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SIMULTANEOUS ESTIMATION OF EIGENVALUES

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Abstract. The problem of simultaneous estimation of eigenvalues of covariance matrix is considered for one and two sample problems under a sum of squared error loss. New classes of estimators are obtained which dominate the best multiple of the sample eigenvalues in terms of risk. These estimators shrink or expand the sample eigenvalues towards their geometric mean. Similar results are obtained for the estimation of eigenvalues of the precision matrix and the residual matrix when the original covariance matrix is partitioned into two groups. As a consequence, a new estimator of trace of the covariance matrix is obtained.

The results are extended to two sample problem where two Wishart distributions are independently observed, say, $S_i \sim W_p(\Sigma_i, k_i)$, $i=1, 2$, and eigenvalues of $\Sigma_1\Sigma_2^{-1}$ are estimated simultaneously. Finally, some numerical calculations are done to obtain the amount of risk improvement.

Key words and phrases: Wishart distribution, covariance matrix, eigenvalues, squared error loss.

1. Introduction and summary

In this paper, first we consider the problem of estimating the eigenvalues of the scale matrix Σ, of a Wishart distribution under a squared error loss. Even if the squared error loss does not much penalize negative estimates, we have considered it for simplicity and convenience. Suppose S has a nonsingular Wishart distribution with unknown matrix Σ and k degrees of freedom, i.e.,

$$S \sim W_p(k, \Sigma), \quad k-p-1>0 .$$

Several authors including James and Stein (1961), Olkin and Selliah (1977), Haff (1980, 1983), and Dey and Srinivasan (1985, 1986) considered the problem of estimating Σ directly under several plausible loss functions by perturbing the eigenvalues of S. Here our objective is to estimate the eigenvalues directly. The estimation of precision matrix Σ^{-1} was also considered by several authors. See for reference Dey *et al.* (1986). However, our problem will be the estimation of the eigenvalues of Σ^{-1} directly. In the same spirit, we will consider the estimation of eigenvalues of the residual matrix.

Next, suppose S_1 and S_2 are independent $p\times p$ Wishart matrices with

$$S_i \sim W_p(k_i, \Sigma_i), \quad i = 1, 2 .$$

We consider the problem of estimating the eigenvalues $\eta_1\geq\eta_2\geq\cdots\geq \eta_p>0$ of $\Sigma_1\Sigma_2^{-1}$ under the sum of squared error loss.

Let us now go back to the one sample problem. Suppose $\lambda_1\geq\cdots\geq\lambda_p>0$ are the eigenvalues of Σ. Our problem is to obtain an improved estimate of $\lambda=(\lambda_1, \ldots, \lambda_p)$ under the loss

$$L(a, \lambda) = \sum_{i=1}^{p} (a_i - \lambda_i)^2 . \tag{1.1}$$

These roots are very important since we encounter them in several problems in multivariate statistical analysis, e.g., testing hypothesis, principal components and discriminant analysis problems. In Section 2, we will obtain a new estimator which dominates the best multiple (including maximum likelihood) estimator of λ, based on the sample eigenvalues. These estimators are developed in the spirit of Dey and Gelfand (1986). It is important to note that if $l_1\geq\cdots\geq l_p$ are the sample eigenvalues of S, then $E(l_1/k)\geq\lambda_1$ and $E(l_p/k)\leq\lambda_p$. In fact, the sample eigenvalues of S tend to be more spread out than the population eigenvalues of Σ. This fact suggests that one should shrink or expand the eigenvalues depending on their magnitudes. The estimator of λ obtained in Section 2 indeed shrinks or expands the sample eigenvalues and it does towards the geometric mean. Then we use similar techniques to obtain improved estimates of eigenvalues of the precision and the residual matrices. As a consequence, we also obtain an improved estimator of trace of Σ.

Section 3 is devoted to the two sample problem. In this case, we defined a random matrix F with eigenvalues $f_1\geq\cdots\geq f_p>0$ which have the same joint distribution as that of $S_1S_2^{-1}$. We consider the estimation of $\eta_1\geq\cdots\geq\eta_p>0$, the eigenvalues of $\Sigma_1\Sigma_2^{-1}$, under the loss

$$L(a, \eta) = \sum_{i=1}^{p} (a_i - \eta_i)^2 . \tag{1.2}$$

These roots are also important, for example, in the problem of testing H_0: $\Sigma_1=\Sigma_2$ against H_1: $\Sigma_1\neq\Sigma_2$. In fact, the power functions of tests based on functions of $f_1,\ldots,f_p$ depend on Σ_1 and Σ_2 only through the maximal invariant $(\eta_1,\ldots, \eta_p)$. For details, see Muirhead (1982) and Muirhead and Verathaworn (1985). Finally, in Section 5, numerical results are given which indicate the percentage improvements.

