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MULTIPLICITY DISTRIBUTIONS IN A TWO-COMPONENT BRANCHING PROCESS

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Abstract. Probability (multiplicity) distributions and those densities (KNO scaling functions) are investigated in a two-component (charged and neutral) branching process. It is shown that the two-component KNO scaling functions depend effectively on one variable in two typical cases. A formula for multiplicity correlation between two components (charged and neutral particles) is formulated. It is applied to the analysis of experimental data.

Key words and phrases: Probability distribution, probability density, branching equation, Fokker-Planck equation.

1. Introduction

In hadron-hadron (h-h) collisions with energy more than several ten GeV ($=10^9$ eV) in the laboratory system, many particles are frequently produced through the strong interactions between the colliding particles. See Fig. 1. A hadron means a strongly interacting particle such as proton, neutron, pion and so on. Experiments on high energy h-h collisions are done mainly by big accelerators, for example, the proton synclotrons at Serpukhov (USSR) and

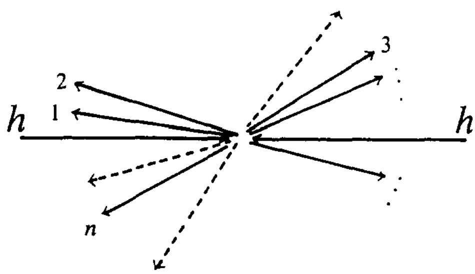

Fig. 1. Illustration of a multiple particle production process in h-h collisions. Charged particles are expressed by solid lines, and neutral particles are by dashed lines.

Fermilab (USA), and ISR and the $\bar{p}p$ collider at CERN (Switzerland). The production process is represented symbolically in the following way,

$$h + h \to n \text{ charged particles} + X \ ,$$

where h is a hadron and X represents missing neutral particles. In many experiments, as the detector sensitive only to charged particles are used, neutral particles are not observed. The number n of charged particles is counted event by event and is sometimes called the charged multiplicity of an event.

A mean number $\langle n \rangle$ of charged particles produced per event is a function of the Lorentz invariant energy squared s of colliding particles. It is defined as $s=(p_1+p_2)^2$, where p_i $(i=1,2)$ is the four momentum of the i-th incident hadron. The observed mean charged multiplicity $\langle n \rangle$ increases with s: roughly speaking $\langle n \rangle \propto s^{1/4}$ (Alner *et al.* (1986)). Number distributions of secondary particles are considered to reflect their underlying production mechanisms.

It is derived theoretically (Koba *et al.* (1972)) that the number distribution $P(n)$ of (charged) secondary particles in high energy h-h collisions satisfies the scaling law,

$$\langle n \rangle P(n) \to \psi(z) \ , \tag{1.1}$$

in the limit of n, $\langle n \rangle \to \infty$ with $z=n/\langle n \rangle$ finite. This is called the KNO scaling. From equation (1.1), the k-th moment of multiplicity is expressed as

$$C_k = \sum_{n=0}^{\infty} n^k P(n)/\langle n \rangle^k \to \int_0^{\infty} z^k \psi(z) dz \ . \tag{1.2}$$

For example, let the probability $P(n)$ be the Poisson distribution, the scaling function and the k-th moment become

$$\psi(z) = \delta(z-1) \tag{1.3a}$$

and

$$C_k = 1 \quad (k = 2, 3, \ldots) \ , \tag{1.3b}$$

in the same limit.

From the analyses of experimental data, the KNO scaling seems to work from the Serpukhov energy region ($\sqrt{s}=11.5$ GeV) (Slattery (1972)) up to the ISR energy region ($\sqrt{s}=60$ GeV) (Breakstone *et al.* (1984)). The distributions $\langle n \rangle P(n)$ plotted with the variable $z=n/\langle n \rangle$ are almost same in shape and $C_k \cong$ constant (>1) $(k=2\sim5)$ are observed in those energy regions.

Experiments at the $\bar{p}p$ collider show that C_k observed at $\sqrt{s}=546$ GeV and 900 GeV (Alpgard *et al.* (1983), Alner *et al.* (1984, 1986)) become larger than those in the ISR energy region, namely, the KNO scaling breaks. However, observed multiplicity distributions are well described by one type of function (negative binomial distribution) (Alner *et al.* (1985)).

It is pointed out (Ellis (1984), Durand and Ellis (1984), Biyajima and Suzuki (1985), Biyajima *et al.* (1987)) that some distribution functions $P(n)$ derived in a single component birth process does not have the scaling form, but $\psi(z, \langle n \rangle)$. This contrasts with equation (1.1). There is a possibility that the observed violation of the KNO scaling is explained by this effect.

In this paper, we mainly consider multiplicity distributions in a two-component (charged and neutral) branching process. In Section 2, a two-component branching equation is rewritten into a differential equation for a generating function, and is solved under a general boundary condition. Multiplicity distributions corresponding to two typical initial conditions are obtained from generating functions in Section 3. A correlation between two-component (charged and neutral) multiplicities is considered in Section 4. The KNO scaling functions for each component multiplicity distribution and those for the two-component distribution are derived in Section 5 where the Fokker-Planck equation corresponding to the two-component branching process is obtained and its solution is investigated. Section 6 is devoted to the analysis of experimental data. Summary and discussions are given in the final section.

2. A two-component branching process

A system composed of two species of particles is considered. One has charge and the other is neutral. Two species will be identified as charged hadron (mainly pion) and neutral hadron (mainly pion) in a hadronic level, or quark and gluon in a sub-hadronic level. In either case, it is assumed that particles with charge are produced in pairs but not the neutral ones. In the following, we assign charged and neutral particle for two species without reference to the hadronic or sub-hadronic level.

