i=0}^{\infty}P[nG_n(n^{-r})\ge 1 \text{ for some } 2^ik\le n<2^{i+1}k]\\
&\quad\le\sum_{i=0}^{\infty}2^{i+1}k(2^ik)^{-r}=2k^{-r+1}/(2^{r-1}-1)\,,
\end{aligned}$$

which goes to zero, as $k\to\infty$. Thus for every $r>1$, $\sup_{0\le t\le n^{-r}} h_n(t)\to 0$ a.s., as $n\to\infty$.

Define $V_{nj}=h_n(jn^{-r}), j=1,\dots,[n^{r-1}]+1$. Now

$$\sup_{n^{-r}\le t\le n^{-1}} h_n(t)\le\sqrt{2}\left\{\max_{1\le j\le[n^{r-1}]+1} V_{nj}+O(n^{-(r-1)/2})\right\}. \tag{2.3}$$

Also,

$$(2.4)\qquad P[V_{nj} > n^{\eta}] \le P\left[\left|\sum_{j=1}^{n}\{u(jn^{-r} - Y_j) - jn^{-r}\}\right| > cn^{\eta+1/2}(jn^{-r})^{1/2}\right]$$
$$\le c\left[\frac{n^{\rho}}{n^{\eta l}} + \frac{n^{\rho}n^{-r+1-(l/2)(-r+1+2\eta)}}{j^{l/2-1}}\right].$$

Pick $l>4$. Summing up over j in (2.4) we obtain

$$P\left[\max_{1\le j\le[n^{r-1}]+1} V_{nj} > n^{\eta}\right] \le cn^{-1-\varepsilon} \quad \text{for some} \quad \varepsilon > 0\ ,$$

by choosing $l>\max\left[\frac{\rho + r}{\eta}, \frac{2(-r+2+\rho)}{-r+1+2\eta}\right]$. Finally, $\sum_{n=1}^{\infty} P\left[\max_{1\le j\le[n^{r-1}]+1} V_{nj}>n^{\eta}\right]$ $<\infty$. By Borel-Cantelli Lemma and (2.3), it follows that $\sup_{n^{-r}\le t\le n^{-1}} h_n(t)=o(n^{\eta})$ a.s., as $n\to\infty$.

Let $W_{nj}=h_n(j/n)$: $j=1,\ldots,[n/2]+1$. Then

$$(2.5)\qquad \sup_{n^{-1}\le t\le 1/2} h_n(t) \le \sqrt{2}\left\{\max_{1\le j\le[n/2]+1} W_{nj} + o(n^{\eta})\right\}.$$

By Lemma 2.2

$$(2.6)\qquad P[W_{nj} > n^{\eta}] \le c\left[\frac{n}{n^{\eta l}} + \frac{n^{\rho}}{n^{\eta l}\, j^{(l/2)-1}}\right].$$

Using (2.6), it is easy to show that $\sum_{n=1}^{\infty} P\left[\max_{1\le j\le[n/2]+1} W_{nj}>n^{\eta}\right]<\infty$. Borel-Cantelli Lemma and (2.5) imply that $\sup_{n^{-1}\le t\le 1/2} h_n(t)=o(n^{\eta})$ a.s., as $n\to\infty$.

Our main tool for the strong mixing case is based on an approximation of $\sum_{i=1}^{\infty} u(t-Y_i)$ by martingales as done in Philipp and Stout (1975) and Philipp (1977).

LEMMA 2.4. *Let δ be an arbitrarily small positive number and $\tau=t(1-t)$. If X_t is strong mixing satisfying* (1.4), *then*

$$E\left[\sum_{i=1}^{n}(u(t - Y_i) - t)\right]^4 \le c[n^2\tau^2 + n\tau]\tau^{-\delta}\ ,$$

Lemma 2.4 can be proved by using similar lines of argument as in the proof of Lemma 2.6 (i) of Mehra and Rao (1975).

