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ESTIMATING COMMON PARAMETERS OF GROWTH CURVE MODELS*

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Abstract. Suppose that we have two independent random matrices X_1 and X_2 having multivariate normal distributions with common unknown matrix of parameters ξ $(q\times m)$ and different unknown covariance matrices Σ_1 and Σ_2, given by $N_{p_1,N_1}(B_1\xi A_1;\Sigma_1, I)$ and $N_{p_2,N_2}(B_2\xi A_2;\Sigma_2, I)$ respectively. Let $\hat{\xi}_1$ $(\hat{\xi}_2)$ be the MLE of ξ based on X_1 (X_2) only. When $q=1$, necessary and sufficient conditions that a combined estimator of $\hat{\xi}_1$ and $\hat{\xi}_2$ has uniformly smaller covariance matrix than those of $\hat{\xi}_1$ and $\hat{\xi}_2$ are given. The k-sample problem as well as one-sample problem is also discussed. These results are extensions of those of Graybill and Deal (1959, *Biometrics*, **15**, 543–550), Bhattacharya (1980, *Ann. Statist.*, **8**, 205–211; 1984, *Ann. Inst. Statist. Math.*, **36**, 129–134) to multivariate case.

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1. Introduction

The problem of estimating the common mean of two univariate normal distributions has been studied in several papers. Of these, Graybill and Deal (1959) are the first who gave necessary and sufficient conditions for the combined estimator having a variance uniformly smaller than that of each sample mean. Recent works by Brown and Cohen (1974), Khatri and Shah (1974) and Bhattacharya (1980) demonstrated a family of combined unbiased estimators with uniformly smaller variance than each sample mean. An extension to multivariate one-sample problem of estimating common components of a mean vector of normal distribution was obtained by

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Halperin (1961), Rustagi and Rohatgi (1974). Related work by Gupta and Rohatgi (1979) is to be noted. Combined estimators in this case can be regarded as a special case of multivariate regression problem, in which the covariance matrix of MLE was obtained by Rao (1967), Williams (1967) and Gleser and Olkin (1972).

In Section 2 of this paper we shall summarize the results in one sample growth curve model which was first formulated by Potthoff and Roy (1964) from our point of view. This will be a basis for the subsequent sections. In Section 3 we discuss two-sample problem of growth curve models with common matrix of unknown parameters and different covariance matrices. When the matrix B_i of internal regressor variables is $p_i\times 1$ vector for $i=1$ and 2, we can give necessary and sufficient conditions for a combined estimator to have uniformly smaller covariance matrix than those of MLE's based on each sample. They are extensions of original results by Graybill and Deal (1959) and of Bhattacharya (1980, 1984) to multivariate case, in that the conditions are free from population parameters and design matrices. When the matrix B_i is $p\times 2$, that is, in estimating a common mean vector of p-variate normal distributions, Chiou and Cohen (1985) showed that combined estimator cannot have uniformly smaller covariance matrix than that of each MLE. Hence our restriction on B_i is inevitable, if population covariance matrices are unknown. If they are known, combined estimator has always uniformly smaller covariance matrix. A class of combined estimators is extended up to $n_i=p_i+1$ where n_i stands for the degrees of freedom for estimating covariance matrix of the i-th population by including the difference of individual MLE's in the weight function. With some restriction on the matrix of external regressor variables, we discuss in Section 4 estimating a matrix of common parameters in k growth curve models with different covariance matrices. Necessary and sufficient conditions are given for a combined estimator having uniformly smaller covariance matrix than that of each MLE, which are generalizations of Norwood and Hinkelmann (1977), Shinozaki (1978) and Bhattacharya (1978, 1984) to multivariate case. An extension of Brown and Cohen type estimator (Brown and Cohen (1974)) is also obtained.

The following lemma due to Bhattacharya (1984) is a basic tool of our proofs in Sections 3 and 4.