A branching process of charged particle (/quark q) and neutral particle (/gluon g) is considered (Giovannini (1979), Anselmino *et al.* (1981), Biyajima and Suzuki (1984), Durand and Sarcevic (1986)). Let $P(n, m; t)$ be a number distribution of n charged particles and m neutral particles at t, where t is a parameter which describes an evolution of the q-g system. (The maximum value of t is determined by the s dependence of the observed mean charged multiplicities.) In an interval $(t, t+dt)$, three types of interactions take place: (i) $q \to q+g$ with a probability $\lambda_0 dt$, (ii) $g \to q+q$ with a probability $\lambda_1 dt$, and (iii) $g \to g+g$ with a probability $\lambda_2 dt$. The elements of branching are illustrated in Fig. 2. It is assumed that λ_0, λ_1 and λ_2 are constants.

Then the distribution $P(n,\ m;\ t)$ satisfies the following difference-differential equation (Giovannini (1979)),

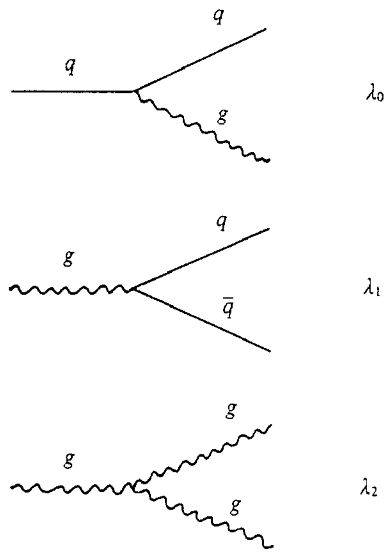

Fig. 2. Three elements of branching. The solid line and wavy line denote charged particle (/quark) and neutral particle (/gluon), respectively.

$$\begin{aligned}(2.1)\qquad \frac{\partial}{\partial t} P(n,m;t) &= n\lambda_0 P(n,m-1;t) - n\lambda_0 P(n,m;t) \\ &+ (m+1)\lambda_1 P(n-2,m+1;t) - m\lambda_1 P(n,m;t) \\ &+ (m-1)\lambda_2 P(n,m-1;t) - m\lambda_2 P(n,m;t)\ .\end{aligned}$$

In order to solve equation (2.1), we define the generating function:

$$(2.2)\qquad \Pi(x,y;t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P(n,m;t)x^n y^m\ .$$

The two-component multiplicity distribution, that of charged particles and that of neutral ones are derived from equation (2.2), respectively, as

$$(2.3a)\qquad P(n,m;t) = \frac{1}{n!m!}\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}\Pi(x,y;t)|_{x=y=0}\ ,$$

$$(2.3b)\qquad P_q(n;t) \equiv \sum_{m=0}^{\infty} P(n,m;t) = \frac{1}{n!}\frac{\partial^n}{\partial x^n}\Pi(x,y=1)|_{x=0}\ ,$$

$$(2.3c)\qquad P_g(m;t) = \frac{1}{m!}\frac{\partial^m}{\partial y^m}\Pi(x=1,y;t)|_{y=0}\ .$$

We can also derive the j-th cumulant of charged multiplicity ($j=1,2,\ldots$) and that of neutral one from equation (2.2):

$$\text{(2.4a)}\quad \langle n_q(n_q-1)\cdots(n_q-j+1)\rangle_T \equiv \sum_{n=0}^{\infty} n(n-1)\cdots(n-j+1)\,P_q(n;t) = \frac{\partial^j}{\partial x^j}\,\Pi(x,y=1)|_{x=1}\ ,$$

$$\text{(2.4b)}\quad \langle n_g(n_g-1)\cdots(n_g-j+1)\rangle = \frac{\partial^j}{\partial y^j}\,\Pi(x=1,y;t)|_{y=1}\ .$$

By the use of equation (2.2), equation (2.1) is rewritten into the following differential equation

$$\text{(2.5a)}\quad \frac{\partial}{\partial t}\,\Pi(x,y;t) = f_1(x,y)\,\frac{\partial}{\partial x}\,\Pi(x,y;t) + f_2(x,y)\,\frac{\partial}{\partial y}\,\Pi(x,y;t)\ ,$$

$$\text{(2.5b)}\quad \begin{aligned} f_1(x,y) &= \lambda_0 x(y-1)\ , \\ f_2(x,y) &= \lambda_2 y^2 - (\lambda_1+\lambda_2)y + \lambda_1 x^2\ . \end{aligned}$$

An initial condition for equation (2.1) and a boundary condition for equation (2.5) are generally written as

$$\text{(2.6a)}\quad P(n,m;t=0) = f(n,m)\ ,$$

$$\text{(2.6b)}\quad \Pi(x,y;t=0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m)x^n y^m = F(x,y)\ .$$

In general, equation (2.5) cannot be solved analytically. Under the condition $\lambda_2=2\lambda_0$ (Giovannini (1979)), we can obtain a solution. In the quantum chromodynamics (QCD), the relation $\lambda_2=2\lambda_0$ holds in the limit of $N_c\to\infty$, where N_c is the number of color states.