LEMMA 2.5. *Let* $\delta>0$ *and* $\tau=t(1-t)$. *Assume* X_t *is strong mixing satisfying* (1.4). *Then*

$$E\left[\sum_{i=1}^{n}(u(t-Y_i)-t)\right]^2 \le cn\tau^{1-\delta}.$$

PROOF. Apply (2.2) with $a=b=2(1-\delta)^{-1}$, we obtain

$$\begin{aligned}E\left[\sum_{i=1}^{n}(u(t-Y_i)-t)\right]^2 &= \sum_{i=1}^{n}E(u(t-Y_i)-t)^2 \\ &\quad + 2\sum_{i<j}\sum E(u(t-Y_i)-t)(u(t-Y_j)-t) \\ &\le cn\tau + c\tau^{1-\delta}\sum_{i<j}\sum e^{-\theta(j-i)\delta} \\ &\le cn\tau^{1-\delta}.\end{aligned}$$

Let γ be a small positive number to be specified later. Define blocks H_i and I_i of consecutive integers inductively as follows: H_i consists of $[i^\gamma]$ and I_i consists of $[i^\gamma]$ consecutive integers, respectively. No gaps are allowed between the blocks. The order is $H_1, I_1, H_2, I_2, \ldots$. Let

$$A_{i,t} = \sum_{j\in H_i}[u(t-Y_j)-t], \qquad B_{i,t} = \sum_{j\in I_i}[u(t-Y_j)-t].$$

Let $M=M_n$ be the index of the block H_i or I_i containing n and let h_i be the smallest member of H_i.

LEMMA 2.6. *Let* $\eta>0$, $1<r<1+\eta$ *and* $\gamma<\eta(2-n)^{-1}$. *Then for any fixed* $t\in[n^{-r}, 1]$

$$n^{-1/2}\sum_{i=h_M}^{h_{M+1}-1}|u(t-Y_i)-t|[t(1-t)]^{-1/2} = o(n^\eta) \quad a.s. \quad as \quad n\to\infty.$$

PROOF. Since $M^\gamma\le cn^{\gamma/(\gamma+1)}$ and $t\ge n^{-r}$, we have

$$\begin{aligned}&n^{-1/2}\sum_{i=h_M}^{h_{M+1}-1}|u(t-Y_i)-t|[t(1-t)]^{-1/2} \\ &\qquad \le cM^\gamma n^{-1/2}n^{r/2} = o(n^\eta) \quad \text{as} \quad n\to\infty.\end{aligned}$$

Denote by $\mathscr{F}_{i,t}$ the σ-field generated by $A_{1,t},\ldots, A_{i,t}$.

LEMMA 2.7. *Let* $1<a<2$. *Assume* X_t *is strong mixing and* (1.4) *holds.*

Then

$$\| E(A_{i,t} | \mathscr{F}_{i-1,t}) \|_{2a} \le c \| A_{i,t} \|_4 \exp\left(-\theta i^{\gamma}\left(\frac{1}{2a} - \frac{1}{4} \right)\right).$$

PROOF. By (2.2)

$$\begin{aligned} E[E(A_{i,t} | \mathscr{F}_{i-1,t})^{2a}] &= E[A_{i,t}(E(A_{i,t}|\mathscr{F}_{i-1,t}))^{2a-1}] \\ &\le c \| A_{i,t} \|_4 [E | E(A_{i,t} | \mathscr{F}_{i-1,t}) |^{2a}]^{(2a-1)/2a} \\ &\quad \cdot \exp\left(-\theta i^{\gamma}\left(\frac{1}{2a} - \frac{1}{4} \right)\right). \end{aligned} \tag{2.7}$$

The lemma is then obtained by dividing both sides of (2.7) by $E[|E(A_{i,t}/\mathscr{F}_{i-1,t})|^{2a}]^{(2a-1)/2a}$.