LEMMA 1.1. *Let $X_i>0$, $i=1,2,\dots,k$ be mutually independent random variables having $E(X_i^{-2})<\infty$. Then for any positive numbers p_i satisfying $\sum_{i=1}^{k}p_i=1$,*

$$\frac{E\left(1\Big/\sum_{i=1}^{k} p_iX_i\right)}{E\left(1\Big/\left(\sum_{i=1}^{k} p_iX_i\right)^2\right)} \ge \min_{1\le i\le k}\left\{\frac{E(1/X_i)}{E(1/X_i^2)}\right\}.$$

2. One-sample problem

Let observed random matrix $X(p\times N)$ have normal distribution $N_{p,N}(B\xi A;\ \Sigma, I)$ where $B(p\times q)$ and $A\ (m\times N)$ are known matrices of ranks q and m respectively and $\xi(q\times m)$ is a matrix of unknown parameters; each column of X is independently distributed according to p-variate normal distribution with common covariance matrix $\Sigma(p\times p)$ which is assumed to be positive definite. This is called a growth curve model by Potthoff and Roy (1964) and practical meaning and applications can be seen in their paper. If we put $q=p$ and $B=I$, we get ordinary multivariate regression model.

When Σ is known, MLE of ξ is given by

$$\hat{\xi} = (B'\Sigma^{-1}B)^{-1}B'\Sigma^{-1}XA'(AA')^{-1}\ , \tag{2.1}$$

which is an unbiased estimator of ξ. Equivalently we can write $\hat{\xi}$ as in Lee (1974)

$$\text{vec}\ \hat{\xi} = \{(AA')^{-1}A \otimes (B'\Sigma^{-1}B)^{-1}B'\Sigma^{-1}\}\ \text{vec}\ X\ , \tag{2.2}$$

where vec X is defined by $pN\times 1$ vector $(X_1', X_2',\dots, X_N')'$ for $X=(X_1,\dots, X_N)$ and $A\otimes B$ stands for Kronecker product defined by $(a_{ij}B)$ for $A=(a_{ij})$. Here we used the identity vec $(B\xi A)=(A'\otimes B)$vec ξ. A good account can be seen in Muirhead (1982) for the relation between Kronecker product and vec operator. We easily get

$$\text{Cov}\ (\text{vec}\ \hat{\xi}) = (AA')^{-1} \otimes (B'\Sigma^{-1}B)^{-1}\ . \tag{2.3}$$

Partition X and B according as first p' components and remaining $p-p'$ components $(p'\geq q)$ and put $X=(X_1', X_2')'$, $B=(B_1', B_2')'$ where X_1 is $p'\times N$ and B_1 is $p'\times q$. Similarly, partition Σ and put $(\Sigma_{ij})_{i,j=1,2}$. The MLE of ξ based on X_1 only is given by

$$\hat{\xi}_1 = (B_1'\Sigma_{11}^{-1}B_1)^{-1}B_1'\Sigma_{11}^{-1}X_1A'(AA')^{-1}\ , \tag{2.4}$$

which is also an unbiased estimate of ξ. We get

$$\text{Cov}\ (\text{vec}\ \hat{\xi}_1) = (AA')^{-1} \otimes (B_1'\Sigma_{11}^{-1}B_1)^{-1}\ . \tag{2.5}$$

Noting the decomposition formula

$$B'\Sigma^{-1}B = (\bar{B}_2' - B_1'\Sigma_{11}^{-1}\Sigma_{12})\Sigma_{22\cdot1}^{-1}(B_2 - \Sigma_{21}\Sigma_{11}^{-1}B_1) + B_1'\Sigma_{11}^{-1}B_1\ ,$$

we can see that

$$\text{Cov}(\text{vec}\,\hat{\xi}) \leq \text{Cov}(\text{vec}\,\hat{\xi}_1)\,, \tag{2.6}$$

for all Σ and ξ. Here the ordering between two positive definite matrices is defined by the nonnegative definiteness of their difference. In fact Cov(vec $\hat{\xi}$) attains the lower bound of Cramér-Rao inequality. We can say that if Σ is known, the estimator $\hat{\xi}$ utilizing all the information given by the data is always better than the estimator $\hat{\xi}_1$ utilizing only some components of the data. We will see that this is not the case when Σ is not known.

When Σ is unknown, MLE of ξ is obtained by Khatri (1966) as

$$\hat{\xi} = (B'S^{-1}B)^{-1}B'S^{-1}XA'(AA')^{-1}\,, \tag{2.7}$$

where $S=X(I-A'(AA')^{-1}A)X'$. This is obtainable by substituting S for Σ in (2.1). We can show that $\hat{\xi}$ is the unique solution of the likelihood equation by generalizing Halperin (1961). Note that XA' and S are independent. We can see that $\hat{\xi}$ is an unbiased estimator of ξ. As for the dispersion of $\hat{\xi}$, we get the following theorem.