The auxiliary equations for equation (2.5) are given as

$$\text{(2.7a)}\quad \frac{dx}{dt} = -f_1(x,y)\ ,$$

$$\text{(2.7b)}\quad \frac{dy}{dt} = -f_2(x,y)\ .$$

Changing the variables x and y with the new ones, $u=x^2$ and $w=y/x^2$, we get

$$\text{(2.8a)}\quad \frac{du}{dt} = -\,2\lambda_0 u(uw-1)\ ,$$

(2.8b) $$\frac{dw}{dt} = -(\lambda_2 - 2\lambda_0)uw^2 + (\lambda_1 + \lambda_2 - 2\lambda_0)w - \lambda_1 .$$

By making use of $\lambda_0 = \lambda_2/2$, we obtain

(2.9a) $$w - 1 = A\exp[\lambda_1 t] ,$$

(2.9b) $$u = 1\Big/\left(\frac{\lambda_2}{\lambda_1 + \lambda_2} A\exp[\lambda_1 t] + 1 + B\exp[-\lambda_2 t]\right) ,$$

where A and B are constants. The solution of equation (2.5), which satisfies the boundary condition (2.6b), is given as follows,

(2.10a) $$\Pi(x, y; t) = F(g_1(x, y; t), g_2(x, y; t)) ,$$

(2.10b) $$g_1(x, y; t) = x[1 - \alpha_1(x^2 - 1) - \alpha_2(y - 1)]^{-1/2} ,$$
$$g_2(x, y; t) = \frac{1 + \beta_1(x^2 - 1) + \beta_2(y - 1)}{1 - \alpha_1(x^2 - 1) - \alpha_2(y - 1)} ,$$

where

(2.11) $$\alpha_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}(e^{\lambda_2 t} - 1), \quad \alpha_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}(e^{\lambda_2 t} - e^{-\lambda_1 t}) ,$$
$$\beta_1 = 1 - e^{-\lambda_1 t}, \quad \text{and} \quad \beta_2 = e^{-\lambda_1 t}$$

Note that $g_1(x, y; t=0) = x$ and $g_2(x, y; t=0) = y$. The generating function for the two-component probability distribution is obtained under the condition $\lambda_0 = \lambda_2/2$.

3. Typical solutions

In this section, we consider solutions for the following initial conditions:

(3.1a) $$f_a(n, m) = \delta_{n,n_0}\delta_{m,m_0} ,$$

(3.1b) $$f_b(n, m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \delta_{n,2i} \frac{\lambda^i}{i!} \exp(-\lambda)\delta_{m,j} \frac{\mu^j}{j!} \exp(-\mu) .$$

The initial condition (2.6a) is in general written by the sum of the function $f_a(n, m)$ with adequate weights. Then the solution (2.10a) can be derived from the solution corresponding to $f_a(n, m)$. The function $f_b(n, m)$ is connected to the boundary condition of the Fokker-Planck equation derived from equation (2.1).

(a) $f_a(n,m)=\delta_{n,n_0}\,\delta_{m,m_0}$: From equations (2.6) and (2.10), the boundary condition and the generating function are derived, respectively, as

$$F_a(x,y) = x^{n_0}y^{m_0}\ , \tag{3.2a}$$

$$\Pi_a(x,y;t) = x^{n_0}[1-\alpha_1(x^2-1)-\alpha_2(y-1)]^{-(\lambda+m_0)} \cdot [1+\beta_1(x^2-1)+\beta_2(y-1)]^{m_0}\ , \tag{3.2b}$$

where $\lambda=n_0/2$. For simplicity, we define the function as

$$G_a(u,v;t) = \Pi_a(\sqrt{u},v;t)/x^{n_0}\ , \tag{3.3}$$

where $u=x^2$ and $v=y$. $G_a(u,v;t)$ is also interpreted as the generating function, because $G_a(1,1;t)=1$. Hereafter, subscript a is abbreviated.

The multiplicity distribution of charged particles and that of neutral ones are expressed by $G(u,v;t)$, respectively, as

$$\begin{aligned} &P_q(n_0+2n;t) = \frac{1}{n!}\frac{\partial^n}{\partial u^n}G(u,1;t)|_{u=0} = p_r(n;t) \quad (n=0,\ 1,\ldots)\ , \\ &P_q(n)=0 \quad \text{for} \quad n<n_0\ , \end{aligned} \tag{3.4a}$$

and

$$P_g(m;t) = \frac{1}{m!}\frac{\partial^m}{\partial v^m}G(1,v;t)|_{v=0} \quad (m=0,1,\ldots)\ . \tag{3.4b}$$

In equation (3.4a), n denotes the number of produced charged-pairs within an interval $(0,t)$, and $p_r(n;t)$ represents the probability that n charged-pairs exist at t. Both distributions reduce to the function (Biyajima and Suzuki (1984))

$$P(n;t) = \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\Gamma(\lambda)}\left(\frac{1-\beta}{1-\alpha}\right)^{m_0} \cdot \frac{\alpha^n}{(1+\alpha)^{n+\lambda}}\,F\left(-m_0,-n;\lambda;\frac{\alpha+\beta}{\alpha(1-\beta)}\right)\ , \tag{3.5}$$

where $F(a,b;c;x)$ is the hypergeometric function. The charged distribution $p_r(n;t)$ is given by the substitution, $\alpha=\alpha_1$ and $\beta=\beta_1$, and $P_q(n;t)$ is by $\alpha=\alpha_2$ and $\beta=\beta_2$.