LEMMA 2.8. *Let $1<a<2$ be chosen so that $x^a \ge 0$ for all $x \in (-\infty, \infty)$ and X_t be strong mixing satisfying* (1.4). *Then*

$$\| E([A_{i,t} - E(A_{i,t}/\mathscr{F}_{i-1,t})]^2 / \mathscr{F}_{i-1,t}) - EA_{i,t}^2 \|_a$$
$$\le c(EA_{i,t}^4)^{1/2} \exp\left(-\theta i^{\gamma}\left(\frac{1}{a} - \frac{1}{2} \right)\right).$$

PROOF. As in the proof of Lemma 2.7, we have

$$\begin{aligned} \| E(A_{i,t}^2 / \mathscr{F}_{i-1,t}) - EA_{i,t}^2 \|_a &= \| E((A_{i,t}^2 - EA_{i,t}^2) / \mathscr{F}_{i-1,t}) \|_a \\ &\le c(EA_{i,t}^4)^{1/2} \exp\left(-\theta i^{\gamma}\left(\frac{1}{a} - \frac{1}{2} \right)\right). \end{aligned} \tag{2.8}$$

By (2.8) and Lemma 2.7

$$\begin{aligned} &\| E([A_{i,t} - E(A_{i,t}/\mathscr{F}_{i-1,t})]^2 / \mathscr{F}_{i-1,t}) - EA_{i,t}^2 \|_a \\ &\qquad \le c(EA_{i,t}^4)^{1/2} \exp\left(-\theta i^{\gamma}\left(\frac{1}{a} - \frac{1}{2} \right)\right). \end{aligned}$$

LEMMA 2.9. *Let*

$$U_{k,t} = \begin{cases} \sum_{i \le k} [A_{i,t} - E(A_{i,t}/\mathscr{F}_{i-1,t})] & \text{for} \quad k \le M, \\ U_{M,t} & \text{for} \quad k > M, \end{cases}$$

$$s_{k,t}^2 = \begin{cases} \sum_{i \le k} E([A_{i,t} - E(A_{i,t}/\mathscr{F}_{i-1,t})]^2 / \mathscr{F}_{i-1,t}) & \text{for} \quad k \le M, \\ s_{M,t}^2 & \text{for} \quad k > M. \end{cases}$$

Let $\lambda>0$ *with* $\lambda M^{\gamma}\leq 1/2$. *Define*

$$T_{k,t} = \exp\left(\lambda U_{k,t} - \frac{1}{2}\lambda^2(1+\lambda M^{\gamma})s_{k,t}^2\right), \quad k \geq 1 .$$

Then the sequence $\{T_{k,t}, \mathscr{F}_{k,t}\}_{k=1}^{\infty}$ *is a non-negative supermartingale satisfying* $P\left[\sup_{k\geq 0} T_{k,t}>\beta\right]\leq 1/\beta$ *for each* $\beta>0$.

PROOF. It is easy to check that $\{U_{k,t}: k\geq 1\}$ is a martingale. Observe that $EU_{1,t}=0$ and $U_{k,t}-U_{k-1,t}\leq 2M^{\gamma}$ for each k. Lemma 2.9 follows from Stout ((1974), p. 299).

LEMMA 2.10. *Assume* X_t *is strong mixing satisfying* (1.4). *Then the conclusion of Lemma* 2.3 *continues to hold.*

PROOF. It is sufficient to show that for some $r>1$

$$\sup_{n^{-r}\leq t\leq 1/2} h_n(t) = o(n^{\eta}) \quad \text{a.s.} \quad \text{as} \quad n\to\infty . \tag{2.9}$$

Let $\mathscr{L}_{i,t}$ be the σ-field generated by $B_{1,t},\ldots, B_{i,t}$. Clearly

$$\begin{aligned}\left|\sum_{i=1}^{n}(u(t-Y_i)-t)\right| \leq & \sum_{i=h_M}^{h_{M+1}-1}|(u(t-Y_i)-t)| \\ & + \sum_{i=1}^{M}|E(A_{i,t}/\mathscr{F}_{i-1,t})| + \sum_{i=1}^{M}|E(B_{i,t}/\mathscr{L}_{i-1,t})| \\ & + \left|\sum_{i=1}^{M}[A_{i,t} - E(A_{i,t}/\mathscr{F}_{i-1,t})]\right| \\ & + \left|\sum_{i=1}^{M}[B_{i,t} - E(B_{i,t}/\mathscr{L}_{i-1,t})]\right| .\end{aligned} \tag{2.10}$$

Note that (2.9) is implied by

$$\sup_{n^{-r}\leq t\leq 1/2} n^{-1/2}\left|\sum_{i=1}^{n}(u(t-Y_i)-t)\right|[t(1-t)]^{-1/2} = o(n^{\eta}) \quad \text{a.s.} \tag{2.11}$$

Set $jn^{-r}=t_j$, and drop the subscripts in A_{i,t_j}, $\mathscr{F}_{i-1,t_j}$, B_{i,t_j}, $\mathscr{L}_{i-1,t_j}$.