THEOREM 2.1. *Put $n=N-m$ where $A\,(m\times N)$ is of rank m and assume that $n\geq p+1$. We get*

$$\text{Cov}(\text{vec}\,\hat{\xi}) = (AA')^{-1} \otimes (B'\Sigma^{-1}B)^{-1}\,\frac{n-1}{n-(p-q)-1}\,. \tag{2.8}$$

PROOF. Noting that S has Wishart distribution $Wp(n, \Sigma)$, we get

$$\begin{aligned}\text{Cov}&(\text{vec}\,\hat{\xi})\\ &= (AA')^{-1} \otimes E[(B'S^{-1}B)^{-1}B'S^{-1}\Sigma S^{-1}B(B'S^{-1}B)^{-1}]\\ &= (AA')^{-1} \otimes (B'\Sigma^{-1}B)^{-1/2}E[I + W_{12}W_{22}^{-2}W_{21}](B'\Sigma^{-1}B)^{-1/2}\,,\end{aligned}$$

where $W=(W_{ij})_{i,j=1,2}$ has $W_p(n, I)$ and W_{11} is $q\times q$. This is obtained by transformation $S\to V=\Sigma^{-1/2}S\Sigma^{-1/2}$ and $V\to W=H'VH$ for some orthogonal matrix H. Noting that the conditional distribution of W_{12}, given W_{22} is $N_{q,p-q}(0;\, I,\, W_{22})$ from Srivastava and Khatri (1979, p. 79), we get $E(W_{12}W_{22}^{-2}W_{21})=(p-q)I/(n-p+q-1)$, giving the desired result.

The fact that $E[(B'S^{-1}B)^{-1}B'S^{-1}\Sigma S^{-1}B(B'S^{-1}B)^{-1}]=(n-1)(B'\Sigma^{-1}B)^{-1}/(n-p+q-1)$ was already obtained by Rao (1967), Williams (1967) and Gleser and Olkin (1972). However we believe that our proof is simpler and more direct. As a byproduct we get the following lemma which will be used in Sections 3 and 4. This is obtained by noting $(B'S^{-1}B)^{-1}=W_{11\cdot 2}$.

LEMMA 2.1. *Let S have $W_p(n, \Sigma)$ distribution and let $B(p\times q)$ be of rank q. Then $(B'S^{-1}B)^{-1}B'S^{-1}\Sigma S^{-1}B(B'S^{-1}B)^{-1}$ and $B'S^{-1}B$ are independent.*

The MLE of ξ based on $X_1(p' \times N)$ only is given by

$$\hat{\xi}_1 = (B_1' S_{11}^{-1} B_1)^{-1} B_1' S_{11}^{-1} X_1 A' (AA')^{-1} , \tag{2.9}$$

where S_{11} is the partitioned matrix of S similar to (2.4). From Theorem 2.1, we get the necessary and sufficient condition for Cov(vec $\hat{\xi}$)$\leq$Cov(vec $\hat{\xi}_1$) as

$$\frac{n-(p'-q)-1}{n-(p-q)-1} B_1' \Sigma_{11}^{-1} B_1 \leq B' \Sigma^{-1} B . \tag{2.10}$$

Note that $(n-(p'-q)-1)/(n-(p-q)-1)>1$. We can always find Σ such that reverse inequality of (2.10) holds. For such Σ, the MLE $\hat{\xi}$ is not better than $\hat{\xi}_1$.

Take $B=e_p$ and $A=e_N'$ with $e_p=(1,\dots, 1)'$ of length p. Then the MLE becomes $\hat{\xi}=e_p' S^{-1} \bar{X} / e_p' S^{-1} e_p$ where $\bar{X}$ stands for the sample mean vector and the problem reduces to estimate mean vector $(\xi, \xi,\dots, \xi)'$ based on a random sample of size N from p-variate normal distribution $N_p(\xi e_p, \Sigma)$. This is the simplest case of growth curve model as Potthoff and Roy (1964) stated. Halperin (1961) considered this case and compared $\hat{\xi}$ with $\hat{\xi}_1=e_p'\bar{X}/p$. He already observed the possibility of losing precision by using all the data. When $A=e_N'$ and B is arbitrary, Rao (1967), Williams (1967) and Gleser and Olkin (1972) discussed the properties of $\hat{\xi}$.