2. Improved estimators of eigenvalues and trace of Σ

Consider the estimator $\delta^c(L)=(\delta_1^c(L),\dots,\delta_p^c(L))$ where $\delta_i^c(L)=cl_i$, $c>0$, is a constant. For example, $c=k^{-1}$, $i=1,\dots,p$, gives rise to the maximum likelihood estimator (MLE) of λ_i. In fact, c can also be chosen to give rise to a minimum mean square estimator.

Let us consider the rival estimator given componentwise as

$$\delta_i(L) = cl_i - b\Pi l_i^{1/p} , \tag{2.1}$$

where b will be appropriately chosen. The following theorem will show that the estimator (2.1) dominates $\delta^c(L)$ in terms of risk under the loss (1.1), but first we note

LEMMA 2.1. *For any real α for which $k+2\alpha>0$*

$$E_\Sigma|S|^\alpha = |\Sigma|^\alpha c_{p,k+2\alpha}/c_{p,k} ,$$

where

$$c_{p,k} = 2^{pk/2}\,\Gamma_p(k/2)$$

and

$$\Gamma_p(a) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(a-\frac{i-1}{2}\right) .$$

PROOF. If $S\sim W_p(k,\Sigma)$, the pdf of S is given as

$$f(S) = \frac{|\Sigma|^{-k/2}}{c_{p,k}}\, e^{(-\mathrm{tr}\Sigma^{-1}S)/2}\,|S|^{(k-p-1)/2} , \qquad S>0 .$$

Thus,

$$E_\Sigma|S|^\alpha = |\Sigma|^\alpha E_I|S|^\alpha = \frac{|\Sigma|^\alpha}{c_{p,k}}\int_{S>0} e^{-(\mathrm{tr}\,S)/2}|S|^{(k-p-1+2\alpha)/2}\,ds$$
$$= |\Sigma|^\alpha\frac{c_{p,k+2\alpha}}{c_{p,k}} .$$

This completes the proof of the lemma.

LEMMA 2.2. *For any real α such that $k+2\alpha>0$*

$$E_\Sigma[|S|^\alpha S] = |\Sigma|^\alpha \frac{c_{p,k+2\alpha}}{c_{p,k}}(k+2\alpha)\Sigma .$$

PROOF.

$$\begin{aligned} \text{L.H.S.} &= \frac{|\Sigma|^{-k/2}}{c_{p,k}}\int_{S>0} Se^{(-\mathrm{tr}\Sigma^{-1}S)/2}\,|S|^{(k-p-1+2\alpha)/2}\,ds \\ &= \frac{|\Sigma|^{-k/2}|\Sigma|^{k/2+\alpha}}{c_{p,k}}\,c_{p,k+2\alpha}(k+2\alpha)\Sigma \\ &= \text{R.H.S.} \end{aligned}$$

Now consider the following theorem which gives rise to the improved estimator of λ.

THEOREM 2.1. *Consider the estimator $\delta(L)$ given as in (2.1). Then, $\delta(L)$ dominates $\delta^c(L)$ under the loss (1.1), if one of the following conditions hold:*

(1) $d>0$ *and* $0<b<2dc_{p,k+2/p}/c_{p,k+4/p}$,

(2) $d<0$ *and* $2dc_{p,k+2/p}/c_{p,k+4/p}<b<0$,

where

$$d=c(k+2/p)-1\neq 0\ , \quad c>0\ .$$

PROOF. Let $\Delta(\lambda)=R(\delta(L),\lambda)-R(\delta^c(L),\lambda)$ be the risk difference. Sufficient to show that $\Delta(\lambda)<0$, it follows that

$$\begin{aligned} \Delta(\lambda) &= pb^2E|S|^{2/p} - 2bc\,E\,\mathrm{tr}(S|S|^{1/p}) + 2b(\Sigma\lambda_i)\,E|S|^{1/p} \qquad (2.2)\\ &= pb^2|\Sigma|^{2/p}\frac{c_{p,k+4/p}}{c_{p,k}} + 2b(\mathrm{tr}\Sigma)|\Sigma|^{1/p}\frac{c_{p,k+2/p}}{c_{p,k}} \\ &\quad - 2bc(\mathrm{tr}\Sigma)|\Sigma|^{1/p}(k+2/p)\frac{c_{p,k+2/p}}{c_{p,k}} \end{aligned}$$