The j-th multiplicity moment $\langle n^j\rangle_T$ of charged particles is related to that $\langle n^j_{qq}\rangle$ of produced charged-pairs:

$$\langle n^j\rangle_T = \sum_{n=0}^{\infty}(n_0 + 2n)^j p_r(n;t) = \langle (n_0 + 2n_{qq})^j\rangle .$$

The j-th cumulant of charged-pairs and that of neutral particles are expressed, respectively, as

(3.6a) $\langle n_{qq}(n_{qq} - 1)\cdots(n_{qq} - j + 1)\rangle$

$$= \frac{\partial^j}{\partial u^j} G(u,1;t)|_{u=1} = \frac{\Gamma(j+\lambda)}{\Gamma(\lambda)} \alpha_1^j F\left(-m_0, -j; \lambda; \frac{\alpha_1+\beta_1}{\alpha_1}\right),$$

(3.6b) $\langle n_g(n_g - 1)\cdots(n_g - j + 1)\rangle$

$$= \frac{\partial^j}{\partial v^j} G(1,v;t)|_{v=1} = \frac{\Gamma(j+\lambda)}{\Gamma(\lambda)} \alpha_2^j F\left(-m_0, -j; \lambda; \frac{\alpha_2+\beta_2}{\alpha_2}\right).$$

From equation (3.6), we obtain the mean multiplicity of charged-pairs $\langle n_{qq}\rangle = \lambda\alpha_1 + m_0(\alpha_1+\beta_1)$ and that of neutral particles $\langle n_g\rangle = \lambda\alpha_2 + m_0(\alpha_2+\beta_2)$.

By the use of $G(u,v;t)$, the two-component probability is expressed as,

$$P(n_0 + 2n, m; t) = \frac{1}{n!m!}\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m} G(u,v;t)|_{u=v=0} = p_r(n,m;t),$$

where

(3.7)

$$p_r(n,m;t) = 0 \quad \text{for} \quad n + m < m_0,$$

$$p_r(n,m;t) = \frac{\Gamma(m_0+1)\Gamma(n+m+\lambda)}{n!m!\Gamma(m_0+\lambda)} \sum_{i=0}^{n}\sum_{j=0}^{m} \delta_{m_0,i+j} \binom{n}{i}\binom{m}{j} \alpha_1^{n-i}\beta_1^i$$

$$\cdot\, \alpha_2^{m-j}\beta_2^j/(1+\alpha_1+\alpha_2)^{\lambda+n+m} \quad \text{for} \quad m_0 \le n + m .$$

Equation (3.7) reduces to a distribution derived in Anselmino *et al.* (1981) with $(\lambda, m_0) = (1/2, 0)$ or $(0, 1)$.

(b) $f_b(n,m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{n,2i}(\lambda^i/i!)\exp(-\lambda)\ \delta_{m,j}(\mu^j/j!)\exp(-\mu)$: λ and μ in $f_b(n,m)$ denote the mean number of charged-pairs and that of neutral particles, respectively, at the initial stage of the evolution (at $t=0$). The boundary condition and the generating function become,

(3.8a) $$F_b(x,y) = \exp[\lambda(x^2 - 1)] \cdot \exp[\mu(y - 1)],$$

(3.8b) $$\Pi_b(x,y;t) = \exp\left[\lambda\left\{\frac{x^2}{1 - \alpha_1(x^2-1) - \alpha_2(y-1)} - 1\right\}\right]$$

$$\cdot \exp\left[\mu\left\{\frac{1-\beta_1(x^2-1)+\beta_2(y-1)}{1-\alpha_1(x^2-1)-\alpha_2(y-1)}-1\right\}\right].$$

The generating function for charged-pairs and neutral particles is defined as,

$$G_b(u,v;t)=\Pi_b(\sqrt{u},v;t)\ , \tag{3.9}$$

where $u=x^2$ and $v=y$. In the following, subscript b in equations (3.8) and (3.9) is abbreviated.

The multiplicity distribution of charged particles and that of neutral ones are expressed by $G(u,v;t)$:

$$P_q(2n;t)=\frac{1}{n!}\frac{\partial^n}{\partial u^n}G(u,1;t)|_{u=0}=p_r(n;t)\ , \tag{3.10}$$

$$P_g(m;t)=\frac{1}{m!}\frac{\partial^m}{\partial v^m}G(1,v;t)|_{v=0}\ . \tag{3.11}$$

The moment of charged particles is related to that of charged-pairs:

$$\langle n^j\rangle_T=\sum_{n=0}^{\infty}(2n)^j p_r(n;t)=2^j\langle n_{qq}^j\rangle\ .$$

The j-th cumulant of charged-pairs is expressed as,

$$\begin{aligned}\text{(3.12a)}\quad \langle n_{qq}(n_{qq}-1)\cdots(n_{qq}-j+1)\rangle &=\frac{\partial^j}{\partial u^j}G(u,1;t)|_{u=1}\\ &=\Gamma(j)\langle n_1\rangle\alpha_1^{j-1}\,L_{j-1}^{(1)}\left(-\frac{\langle n_1\rangle}{\alpha_1}\right),\end{aligned}$$

where $\langle n_1\rangle=\langle n_{qq}\rangle=\lambda(1+\alpha_1)+\mu(\alpha_1+\beta_1)$.

The j-th cumulant of neutral particles is given by,

$$\begin{aligned}\text{(3.12b)}\quad \langle n_g(n_g-1)\cdots(n_g-j+1)\rangle &=\frac{\partial^j}{\partial v^j}G(1,v;t)|_{v=1}\\ &=\Gamma(j)\langle n_2\rangle\alpha_2^{j-1}\,L_{j-1}^{(1)}\left(-\frac{\langle n_2\rangle}{\alpha_2}\right),\end{aligned}$$

where $\langle n_2\rangle=\langle n_g\rangle=\lambda\alpha_2+\mu(\alpha_2+\beta_2)$.