3. Two-sample problem

Let $X_1(p_1 \times N_1)$ and $X_2(p_2 \times N_2)$ be independent observed random matrices having normal distribution $N_{p_1,N_1}(B_1 \xi A_1; \Sigma_1, I)$ and $N_{p_2,N_2}(B_2 \xi A_2; \Sigma_2, I)$ respectively where $B_i(p_i \times q)$ and $A_i(m \times N_i)$ are known matrices of ranks q and m respectively. The problem is to estimate common $\xi(q \times m)$. If Σ_1 and Σ_2 are known, the MLE of ξ is given by

$$\text{vec } \hat{\xi} = \left(\sum_{i=1}^{2} A_i A_i' \otimes B_i' \Sigma_i^{-1} B_i \right)^{-1} \sum_{i=1}^{2} (A_i A_i' \otimes B_i' \Sigma_i^{-1} B_i) \text{ vec } \hat{\xi}_i , \tag{3.1}$$

where $\hat{\xi}_i=(B_i'\Sigma_i^{-1}B_i)^{-1}B_i'\Sigma_i^{-1}X_iA_i'(A_iA_i')^{-1}$ is the MLE of ξ based on X_i. The covariance matrix of vec $\hat{\xi}$ is given by

$$\text{Cov (vec } \hat{\xi}) = \left(\sum_{i=1}^{2} A_i A_i' \otimes B_i' \Sigma_i^{-1} B_i \right)^{-1} . \tag{3.2}$$

In view of (2.3), we get Cov(vec $\hat{\xi}$)$\leq$Cov(vec $\hat{\xi}_1$) uniformly for all Σ_1 and Σ_2.

When Σ_1 and Σ_2 are unknown, an extension of Graybill-Deal estimator (Graybill and Deal (1959)) is obtained by substituting Σ_i by an unbiased estimator S_i/n_i in (3.1) as

$$(3.3)\qquad \operatorname{vec} \hat{\xi}_{GD} = \left(\sum_{i=1}^{2} A_iA_i' \otimes B_i'S_i^{-1}B_in_i \right)^{-1} \sum_{i=1}^{2} (A_iA_i' \otimes B_i'S_i^{-1}B_in_i) \operatorname{vec} \hat{\xi}_i \,,$$

where

$$\hat{\xi}_i = (B_i'S_i^{-1}B_i)^{-1}B_i'S_i^{-1}X_iA_i'(A_iA_i')^{-1} \,,$$
$$S_i = X_i(I - A_i'(A_iA_i')^{-1}A_i)X_i' \,,$$

and S_i has $W_{p_i}(n_i, \Sigma_i)$ distribution with $n_i=N_i-m$. A more general class of unbiased estimators is obtained if we put

$$(3.4)\qquad \operatorname{vec} \hat{\xi}_{KS} = \left(\sum_{i=1}^{2} b_in_iA_iA_i' \otimes B_i'S_i^{-1}B_i \right)^{-1} \sum_{i=1}^{2} b_in_i(A_iA_i' \otimes B_i'S_i^{-1}B_i) \operatorname{vec} \hat{\xi}_i \,,$$

for positive constants b_1 and b_2, the univariate case of which was discussed by Khatri and Shah (1974). This class can be further generalized by considering

$$(3.5)\qquad \operatorname{vec} \hat{\xi} = \operatorname{vec} \hat{\xi}_1 + \phi(\operatorname{vec} \hat{\xi}_2 - \operatorname{vec} \hat{\xi}_1) \,,$$

where

$$(3.6)\qquad \begin{aligned} \phi = a(bn_1A_1A_1' &\otimes B_1'S_1^{-1}B_1 \\ &+ n_2A_2A_2' \otimes B_2'S_2^{-1}B_2)^{-1}n_2A_2A_2' \otimes B_2'S_2^{-1}B_2 \,, \end{aligned}$$

for positive constants a and b, the univariate case of which was discussed by Bhattacharya (1980). Putting $a=b=1$, we get $\hat{\xi}_{GD}$ and putting $a=1$ and $b=b_1/b_2$, we get $\hat{\xi}_{KS}$.