By Lemmas 2.4 and 2.7

$$\begin{aligned} & P\left[\max_{1\leq j\leq [n^{r-1}]+1}\sum_{i=1}^{M}|E(A/\mathscr{F})|n^{-1/2}t_j^{-1/2} > cn^{\eta}\right] \\ & \leq c\sum_{j=1}^{[n^{r-1}]+1}\left\{\sum_{i=1}^{M}\frac{(EA^4)^{1/4}\exp\left(-\theta i^{\gamma}\left(\frac{1}{2a}-\frac{1}{4}\right)\right)}{n^{1/2+\eta}\,t_j^{1/2}}\right\}^{2a}\end{aligned} \tag{2.12}$$

$$\le c \sum_{j=1}^{[n^{r-1}]+1} \left\{ \sum_{i=1}^{M} \frac{[(i^{2\gamma} t_j^2 + i^\gamma t_j)\, t_j^{-\delta}]^{1/4} \exp\left(- \theta i^\gamma \left(\frac{1}{2a} - \frac{1}{4} \right) \right)}{n^{1/2+\eta}\, t_j^{1/2}} \right\}^{2a}$$

$$\le cn^{-1-\varepsilon} \quad \text{for some} \quad \varepsilon > 0.$$

By Borel-Cantelli Lemma

$$\max_{1\le j\le [n^{r-1}]+1} \sum_{i=1}^{M} |E(A/\mathscr{F})| n^{-1/2}\, t_j^{-1/2} = o(n^\eta) \quad \text{a.s.} \tag{2.13}$$

as $n \to \infty$. Similarly

$$\max_{1\le j\le [n^{r-1}]+1} \sum_{i=1}^{M} |E(B/\mathscr{L})| n^{-1/2}\, t_j^{-1/2} = o(n^\eta) \quad \text{a.s.} \tag{2.14}$$

Next

$$\begin{aligned} (2.15)\qquad & P\left[\max_{1\le j\le [n^{r-1}]+1} \left| \sum_{i=1}^{M} [A - E(A/\mathscr{F})] \right| n^{-1/2}\, t_j^{-1/2} > cn^\eta \right] \\ & \le \sum_{j=1}^{[n^{r-1}]+1} P\left[\sum_{i=1}^{M} [A - E(A/\mathscr{F})] n^{-1/2}\, t_j^{-1/2} > cn^\eta \right] \\ & \quad + \sum_{j=1}^{[n^{r-1}]+1} P\left[\sum_{i=1}^{M} [\,-A + E(A/\mathscr{F})] n^{-1/2}\, t_j^{-1/2} > cn^\eta \right]. \end{aligned}$$

The first term on the right hand side of (2.15) is bounded by

$$\sum_{j=1}^{[n^{r-1}]+1} P\left[\sup_{k\ge 0} U_{k,t_j} > \lambda K \right], \tag{2.16}$$

where $\lambda = 1/cn^{(\eta+1)/2}\, t_j^{1/2}$, $K = c^2 n^{1+3\eta/2}\, t_j$. Choose $r < 1 + \eta/2$ and γ small enough so that $\gamma(\gamma+1)^{-1} < \eta/4$. Then $\lambda M^\gamma \le 1$. Pick δ small enough so that $r\delta < 3\eta/2$. By Lemmas 2.5, 2.8 and 2.9, we obtain that (2.16) equals