From equations (3.8b), (3.9), (3.10) and (3.11), $p_r(n;t)$ and $P_g(n;t)$ reduce to the equation (Biyajima and Suzuki (1984)),

$$
\begin{aligned}
& P(0;t) = \exp\left[-\frac{\langle n\rangle}{1+\alpha}\right], \\
(3.13)\quad & P(n;t) = \frac{\langle n\rangle}{n}\frac{\alpha^{n-1}}{(1+\alpha)^{n+1}} \\
& \cdot \exp\left[-\frac{\langle n\rangle}{1+\alpha}\right] L_{n-1}^{(1)}\left(-\frac{\langle n\rangle}{\alpha(1+\alpha)}\right) \quad (n = 1, 2,\ldots),
\end{aligned}
$$

where $\langle n\rangle=\langle n_{qq}\rangle$, $\alpha=\alpha_1$, and $\beta=\beta_1$ for $p_r(n;t)$, and $\langle n\rangle=\langle n_g\rangle$, $\alpha=\alpha_2$, and $\beta=\beta_2$ for $P_g(n;t)$.

The two-component probability is expressed by

$$
(3.14)\qquad P(2n,m;t) = \frac{1}{n!m!}\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m} G(u,v;t)|_{u=v=0} = p_r(n,m;t) .
$$

As is known (Biyajima and Suzuki (1985)), $f_b(n,m)$ is the sum of $f_a(n,m)$ with the Poisson weights. Therefore, the two-component probability defined by equation (3.14) can be obtained from equation (3.7).

4. Correlation between charged-pair and neutral particle

Here, we consider a correlation between the number of charged-pairs and that of neutral particles (Csikor *et al.* (1973), Zajc (1986)). The mean number $\langle n_g\rangle_n$ of neutral ones when n charged-pairs are observed, and the mean number $\langle n_{qq}\rangle_m$ of charged-pairs when m neutral particles are observed. It is noticed that we restrict ourselves to the case (3.1b). Those are also derived by the use of formulae,

$$
\langle n_g\rangle_n p_r(n;t) = \frac{1}{n!}\frac{\partial^n}{\partial u^n}\frac{\partial}{\partial v} G(u,v;t)|_{u=0,v=1} ,
$$

$$
\langle n_{qq}\rangle_m P_g(m;t) = \frac{1}{m!}\frac{\partial}{\partial u}\frac{\partial^m}{\partial v^m} G(u,v;t)|_{u=1,v=0} .
$$

Then we obtain

$$
(4.1a)\qquad \langle n_g\rangle_n = r\kappa(n+1)p_r(n+1;t)/p_r(n;t) + r(1-\kappa)n ,
$$

$$
(4.1b)\qquad \langle n_{qq}\rangle_m = \left\{1 + \frac{(1-\kappa)}{r}\right\}(m+1)P_g(m+1;t)/P_g(m;t)
$$

$$-\frac{(1-\kappa)}{r}\,m\;,$$

where $\kappa=\mu(1+\alpha_1+\alpha_2)\beta_2/\langle n_g\rangle$ and $r=\langle n_g\rangle/\langle n_{qq}\rangle$. It is noticed that $\kappa\to 0$ and $r\to\lambda_2/\lambda_1$ in the limit of $t\to\infty$. Equation (4.1a) coincides with a formula in Csikor *et al.* (1973) and Zajc (1986), when $\kappa=1$ and $r=1$.

In the following, we examine asymptotic behaviors of equation (4.1a) in two cases. Substituting equation (3.13) into equation (4.1a), we get

$$\langle n_g\rangle_n = r\kappa n\,\frac{\alpha_1}{1+\alpha_1}\,L_n^{(1)}(-x)/L_{n-1}^{(1)}(-x) + r(1-\kappa)n \qquad (n=1,2,\ldots)\;,$$

where $x=\langle n_{qq}\rangle/\{\alpha_1(1+\alpha_1)\}$.

(i) When $|x|\ll 1$. The generalized Lagurre polynomial is written approximately in the form

$$L_n^{(1)}(-x) \cong \frac{(n+1)!}{n!}\left\{1+\frac{n}{2}\,x+O(x^2)\right\}.$$

Then we have

$$\langle n_g\rangle_n \cong r\left\{\frac{1}{1+\alpha_1}\left(\frac{x}{2}\,\alpha_1-1\right)\kappa+1\right\}n + r\,\frac{\alpha_1\kappa}{1+\alpha_1}\left(1+\frac{x}{2}\right). \tag{4.2}$$

(ii) When $|x|\gg 1$, using the approximation,

$$L_n^{(1)}(-x) \cong \frac{x^n}{n!}\left\{1+\frac{n(n+1)}{x}+O\left(\frac{1}{x^2}\right)\right\},$$

we obtain

$$\langle n_g\rangle_n \cong r\left\{1+\frac{\alpha_1-1}{1+\alpha_1}\,\kappa\right\}n + r\,\frac{\alpha_1\kappa}{1+\alpha_1}\,x\,. \tag{4.3}$$

It is found that $\langle n_g\rangle_n$ depends linearly on n in both limits.

5. KNO scaling functions and Fokker-Planck equation

In this section, the branching equation for $p_r(n,m;t)$ is considered:

$$\frac{\partial}{\partial t}\,p_r(n,m;t) = 2n\lambda_0 p_r(n,m-1;t) - 2n\lambda_0 p_r(n,m;t) \tag{5.1}$$

$$+ (m+1)\lambda_1 p_r(n-1, m+1; t) - m\lambda_1 p_r(n, m; t)$$
$$+ (m-1)\lambda_2 p_r(n, m-1; t) - m\lambda_2 p_r(n, m; t) .$$