It is easily seen that $E(\hat{\xi})=\xi$ and that $\operatorname{Cov}(\operatorname{vec} \hat{\xi}) \leq \operatorname{Cov}(\operatorname{vec} \hat{\xi}_1)$ is equivalent to

$$(3.7)\qquad \begin{aligned} &E[\phi\{\operatorname{Cov}(\operatorname{vec} \hat{\xi}_1 \mid S_1) + \operatorname{Cov}(\operatorname{vec} \hat{\xi}_2 \mid S_2)\}\phi'] \\ &\quad \leq E[\operatorname{Cov}(\operatorname{vec} \hat{\xi}_1 \mid S_1)\phi' + \phi \operatorname{Cov}(\operatorname{vec} \hat{\xi}_1 \mid S_1)] \,. \end{aligned}$$

Note that S_i and X_iA_i' are independent. We get

$$(3.8)\qquad \begin{aligned} &\operatorname{Cov}(\operatorname{vec} \hat{\xi}_1 \mid S_1) \\ &\quad = (A_1A_1')^{-1} \otimes (B_1'S_1^{-1}B_1)^{-1}B_1'S_1^{-1}\Sigma_1S_1^{-1}B_1(B_1'S_1^{-1}B_1)^{-1} \,. \end{aligned}$$

In view of Lemma 2.1, we can write the condition (3.7) as

$$(3.9)\qquad \begin{aligned} &E[\phi\{(A_1A_1')^{-1} \otimes \Sigma_1^*k_1 + (A_2A_2')^{-1} \otimes \Sigma_2^*k_2\}\phi'] \\ &\quad \leq E[\{(A_1A_1')^{-1} \otimes k_1\Sigma_1^*\}\phi' + \phi\{(A_1A_1')^{-1} \otimes k_1\Sigma_1^*\}] \,, \end{aligned}$$

where $\Sigma_i^*=(B_i'\Sigma_i^{-1}B_i)^{-1}$ and $k_i=(n_i-1)/(n_i-p_i+q-1)$. Put $C=(A_2A_2')^{-1/2}(A_1A_1')$

$\cdot(A_2A_2')^{-1/2}$ and $V_i=(B_i'S_i^{-1}B_i)^{-1}$. Then we get

$$(3.10)\qquad \phi = a\{(A_2A_2')^{-1/2}\otimes I_q\}\left\{I_m\otimes I_q+\frac{bn_1}{n_2}C\otimes V_2V_1^{-1}\right\}^{-1} \cdot\{(A_2A_2')^{1/2}\otimes I_q\}\,,$$

and the condition (3.9) can be expressed by

$$(3.11)\qquad \begin{aligned} &aE[\psi(I_m\otimes k_2\Sigma_2^*+C^{-1}\otimes k_1\Sigma_1^*)\psi'] \\ &\quad\le E[(C^{-1}\otimes k_1\Sigma_1^*)\psi'+\psi(C^{-1}\otimes k_1\Sigma_1^*)]\,, \end{aligned}$$

where $\psi=(I_{mq}+(bn_1/n_2)C\otimes V_2V_1^{-1})^{-1}$.

From here we should assume that $q=1$, in order to get definite condition for a and b. In this case ξ is a row vector and V_i and Σ_i^* are scalars. The condition (3.11) is further simplified as

$$(3.12)\qquad \begin{aligned} &aE\left[\left(I_m+\frac{bn_1}{n_2}\frac{V_2}{V_1}C\right)^{-1}\left(\frac{k_2\Sigma_2^*}{k_1\Sigma_1^*}I_m+C^{-1}\right)\left(I_m+\frac{bn_1}{n_2}\frac{V_2}{V_1}C\right)^{-1}\right] \\ &\quad\le E\left[C^{-1}\left(I_m+\frac{bn_1}{n_2}\frac{V_2}{V_1}C\right)^{-1}+\left(I_m+\frac{bn_1}{n_2}\frac{V_2}{V_1}C\right)^{-1}C^{-1}\right]. \end{aligned}$$