(using Lemmas 2.1 and 2.2)

$$\begin{aligned} &= pb^2|\Sigma|^{2/p}\frac{c_{p,k+4/p}}{c_{p,k}} + 2bd(\mathrm{tr}\Sigma)|\Sigma|^{1/p}\frac{c_{p,k+2/p}}{c_{p,k}} \\ &= pb|\Sigma|^{2/p}c_{p,k}^{-1}\left[bc_{p,k+4/p} - 2dc_{p,k+2/p}\frac{p^{-1}\,\mathrm{tr}\Sigma}{|\Sigma|^{1/p}}\right]. \end{aligned}$$

Now using $p^{-1}\mathrm{tr}\Sigma/|\Sigma|^{1/p}\geq 1$, conditions (1) and (2) of Theorem 2.1, the proof follows from (2.4).

Remark 2.1. If for some c and b the estimator (2.1) is negative, one should naturally take the positive part version of it.

Let us now consider the estimation of eigenvalues of the precision matrix Σ^{-1}. Dey *et al.* (1986) considered the problem of Σ^{-1} under several plausible loss functions. However, we consider the problem of the estimation of eigenvalues directly. It is easy to observe that $(k-p-1)S^{-1}$ is an unbiased estimate of Σ^{-1} for $k-p-2\geq 0$ and kS^{-1} is the MLE of Σ^{-1}. Therefore, two natural estimators of eigenvalues of Σ^{-1} are cl_i^{-1} where $c=k-p-1$ or k, respectively. The following theorem gives a class of improved estimators of $\hat{\delta}^c(L)$ given componentwise as $\hat{\delta}_i^c(L)=cl_i^{-1}$, $i=1,\ldots,p$. But, first, we have the following lemma.

LEMMA 2.3. *For any real α such that $k-p-1-2\alpha>0$,*

$$E_\Sigma|S|^{-\alpha}S^{-1}=\frac{c_{p,k-2\alpha}}{c_{p,k}}|\Sigma|^{-\alpha}\frac{\Sigma^{-1}}{k-2\alpha-p-1}\,.$$

PROOF.

$$\begin{aligned}\text{L.H.S.}&=\frac{|\Sigma|^{-k/2}}{c_{p,k}}\int_{S>0}S^{-1}e^{-\mathrm{tr}\Sigma^{-1}S/2}|S|^{(k-p-1-2\alpha)/2}ds\\&=\frac{|\Sigma|^{-k/2}}{c_{p,k}}|\Sigma|^{k/2-\alpha}c_{p,k-2\alpha}\frac{\Sigma^{-1}}{k-2\alpha-p-1}\\&=\text{R.H.S.}\end{aligned}$$

THEOREM 2.2. *Consider the estimator $\hat{\delta}(L)$ given componentwise as*

$$\hat{\delta}_i(L)=\hat{\delta}_i^c(L)-b\,\Pi\, l_i^{-1/p},\quad i=1,\ldots,p\,. \tag{2.3}$$

Then $\hat{\delta}(L)$ dominates $\hat{\delta}^c(L)$ under squared error loss if one of the following conditions hold:

(1) $d>0$ *and* $0<b<2dc_{p,k-2/p}/c_{p,k-4/p}$,

(2) $d<0$ *and* $2dc_{p,k-2/p}/c_{p,k-4/p}<b<0$,

where

$$d = c/(k - 2/p - p - 1) - 1 \neq 0 \quad and \quad c > 0.$$

PROOF. Suppose $\Delta(\lambda^{-1}) = R(\hat{\delta}(L), \lambda^{-1}) - R(\hat{\delta}^c(L), \lambda^{-1})$ is the risk difference. Then using similar calculations as in the proof of Theorem 2.1, it follows that

$$\Delta(\lambda^{-1}) = pb^2 E|S|^{-2/p} - 2bcE(\mathrm{tr}S^{-1}|S|^{-1/p}) + 2b\ \mathrm{tr}\Sigma^{-1}E|S|^{-1/p}.$$

Now, using Lemmas 2.1 and 2.3, and the definition of d, it follows that

$$\Delta(\lambda^{-1}) = \frac{pb^2|\Sigma|^{-2/p}}{c_{p,k}}\left\{bc_{p,k-4/p} - 2bd\frac{p^{-1}\mathrm{tr}\Sigma^{-1}}{|\Sigma|^{-1/p}}c_{p,k-2/p}\right\}.$$

Finally, using $p^{-1}\mathrm{tr}\Sigma^{-1} \geq |\Sigma|^{-1/p}$, it follows that $\Delta(\lambda^{-1}) < 0$, which completes the proof of the theorem.