We get equation (5.1) from equation (2.1) by replacing n by $2n$, and $P(2n, m; t)$ by $p_r(n, m; t)$. By the use of the generating function

$$G(u, v; t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_r(n, m; t) u^n v^m , \tag{5.2}$$

equation (5.1) is reduced to the differential equation,

$$\frac{\partial}{\partial t} G(u, v; t) = f_1(u, v) \frac{\partial}{\partial u} G(u, v; t) + f_2(u, v) \frac{\partial}{\partial v} G(u, v; t) , \tag{5.3a}$$

$$f_1(u, v) = 2\lambda_0 u(v-1) , \tag{5.3b}$$

$$f_2(u, v) = \lambda_2 v^2 - (\lambda_1 + \lambda_2)v + \lambda_1 u . \tag{5.3c}$$

The distribution function $p_r(n, m; t)$ of discrete variables n and m is connected to the continuous function $\psi(z_1, z_2; t)$ of z_1 and z_2 by the Poisson transform,

$$p_r(n, m; t) = \frac{\langle n\rangle^n}{n!} \frac{\langle m\rangle^m}{m!} \int_0^\infty \int_0^\infty z_1^n z_2^m \cdot \exp[-\langle n\rangle z_1 - \langle m\rangle z_2]\, \psi(z_1, z_2; t) dz_1 dz_2 . \tag{5.4}$$

In equation (5.4), $\langle n\rangle$ and $\langle m\rangle$ do not depend on t, but are to be identified with the mean number of charged-pairs, $\langle n_{qq}\rangle$ and that of neutral particles, $\langle n_g\rangle$, respectively. The Laplace transform of $\psi(z_1, z_2; t)$ and its inverse transform are defined by the following equations,

$$F(s_1, s_2; t) = \int_0^\infty \int_0^\infty \psi(z_1, z_2; t) \exp[-s_1 z_1 - s_2 z_2] dz_1 dz_2 , \tag{5.5a}$$

$$\psi(z_1, z_2; t) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} F(s_1, s_2; t) \exp[s_1 z_1 + s_2 z_2] ds_1 ds_2 , \tag{5.5b}$$

where c_1 and c_2 are some real constants. Then we have

$$F(s_1, s_2; t) = G(1 - s_1/\langle n\rangle, 1 - s_2/\langle m\rangle; t) , \tag{5.6}$$

from equations (5.2), (5.4) and (5.5). Equation (5.6) means that the two-component scaling function $\psi(z_1, z_2; t)$ can be derived from the generating function of $p_r(n, m; t)$ by the inverse Laplace transform.

In the following, we consider two cases corresponding to the initial conditions (3.1a) and (3.1b) in Section 3.

(a) An adequate initial condition for equation (5.1) cannot be derived from equation (3.1a). However, we can obtain

$$F(s_1, s_2; t) = \left[1 + \frac{\alpha_1}{\langle n \rangle} s_1 + \frac{\alpha_2}{\langle m \rangle} s_2 \right]^{-\lambda - m_0} \left[1 - \frac{\beta_1}{\langle n \rangle} s_1 - \frac{\beta_2}{\langle m \rangle} s_2 \right]^{m_0}, \tag{5.7}$$

from equations (3.2b), (3.3) and (5.6).

First of all, we will derive the scaling function of charged-pairs and that of neutral particles from equation (5.7):

$$\psi_r(z_1; t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s_1, s_2 = 0; t) e^{s_1 z_1} ds_1 \ , \tag{5.8a}$$

$$\psi_g(z_2; t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s_1 = 0, s_2; t) e^{s_2 z_2} ds_2 \ . \tag{5.8b}$$

Both functions reduce to

$$\psi(z; t) = \frac{m_0! \xi}{\Gamma(\lambda + m_0)} (\xi z)^{\lambda - 1} \exp[-\xi z] \left(- \frac{\beta}{\alpha} \right)^{m_0} L_{m_0}^{(\lambda-1)} \left(\frac{\alpha + \beta}{\beta} \xi z \right), \tag{5.9}$$

where $\xi = \langle n \rangle / \alpha_1$, $\alpha = \alpha_1$ and $\beta = \beta_1$ for $\psi_r(z; t)$, and $\xi = \langle m \rangle / \alpha_2$, $\alpha = \alpha_2$ and $\beta = \beta_2$ for $\psi_g(z; t)$.

We can also derive the two-component scaling function from equation (5.7):

$$\psi(z_1, z_2; t) = \delta(\xi_2 z_2 - \xi_1 z_1) \frac{1}{2} [\xi_2 \psi_r(z_1; t) + \xi_1 \psi_g(z_2; t)] \ , \tag{5.10}$$

where $\xi_1 = \langle n \rangle / \alpha_1$ and $\xi_2 = \langle m \rangle / \alpha_2$. The function $\psi(z_1, z_2; t)$ depends only on a single variable z_1 or z_2, because of the term $\delta(\xi_2 z_2 - \xi_1 z_1)$ on the r.h.s. of equation (5.10), namely, two components are not independent. Furthermore, the two-component scaling function is expressed by two single component scaling functions.

Here, let's consider the relation between $\langle n \rangle$ and $\langle n_{qq} \rangle$, and also $\langle m \rangle$ and $\langle n_g \rangle$. Equation (1.2) implies

$$C_0 = \sum_{n=0}^{\infty} \langle n \rangle P(n) / \langle n \rangle = 1 \quad \text{and} \quad C_1 = \sum_{n=0}^{\infty} n P(n) / \langle n \rangle = 1 \ .$$

Therefore, each single component scaling function should satisfy

$$\int_0^\infty \psi(z;t)dz = 1 \, , \tag{5.11a}$$

$$\int_0^\infty z\psi(z;t)dz = 1 \, . \tag{5.11b}$$

The following relations are derived from equation (5.11b):

$$\langle n\rangle = \langle n_{qq}\rangle \quad \text{and} \quad \langle m\rangle = \langle n_g\rangle \, . \tag{5.12}$$