Let $C=H\operatorname{diag}(c_1,\dots,c_m)H'$ for some orthogonal matrix H. Then c_i's are all positive. Diagonalizing the matrix C in (3.12) by multiplying H' and H from left and right, we get an equivalent condition to (3.12) as

$$(3.13)\qquad \frac{a}{2}\le E\left[\frac{1+\tau_i}{\left(1+\dfrac{bn_1}{n_2}\dfrac{V_2}{V_1}c_i\right)}\right]\Bigg/E\left[\frac{(1+\tau_i)^2}{\left(1+\dfrac{bn_1}{n_2}\dfrac{V_2}{V_1}c_i\right)^2}\right], \qquad i=1,2,\dots,m\,,$$

where $\tau_i=k_2\Sigma_2^*c_i/(k_1\Sigma_1^*)$. By Lemma 1.1, a sufficient condition for (3.13) is given by

$$(3.14)\qquad \frac{a}{2}\le\min\left\{1,\ \frac{bn_1}{n_2}\frac{c_i}{\tau_i}\frac{E(V_1/V_2)}{E((V_1/V_2)^2)}\right\}\quad\text{for all}\quad i\,.$$

Note that V_j/Σ_j^* has χ^2-distribution with n_j-p_j+1 degrees of freedom and that $V_2/(\tau_iV_1)$ has a distribution free from parameters Σ_1^* and Σ_2^*. Putting $\tau_i=0$ in (3.13) yields $a\le 2$. Letting τ_i tend to infinity in (3.13) yields $a\le 2\{bn_1c_i/(n_2\tau_i)\}\cdot E(V_1/V_2)/E(V_1^2/V_2^2)$. Hence the sufficient condition (3.14) is also necessary, which gives the following theorem.

THEOREM 3.1. *Let $n_i=N_i-m$ and assume that $q=1$, $n_1\ge p_1+1$ and*

$n_2 \geq p_2+4$. *Then a necessary and sufficient condition for the combined estimator* $\hat{\xi}$ *of* ξ *to have covariance matrix uniformly less than or equal to the covariance matrix of* $\hat{\xi}_1$ *is*

$$a \leq 2 \min\left\{1,\ b\,\frac{n_1(n_1-1)(n_2-p_2)(n_2-p_2-3)}{n_2(n_2-1)(n_1-p_1)(n_1-p_1+3)}\right\}. \tag{3.15}$$

Put $a=b=1$ in Theorem 3.1, and symmetry consideration yields the following corollary.

COROLLARY 3.1. *Assume that* $q=1$ *and* $n_i \geq p_i+4$ *for* $i=1, 2$. *Then a necessary and sufficient condition for* $\mathrm{Cov}(\hat{\xi}_{GD}) \leq \mathrm{Cov}(\hat{\xi}_1)$ *and* $\mathrm{Cov}(\hat{\xi}_2)$ *is*

$$\begin{aligned} &2n_1(n_1-1)(n_2-p_2)(n_2-p_2-3) \\ &\qquad \geq n_2(n_2-1)(n_1-p_1)(n_1-p_1+3)\,, \\ &2n_2(n_2-1)(n_1-p_1)(n_1-p_1-3) \\ &\qquad \geq n_1(n_1-1)(n_2-p_2)(n_2-p_2+3)\,. \end{aligned} \tag{3.16}$$

If we put $p_1=p_2=1$ in Corollary 3.1, we get $(n_1-2)(n_2-8) \geq 16$ and $(n_2-2)(n_1-8) \geq 16$. Further put $m=1$, we get the results by Graybill and Deal (1959) and corrected by Norwood and Hinkelmann (1977). If we put $a=1$ and $b=b_1/b_2$ for $b_i>0$ in Theorem 3.1, we get a result for $\hat{\xi}_{KS}$ discussed by Khatri and Shah (1974). Symmetry consideration yields the following corollary.

COROLLARY 3.2. *Assume that* $q=1$ *and* $n_i \geq p_i+4$ *for* $i=1, 2$. *Then a necessary and sufficient condition for* $\mathrm{Cov}(\hat{\xi}_{KS}) \leq \mathrm{Cov}(\hat{\xi}_1)$ *and* $\mathrm{Cov}(\hat{\xi}_2)$ *is that*

$$\frac{1}{2}\,\frac{n_1-p_1+3}{n_2-p_2-3} \leq \frac{b_1 n_1(n_1-1)(n_2-p_2)}{b_2 n_2(n_2-1)(n_1-p_1)} \leq 2\,\frac{n_1-p_1-3}{n_2-p_2+3}\,. \tag{3.17}$$