Remark 2.2. If for some choice of c and b, the estimator (2.3) is negative, we again take the positive part version of (2.3).

Next consider the problem of the estimation of eigenvalues of the residual matrix. Suppose $S \sim W_p(k, \Sigma)$ and Σ is partitioned as $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. The residual matrix is defined as $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. The natural estimates are constant multiples of $S_{11.2} = S_{11} - S_{12}S_{22}^{-1}S_{21}$ where S is also partitioned as $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. In this case, it is well known that $S_{11.2} \sim W_{p_1}(K - p_2, \Sigma_{11.2})$, where S_{11} is $p_1 \times p_1$ and $p_2 = p - p_1$. The inadmissibility of the usual estimates of the eigenvalues of $S_{11.2}$ (i.e., multiple of eigenvalues of $S_{11.2}$) follows immediately from Theorem 2.1. Similar result holds for the estimation of eigenvalues of the other residual matrix $S_{22.1} = S_{22} - S_{21}S_{11}^{-1}S_{12}$. The estimation of eigenvalues of the inverse of residual matrices similarly can be done by using Theorem 2.2.

Now we consider the estimation of $\mathrm{tr}\Sigma$. Olkin and Selliah (1977) considered this problem under weighted squared error loss and they estimated $\mathrm{tr}\Sigma$ directly. However, we consider this problem through the improved estimates of the eigenvalues of Σ. The following theorem gives an improved estimate over the best scalar multiple of $\mathrm{tr}S$.

THEOREM 2.3. *Consider the estimator $\delta_c(S) = c\,\mathrm{tr}S$, the best multiple of $\mathrm{tr}S$ and define the estimator $\delta(S)$ as*

$$\delta(S) = \delta_c(S) - b|S|^{1/p}. \tag{2.4}$$

Then $\delta(S)$ dominates $\delta_c(S)$ under squared error loss, if one of the following conditions hold:

(1) $d > 0$ *and* $0 < b < 2dc_{p,k+2/p}/ c_{p,k+4/p}$,

(2) $d < 0$ *and* $2dc_{p,k+2/p}/ c_{p,k+4/p} < b < 0$,

where

$$d = c(k + 2/p) - 1 \neq 0, \quad c > 0 .$$

PROOF. The proof is similar to that of Theorem 2.1.

Remark 2.3. It is to be noted that the estimate (2.3) is always positive. For $d<0$, we have $b<0$, thus it follows immediately that the estimator is positive. For $d>0$, $c>p/(2+pk)$ and $b<2dc_{p,k+2/p}/ c_{p,k+4/p}<2d=2c(k+2/p)-2$. Thus

$$\begin{aligned} cp - b > cp - 2c(k + 2/p) + 2 &= c(p - 2k - 4/p) + 2 \\ &> \frac{p}{pk + 2} \frac{p^2 - 2pk - 4}{p} + 2 \\ &= \frac{p^2}{(2 + pk)} > 0 . \end{aligned}$$

Finally,

$$\delta(S) = c \operatorname{tr} S - b|S|^{1/p} \geq cp|S|^{1/p} - b|S|^{1/p} = (cp - b)|S|^{1/p} > 0 .$$

3. Improved estimators of eigenvalues of $\Sigma_1\Sigma_2^{-1}$

Suppose S_1 and S_2 are independent $p\times p$ Wishart matrices with

$$S_i \sim W_p(k_i, \Sigma_i), \quad i = 1, 2 ,$$

so that $E(S_i)=k_i\Sigma_i$ with $k_i>p+1$, $i=1,2$. The problem considered in this section is essentially that of estimating the eigenvalues $\eta_1,\ldots, \eta_p$ ($\eta_1\geq\eta_2\geq\cdots\geq\eta_p>0$) of $\Sigma_1\Sigma_2^{-1}$ under squared error loss function. Muirhead and Verathaworn (1985) studied this problem by considering a random $p\times p$ positive definite matrix F such that the distribution of the eigenvalues $f_1>\cdots>f_p>0$ for F is the same as the distribution of the eigenvalues of $S_1S_2^{-1}$ and depends only on $\eta_1,\ldots, \eta_p$, the eigenvalues of $\Sigma_1\Sigma_2^{-1}$. Muirhead and Verathaworn (1985) estimated the scale matrix of eigenvalues $\eta_1,\ldots, \eta_p$ directly, using a loss function based on "entropy" measure and hence obtained estimates of $\eta_1,\ldots, \eta_p$. We, however, estimate the eigenvalues directly as mentioned in the introduction.