In the limit of $\langle n_{qq}\rangle$, $\langle n_g\rangle \to \infty$ ($t\to\infty$), we have the following asymptotic distributions,

$$\psi(z;t) \to \psi_s(z) = \frac{\lambda + m_0}{\Gamma(\lambda + m_0)} \{(\lambda + m_0)z\}^{\lambda+m_0-1} \exp[-(\lambda + m_0)z] \, ,$$

$$\psi(z_1, z_2; t) \to \psi_s(z_1)\delta(z_2 - z_1) \, .$$

(b) The initial condition $p_r(n,m;t=0)=(\lambda^n/n!)e^{-\lambda}(\mu^m/m!)e^{-\mu}$ is obtained from $f_b(n,m)$ in Section 3. Then the inverse Laplace transform of $\psi(z_1,z_2;t)$ is given as follows,

$$F(s_1, s_2; t = 0) = \exp\left[-\frac{\lambda}{\langle n\rangle} s_1 - \frac{\mu}{\langle m\rangle} s_2\right], \tag{5.13a}$$

$$F(s_1, s_2; t) = \exp\left[\lambda\left(\frac{1 - s_1/\langle n\rangle}{1 + \alpha_1 s_1/\langle n\rangle + \alpha_2 s_2/\langle m\rangle} - 1\right)\right] \cdot \exp\left[\mu\left(\frac{1 - \beta_1 s_1/\langle n\rangle - \beta_2 s_2/\langle m\rangle}{1 + \alpha_1 s_1/\langle n\rangle + \alpha_2 s_2/\langle m\rangle} - 1\right)\right]. \tag{5.13b}$$

Two single component scaling functions, $\psi_r(z;t)$ and $\psi_g(z;t)$ are derived in a similar way as equation (5.9). They reduce to

$$\psi(z;t) = \xi(z/\xi')^{-1/2} e^{-\xi(\xi'+z)} I_1(2\xi\sqrt{\xi' z}) \, , \tag{5.14}$$

where $\xi=\langle n\rangle/\alpha_1$ and $\xi'=\langle n_{qq}\rangle/\langle n\rangle$ for $\psi_r(z;t)$, and $\xi=\langle m\rangle/\alpha_2$ and $\xi'=\langle n_g\rangle/\langle m\rangle$ for $\psi_g(z;t)$.

Next, we consider about the two-component scaling function. The boundary condition for $\psi(z_1,z_2;t)$ is given by

$$\psi(z_1, z_2; t = 0) = \delta(z_1 - \lambda/\langle n\rangle)\delta(z_2 - \mu/\langle m\rangle) \, , \tag{5.15a}$$

from equations (5.5b) and (5.13a). The two-component scaling function is also derived by the inverse Laplace transform of equation (5.13b):

$$(5.15b)\qquad \psi_1(z_1,z_2;t) = \delta(\xi_2 z_2 - \xi_1 z_1)\,\frac{1}{2}\,[\xi_2\psi_r(z_1;t) + \xi_1\psi_g(z_2;t)]\ ,$$

where $\xi_1=\langle n\rangle/\alpha_1$ and $\xi_2=\langle m\rangle/\alpha_2$.

From equations (5.3), (5.5) and (5.6), we obtain the following Fokker-Planck equation,

$$(5.16)\qquad \frac{\partial}{\partial t}\psi = \frac{2\lambda_0}{\langle m\rangle}\left(\frac{\partial^2}{\partial z_1\partial z_2} + \langle n\rangle\frac{\partial}{\partial z_2}\right)(z_1\psi) + \left[\frac{\lambda_2}{\langle m\rangle}\frac{\partial^2}{\partial z_2^2} + (\lambda_1-\lambda_2)\frac{\partial}{\partial z_2} - \frac{\langle m\rangle\lambda_1}{\langle n\rangle}\frac{\partial}{\partial z_1}\right](z_2\psi)\ .$$

Equation (5.15b) becomes the solution for equation (5.16) with the boundary condition (5.15a) and $\lambda_2=2\lambda_0$.

In the limit of $\langle n_{qq}\rangle$, $\langle n_g\rangle\to\infty$, we obtain the following asymptotic expressions,

$$\psi(z;t) \to \psi_s(z) = (\lambda+\mu)z^{-1/2}e^{-(\lambda+\mu)(1+z)}I_1(2(\lambda+\mu)\sqrt{z})\ ,$$

$$\psi(z_1,z_2;t) \to \psi_s(z_1)\delta(z_2-z_1)\ .$$

6. Analysis of experimental data

Equations (3.13) and (4.1a) are applied to the analysis of experimental data in *pp* collisions (Dao *et al.* (1972, 1973), Dao and Whitmore (1973)). We assume that produced secondary particles are mainly pions. We interpret charged-pairs as charged pion-pairs and neutral particles as neutral pions. Equation (3.13) is used for the probability distribution of negative pions. Equation (4.1a) is for the mean multiplicity of neutral pions $\langle n_0\rangle_n$, when n negative pions are observed in the final states.

The observed multiplicity distribution of negative pions at $\sqrt{s}=24$ GeV (Dao *et al.* (1972)) is shown in Fig. 3 with the theoretical curve. Observed values of $\langle n\rangle=3.428$ and $C_2=1.408$ are used to fix the parameter $\langle n\rangle$ and α_1.

In analyzing the correlation between neutral pions and negative pions, we put $r=1$, namely, we assume the mean number of neutral pions is equal to that of negative pions. Calculated results of $\langle n_0\rangle_n$ with $\kappa=0$, 0.4 and 1 are compared with the experimental data at $\sqrt{s}=24$ GeV (Dao *et al.* (1972)) in Fig. 4. Data are well described by the linear equation,

$$(6.1)\qquad \langle n_0\rangle_n = 0.858\ n + 1.19\ .$$

Equation (6.1) is determined by the method of linear regression from the data point of $n=0$ to that of $n=8$ and the correlation coefficient (c.c.) is 0.973.