$$\begin{aligned} (2.17)\qquad & \sum_{j=1}^{[n^{r-1}]+1} P\left[\sup_{k\ge 0} T_{k,t_j} > \exp(\lambda^2 K - \tfrac{1}{2} \lambda^2 (1 + \lambda M^\gamma)\, s_{M,t_j}^2 \right] \\ & \le \sum_{j=1}^{[n^{r-1}]+1} \left\{ P\left[\sup_{k\ge 0} T_{k,t_j} > \exp(2^{-1}\lambda^2 K) \right] + P[s_{M,t_j}^2 > 2^{-1} K] \right\} \\ & \le cn^{r-1} \exp(-2^{-1} n^{\eta/2}) \\ & \quad + \sum_{j=1}^{[n^{r-1}]+1} \left[\frac{\sum_{i=1}^{M} \| E([A - E(A/\mathscr{F})]^2/\mathscr{F}) - EA^2 \|_a}{cnt_j(n^{3\eta/2} - t_j^{-\delta})} \right]^a \end{aligned}$$

$$\le cn^{r-1}\exp(-2^{-1}n^{\eta/2})$$
$$+\sum_{j=1}^{[n^{r-1}]+1}\left[\frac{\sum_{i=1}^{M}[(i^{2\gamma}t_j^2+i^{\gamma}t_j)\,t_j^{-\delta}]^{1/2}\exp\left(-\theta^{\gamma}\,i\left(\frac{1}{2}-\frac{1}{2}\right)\right)}{cn^{1+3\eta/2}\,t_j}\right]^a$$
$$\le cn^{r-1}\exp\left(-\frac{1}{2}n^{\eta/2}\right)+cn^{-1-\varepsilon}\quad\text{for some}\quad\varepsilon>0,$$

by choosing δ small, then r close to 1 and a close to 2. A bound for the second term on the right hand side of (2.15) can be found similarly. By Borel-Cantelli Lemma, (2.15), (2.16) and (2.17)

$$\text{(2.18)}\qquad \max_{1\le j\le[n^{r-1}]+1}\left|\sum_{i=1}^{M}[A-E(A/\mathscr{F})]\right| n^{-1/2}\,t_j^{-1/2}=o(n^{\eta})\quad\text{a.s.}$$

Similarly

$$\text{(2.19)}\qquad \max_{1\le j\le[n^{r-1}]+1}\left|\sum_{i=1}^{M}[B-E(B/\mathscr{L})]\right| n^{-1/2}\,t_j^{-1/2}=o(n^{\eta})\quad\text{a.s.}$$

From (2.10), it is easily seen with Lemma 2.6 and (2.3), (2.13), (2.14), (2.18) and (2.19) that (2.11) holds for $n^{-r}\le t\le n^{-1}$.

The proof that (2.11) holds for $n^{-1}\le t\le 1/2$ is similar and hence is omitted.

Remark 2.2. It is not hard to verify that the conclusion of Lemma 2.10 remains valid if (1.3) is weakened to $\alpha(n)=O(e^{-\theta(n)\log n})$ where $\theta(n)\uparrow\infty$ arbitrarily slowly.

LEMMA 2.11. *Let* $0\le\eta$. *If* X_t *is* Φ*-mixing satisfying* (1.3) *or strong mixing satisfying* (1.4), *then*

$$H_n(a_n)-H(a_n)=o(n^{-1+2^{-1}\mu+\eta})\quad a.s.\quad as\quad n\to\infty,$$

where a_n *is defined by* $H(a_n)=1-n^{-1+\mu}$.

PROOF. By Lemma 2.2 in the Φ-mixing case and martingale approximation in the strong mixing case, Lemma 2.11 follows.

LEMMA 2.12. (i) *If* X_t *is* Φ*-mixing satisfying* $\sum_{n=1}^{\infty}\Phi^{1/2}(n)<\infty$, *then*

$$\sup_{0<u<1,\,t:\,|t-u|<n^{-1/2}}\{|F_n(t)-F_n(u)-t+u|\}$$
$$=o(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4})\quad a.s.$$

(ii) *If X_t is strong mixing satisfying* (1.4), *then*

$$\sup_{0<u<1,t:\,|t-u|<n^{-1/2}}\sup \{|F_n(t) - F_n(u) - t + u|\}$$
$$= o(n^{-3/4} \log n(\log\log n)^{1/4}) \quad a.s.$$

Lemma 2.12 can be obtained by using the same line of arguments as in Theorems 1, 2, 3 and 4 of Babu and Singh (1978).