It follows immediately that the best unbiased estimate of $\Delta=\Sigma_1\Sigma_2^{-1}$ is

$$\hat{\Delta}_U = \frac{k_2 - p - 1}{k_1} F , \tag{3.1}$$

but the eigenvalues of $\hat{\Delta}_U$ either overestimate or underestimate the population eigenvalues. In fact, $E(((k_2-p-1)/k_1)f_p) \leq \eta_p$ and $E(((k_2-p-1)/k_1)f_1) \geq \eta_1$. Now, the maximum likelihood estimates of η_i's are $\hat{\eta}_i = (k_2/k_1)f_i$, $i=i,\ldots, p$. Thus we can use our starting estimator of η as $\delta^c(F)$ given componentwise as

$$\delta_i^c(F) = cf_i , \tag{3.2}$$

where c is an appropriate constant depending on k_1, k_2 and p. Consider the rival estimator

$$\delta(F) = \delta^c(F) - b \prod_{i=1}^{p} f_i^{1/p} \cdot \mathbf{1} , \tag{3.3}$$

where $\mathbf{1}=(1,\ldots, 1)'$ and b is a constant which will be chosen later. Now, we need the following lemmas to calculate the risk difference of $\delta(F)$ and $\delta^c(F)$.

LEMMA 3.1. *For any real α such that $k_2-2\alpha-p-1>0$,*

$$\operatorname{tr} E[|F|^\alpha F] = \frac{c_{p,k_1+2\alpha}}{c_{p,k_1}} \cdot \frac{c_{p,k_2-2\alpha}}{c_{p,k_2}} \cdot \frac{k_1 + 2\alpha}{k_2 - 2\alpha - p - 1} |\Delta|^\alpha \operatorname{tr}\Delta . \tag{3.4}$$

PROOF. L.H.S.$=\operatorname{tr}\{E|S_1|^\alpha S_1\}\{E|S_2|^{-\alpha} S_2^{-1}\}$. Now using Lemmas 2.2 and 2.3, the proof follows.

LEMMA 3.2. *For any real α such that $k_i \pm 2\alpha > 0$, $i=1, 2$,*

$$E_\Delta |F|^\alpha = |\Delta|^\alpha \frac{c_{p,k_1+2\alpha}}{c_{p,k_1}} \cdot \frac{c_{p,k_2-2\alpha}}{c_{p,k_2}} .$$

PROOF. The proof follows from Lemma 2.1. The following theorem provides an improved estimator of η.

THEOREM 3.1. *Under squared error loss, the estimator* (3.3) *dominates* (3.2) *if one of the following conditions hold*:

(1) $d>0$ *and* $0 < b < 2dc_{p,k_1+2/p}/c_{p,k_1+4/p} \cdot c_{p,k_2-4/p}$,

(2) $d<0$ *and* $2dc_{p,k_1+2/p}/c_{p,k_1+4/p} \cdot c_{p,k_2-4/p} < b < 0$,

where

$$d = c\,\frac{k_1 + 2/p}{k_2 - 2/p - p - 1} - 1 \neq 0, \qquad c > 0\,.$$

PROOF. Let $\Delta(\eta) = R(\delta, \eta) - R(\delta^c, \eta)$ be the risk difference. Then it follows using Lemmas 3.1 and 3.2 that

$$\begin{aligned}
\Delta(\eta) &= pb^2 E|F|^{2/p} + 2b\,\mathrm{tr}\Delta E|F|^{1/p} - 2bc\,\mathrm{tr}E[|F|^{1/p}F] \\
&= pb^2|\Delta|^{2/p}\,\frac{c_{p,k_1+4/p}}{c_{p,k_1}}\cdot\frac{c_{p,k_2-4/p}}{c_{p,k_2}} \\
&\quad + 2b(\mathrm{tr}\Delta)|\Delta|^{1/p}\,\frac{c_{p,k_1+2/p}}{c_{p,k_1}}\cdot\frac{c_{p,k_2-2/p}}{c_{p,k_2}} \\
&\quad - 2bc(\mathrm{tr}\Delta)|\Delta|^{1/p}\,\frac{c_{p,k_1+2/p}}{c_{p,k_1}}\cdot\frac{c_{p,k_2-2/p}}{c_{p,k_2}}\cdot\frac{k_1 + 2/p}{k_2 - 2/p - p - 1} \\
&= pb^2|\Delta|^{2/p}\,\frac{c_{p,k_1+4/p}}{c_{p,k_1}}\cdot\frac{c_{p,k_2-4/p}}{c_{p,k_2}} \\
&\quad - 2bd\,\frac{c_{p,k_1+2/p}c_{p,k_2-2/p}}{c_{p,k_1}c_{p,k_2}}(\mathrm{tr}\Delta)|\Delta|^{1/p}\,.
\end{aligned}$$