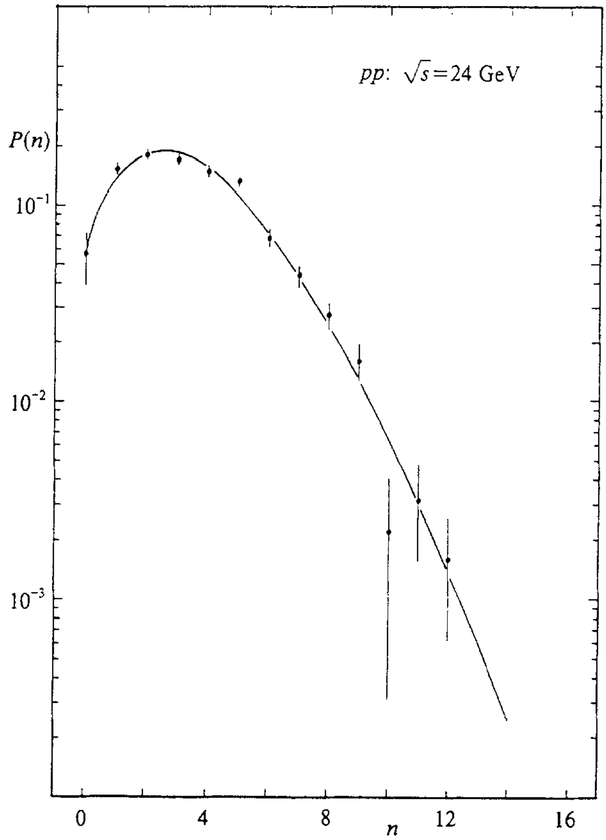

Fig. 3. Multiplicity distribution of negative pions at $\sqrt{s}=24$ GeV (Dao *et al.* (1972)). Solid lines represent the theoretical results obtained by equation (3.13).

Our calculated results are also expressed by the linear equation, $\langle n_0\rangle_n=an+b$. The slope parameter a is a decreasing function of κ; $a=1$ at $\kappa=0$ and $a=0.277$ at $\kappa=1$. For example, we get

$$\langle n_0\rangle_n = 0.712\ n + 0.982 \qquad (\text{c.c.} = 0.9998)\ .$$

with $r=1$ and $\kappa=0.4$. Our results with $\kappa=0\sim0.4$ are consistent with the data.

7. Summary and discussions

The two-component branching equation is investigated under the condition $\lambda_0=\lambda_2/2$. It is proposed by Giovannini, but is only investigated under the simple initial conditions, δ_{n,n_0} δ_{m,m_0} with $(n_0, m_0)=(1,0)$ or $(0,1)$ (Giovannini (1979)). Even in these simple cases, two-component probability distributions or those scaling functions are not derived (Anselmino *et al.* (1981), Durand and Sarcevic (1986)).

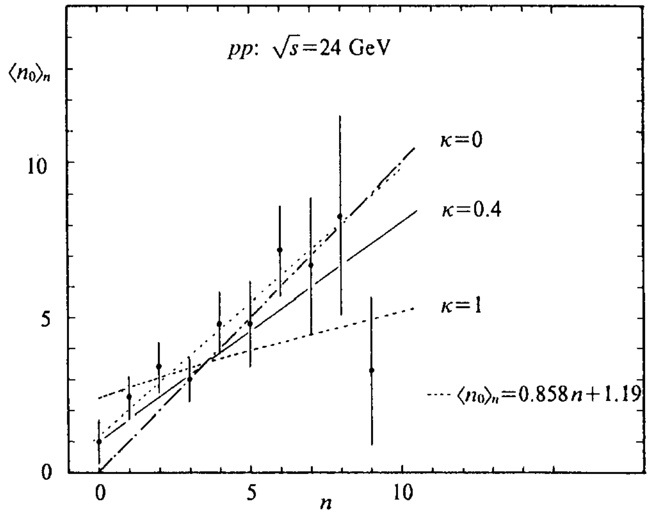

Fig. 4. Mean multiplicity of neutral pions, when n negative pions are observed in the final states. Data are taken from Dao *et al.* (1973). Theoretical results with $r=1$ and $\kappa=0$, 0.4, 1 are obtained by equation (4.1a).

We obtain solutions under two typical initial conditions, (3.1a) and (3.1b), using the method of the generating function. In each case, the multiplicity distribution of charged-pairs and that of neutral particles are reduced to the same function. This function is also the same as that obtained in a single component birth and death process (Biyajima and Suzuki (1984)). The two-component multiplicity distribution is explicitly obtained under the initial condition (3.1a), $\delta_{n,n_0}\,\delta_{m,m_0}$.

The two-component scaling functions are investigated in two cases. It is found that only one variable is effective in those functions, namely, two components are not independent. The two-component scaling function is given by the single component scaling function multiplied by a δ-function in each case. This result is contrasted to the formula in Durand and Sarcevic (1986).

The Fokker-Planck equation for the two-component branching process is obtained and its solution is found by the use of the inverse Laplace transform for the generating function. Further consideration on the Fokker-Planck equation and its solution will be reported elsewhere (Biyajima and Suzuki (1988)).

As for the correlation between charged-pairs and neutral particles, the formula is derived for the conditional mean multiplicity of neutral particles when a given number of charged-pairs is observed. It is applied to the analysis of experimental data.

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