3. Proof of the theorems

LEMMA 3.1. *Let X_t be Φ-mixing satisfying* (1.3) *or strong mixing satisfying* (1.4). *If* (1.2) *holds, then*

$$n^{1/2}R_n = o(n^{-\eta}) \quad a.s. \quad as \quad n \to \infty \quad for\ some \quad \eta > 0\ .$$

Lemma 3.1 can easily be obtained by a careful analysis of the proof of Theorem 4.1 of Sen and Ghosh (1973). Using Lemmas 2.11 and 2.12 in place of their (4.14) and (4.26) and Lemmas 2.3 and 2.10, R_n can be shown to be of order $n^{-1/2-\eta}$ a.s. for some $\eta>0$.

Sen and Ghosh (1973) have obtained a somewhat weaker version of Lemma 3.1.

PROOF OF THEOREM 1.1. Using (1.2), by a simple computation we have $E|B(X_1)|^2<\infty$. In the Φ-mixing case, Theorem 1.5 of Ibragimov (1962) gives $L\left(n^{-1/2}\left[\sum_{i=1}^{\infty} B(X_i)\Big/\sigma\right]\right)\to N(0,1)$. Since $n^{1/2}R_n=o(n^{-\eta})$ a.s. by Lemma 3.1, Theorem 1.1 follows.

In the strong mixing case, (1.2) gives $E|B(X_1)|^{2+\delta}<\infty$ for some $\delta<\mu$. The theorem then follows from Theorem 1.7 of Ibragimov (1962) and Lemma 3.1.

LEMMA 3.2. *Assume the conditions of Theorem* 1.1 *hold. Let*

$$S(t) = \sum_{k\le t} \frac{B[X_k]}{\sigma}, \quad t \ge 0\ .$$

Then without changing the distribution of $S(t)$, $t>0$, we can define the process $S(t)$ on a richer probability space together with standard Brownian motion $\{W(t)\colon t\ge 0\}$ such that $S(t)-W(t)\le t^{1/2-\lambda}$ a.s. as $t\to\infty$ for some $\lambda\ge 0$.

PROOF. In the Φ-mixing case, since $|EB(X_1)|^{2+\delta}<\infty$ for some $\delta>0$, the conclusion of the lemma is a consequence of Theorem 4.1 of Philipp and Stout (1975).

In the strong mixing case, we will show that the lemma is a consequence of Theorem 7.1 of Philipp and Stout (1975). Note that $E|B(X_1)|^{2+\delta}$ for $\delta<\mu$. By (2.2) with $a=b=2/(1-\delta)$, we have

$$\begin{aligned} E\left[\sum_{k=1}^{n}\frac{B(X_k)}{\sigma}\right]^2 &= \sigma^{-2}[nE(B(X_1))^2 \\ &\quad + 2\sum_{k=1}^{n-1}(n-k)E(B(X_1)B(X_{k+1}))] \\ &= n - 2n\sigma^{-2}\sum_{k=n}^{\infty}E(B(X_1)B(X_{k+1})) \\ &\quad - 2\sigma^{-2}\sum_{k=1}^{n-1}kE(B(X_1)B(X_{k+1})) \\ &\leq n + o(\eta^{1-\delta/30}) \quad \text{as} \quad n\to\infty . \end{aligned}$$

Thus (7.1.7) of Theorem 7.1 of Philipp and Stout (1975) is satisfied. Other conditions of their Theorem 7.1 can easily be verified.

PROOF OF THEOREM 1.2. Theorem 1.2 is a direct consequence of the law of the iterated logarithm for Brownian motion, Lemmas 3.1 and 3.2.