Now, using $p^{-1}\mathrm{tr}\Delta/|\Delta|^{-1/p} \geq 1$, conditions (1) and (2) of Theorem 3.1, it follows that $\Delta(\eta) < 0$ which completes the proof.

Remark 3.1. If for some c and b the estimator (3.3) is negative, we take the positive part version of it.

4. Numerical studies

In this section, we use Monte Carlo simulation method to compute the risk of the maximum likelihood estimates (R_0) and that of the improved estimates (R_1) of the eigenvalues $\lambda_1,\ldots,\lambda_p$ and compute the percentage improvements in risk $PI = (R_0 - R_1)\times 100/R_0$. Then we do the similar calculations for the estimation of trace of Σ. In our calculations, we take several values of p and k and generate 100 Wishart variates for different choices of Σ. The Σ matrix is taken to be diagonal for simplicity and the diagonal elements are selected in such a way that we get a wide spectrum of eigenvalues. Table 1 gives the percentage improvements in risk for the improved estimates of eigenvalues and the trace of Σ for different p, k and Σ values.

Next, we consider the two sample problem. For $p=4$ and $k_1, k_2 = 10$, 15 and 20, a sample of 100 S_1's and 100 S_2's are generated where $S_1 \sim W_4(k_1, I_4)$, $S_2 \sim W_4(k_2, I_4)$, S_1 and S_2 are independent. For each (k_1, k_2), the 100 pairs (S_1, S_2) are transformed into

$$F = \Delta^{1/2}\, S_1^{1/2}\, S_2^{-1}\, S_1^{1/2}\, \Delta^{1/2}\,,$$

Table 1. Percentage improvement for eigenvalues (PI1) and trace (PI2).

$p = 2$

	Σ = diag (1,1)		Σ = diag (2,1)		Σ = diag (25,1)	
k	PI1	PI2	PI1	PI2	PI1	PI2
4	15.9	19.3	17.9	17.5	13.2	12.4
10	12.2	12.5	12.9	12.2	11.2	7.4
25	11.2	12.3	11.7	12.2	6.3	8.7

$p = 3$

	Σ = diag (1,1,1)		Σ = diag (4,2,1)		Σ = diag (25,1,1)	
k	PI1	PI2	PI1	PI2	PI1	PI2
5	11.5	12.6	12.2	12.6	6.3	8.7
10	10.1	11.9	11.8	12.9	4.5	1.8
25	3.9	9.9	6.8	11.9	1.6	0.9

Table 2. Percentage improvements of $\delta(F)$ over $\delta^c(F)$ for $p=4$.

	$c = k_2/k_1$	$c = (k_2 - p - 1)/k_1$
Δ = diag (1,1,1,1)		
$k_1 = k_2 = 10$	57.87	85.58
$k_1 = k_2 = 15$	45.62	79.19
$k_1 = k_2 = 20$	27.56	39.34
Δ = diag (8,4,2,1)		
$k_1 = k_2 = 10$	49.65	77.92
$k_1 = k_2 = 15$	39.48	62.23
$k_1 = k_2 = 20$	16.73	34.14
Δ = diag (25,1,1,1)		
$k_1 = k_2 = 10$	20.68	60.57
$k_1 = k_2 = 15$	17.27	51.72
$k_1 = k_2 = 20$	2.05	11.02

for each of 3 choices of Δ. The choices of Δ are taken from Muirhead and Verathaworn (1985). The eigenvalues of F's are then obtained to form the estimates of η. Finally, the percentage improvements in risk of the improved estimators over $\delta^c(F)$ are computed in Table 2.

Table 1 indicates that for most choices of Σ, the percentage improvements are very significant. Table 2 indicates that for all choices of Δ, the percentage improvements are very significant and are largest when $\Delta = I_4$.

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