4. Applications

4.1 *Estimation of the center of symmetry*

Here assume that the distribution $F(x)$ of X_t satisfies $F(x)=F_\theta(x)=F_0(x-\theta)$ where $-\infty<\theta<\infty$ and F_0 is symmetric about 0. Define $T_n(b)$ as in (1.1) but the X_i being replaced by X_i-b, $1\leq i\leq n$, where $-\infty<b<\infty$. Define

$$\hat{\theta}_{n1} = \sup\{b\colon T_n(b) > \mu_0\}\ , \qquad \hat{\theta}_{n2} = \inf\{b\colon T_n(b) > \mu_0\}\ ,$$

$$\hat{\theta}_n = (\hat{\theta}_{n1} + \hat{\theta}_{n2})/2\ .$$

Here $\mu_0=(1/2)\int_0^1 J(u)du=(1/2n)\sum_{i=1}^{n} J_n(i/(n+1))$ if $J_n(i/(n+1))=EJ(U_{ni})$, $1\leq i\leq n$, where the U_{ni}'s are as defined in Section 1. Consider $\hat{\theta}_n$ as an estimator of θ. Using Theorem 1.1 and following an argument similar to Theorems 1 and 5 of Hodges and Lehmann (1963), one obtains:

THEOREM 4.1. *Let $\Phi(x)=(1/2\pi)\int_{-\infty}^{x} e^{-t^2/2}\,dt$, J^* be as defined in Section 1 and $B(F_0)=\int_{-\infty}^{\infty}[dJ^*(F_0(x)/dx)]f_0(x)dx$ where $f_0=F_0'$. Assume $J(u)$ is a non-decreasing score function and $(d/dx)\,J(F_0(x))$ remains bounded as $x\to\infty$ for $-\infty<x<\infty$. Then under the assumptions of Theorem 1.1 $P[n^{1/2}(\hat{\theta}_n-\theta)\cdot B(F_0)/\sigma\leq x]\to\Phi(x)$ for every $-\infty<x<\infty$.*

As an example, let $X(t)$ be a first order autoregressive time series model given by $X(t)-\theta=a(X(t-1)-\theta)+e(t)$, where $|a|<1$. The density $f(x)$ of $e(t)$ is assumed to be symmetric about the origin and has a heavy tailed non-Gaussian distribution e.g., $(1-\gamma)N(0,1)+\gamma N(0,\sigma^2)$ where $\gamma>0$ and $\sigma^2>1$. This model may fit a time series with outliers. For more information and discussion of this model, see Denby and Martin (1979). Consider the problem of obtaining robust estimates for the center of symmetry θ.

Assume that $\int|x|^\delta f(x)dx<\infty$ for some $\delta>0$ and $\int|f(x)-f(x-\lambda)|dx = O(|\lambda|^\beta)$ as $\lambda\to 0$, for some $\beta>0$. Then $X(t)$ is absolutely regular and hence strong mixing. The mixing rate also satisfies (1.4). See Pham and Tran ((1985), p. 301). The sample means $\bar{X}_n$ may not be a robust estimator of θ when $f(x)$ is heavy tailed. An alternative estimator is $\hat{\theta}_n$ defined above.

4.2 *Rank tests for serial dependence*

Assume the distribution of X_t is symmetric about zero. Let $Z_t=X_tX_{t+k}$ where k is a positive integer. Then a simple way of defining positive (negative) serial dependence at lag k consists in saying that the median of Z_t is positive (negative).

Let $X_1,\ldots, X_n$ be n consecutive observations of X_t. Consider the problem of testing the null hypothesis H_0 that $X_1,\ldots, X_n$ are mutually independent against the alternatives that these variables are positively or negatively serially dependent. Under H_0, the median of Z_t is zero. Dufour (1982) proposed to test H_0 versus serial dependence by applying rank tests for symmetry about zero, applied to the variables Z_t, $t=1,2,\ldots, n-k$.

Note that if X_t is Φ-mixing satisfying (1.3) or strong mixing satisfying (1.4), then Z_t is also Φ-mixing or strong mixing satisfying the same mixing conditions. Thus the statistic considered in (1.1) can be used to test for independence versus serial dependence.

Rank tests are especially useful when there is evidence of non-normality of distributions. Potential applications are found in the studies of stock prices or exchange rates. See for example, Mandelbrot (1967). In terms of power, rank tests compare favorably with well-known alternative tests under a wide range of circumstances. Some discussion on rank tests for serial dependence can be found in Dufour (1982).

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