It is easily seen that such b_1/b_2 exists if and only if $(n_1-p_1-5)(n_2-p_2-5) \geq 16$. Reasonable choice of b_1/b_2 may be $n_2(n_2-1)(n_1-p_1)(n_1-p_1-1)/\{n_1(n_1-1)(n_2-p_2)(n_2-p_2-1)\}$, which satisfies (3.17) whenever $(n_1-p_1-5)\cdot(n_2-p_2-5) \geq 16$. When $p_1=p_2=m=1$, these results are obtained by Khatri and Shah (1974) and Bhattacharya (1980). If $n_1=n_2$ and $p_1=p_2$, the condition (3.16) and the condition $(n_1-p_1-5)(n_2-p_2-5) \geq 16$ are the same, giving $n_1=n_2 \geq p_1+9$.

The assumption that $q=1$ in Theorem 3.1 cannot be avoided, since Chiou and Cohen (1985) has shown that for estimating common mean vector of two bivariate normal distributions with different unknown covariance matrices, the combined estimator cannot have uniformly smaller covariance matrix than that of sample mean of the first sample. This is the case with $q=2$ in our notation. The other assumption that $n_2 \geq p_2+4$ in Theorem 3.1 can be weakened up to $n_2 \geq p_2+1$ by using nonsingular matrix H such that $HA_iA_i'H'=D_i=\mathrm{diag}(d_1^{(i)},\ldots, d_m^{(i)})$ and considering a wider class of unbiased estimators $\mathrm{vec}\,\hat{\xi}=\mathrm{vec}\,\hat{\xi}_1+\phi(\mathrm{vec}\,\hat{\xi}_2-\mathrm{vec}\,\hat{\xi}_1)$ with

$$
\begin{aligned}
(3.18) \qquad \phi = a\,\frac{(A_1A_1')^{-1}}{B_1'S_1^{-1}B_1}\Bigg\{ & \frac{bn_1}{n_2}\,\frac{(A_2A_2')^{-1}}{B_2'S_2^{-1}B_2} + \frac{(A_1A_1')^{-1}}{B_1'S_1^{-1}B_1} \\
& + c(\hat{\xi}_2-\hat{\xi}_1)(H'L_1H)^{-1}(\hat{\xi}_2-\hat{\xi}_1)'H'L_2H\Bigg\}^{-1},
\end{aligned}
$$

where $L_i=\mathrm{diag}(l_1^{(i)},\ldots, l_m^{(i)})$ and $l_j^{(i)}>0$. When $p_1=p_2=1$ and $A_i=e'_{N_i}$, Brown and Cohen (1974), Khatri and Shah (1974) and Bhattacharya (1980) discussed this class of estimates with additional restriction between positive constants a, b, and c. It is easily seen that

$$
(3.19) \qquad E[\phi(\hat{\xi}_2-\hat{\xi}_1)\,|\,(\hat{\xi}_2-\hat{\xi}_1)'(\hat{\xi}_2-\hat{\xi}_1)] = 0\,,
$$

so that $E[\mathrm{vec}\,\hat{\xi}]=\mathrm{vec}\,\xi$. We shall give the result first.

THEOREM 3.2. *Assume that $q=1$ and $n_i \geq p_i+1$ for $i=1, 2$. Let H be a nonsingular matrix such that $HA_iA_i'H'=\mathrm{diag}(d_1^{(i)},\ldots, d_m^{(i)})$ for $i=1, 2$. Take positive constants a, b, c satisfying*

$$
(3.20) \qquad a \leq 2 \min \left\{1, \frac{(n_2-p_2)(n_2-p_2+m-1)}{(n_2-1)(n_1-p_1+3)} \min\left(\frac{bn_1}{n_2}, c\,\frac{(D_2L_2)_{\min}}{(D_2L_1)_{\max}}\right)\right\},
$$

where $(D_2L_1)_{\max}=\max_{1\leq i\leq m} d_i^{(2)}l_i^{(1)}$ and $(D_2L_2)_{\min}=\min_{1\leq i\leq m} d_i^{(2)}l_i^{(2)}$. Then $\mathrm{Cov}(\hat{\xi})\leq \mathrm{Cov}(\hat{\xi}_1)$ uniformly for all Σ_1 and Σ_2.

PROOF. Note that $\hat{\xi}$ is a $1\times m$ vector. We easily get

$$
\begin{aligned}
(3.21) \qquad \mathrm{Cov}\,(\hat{\xi}) = \mathrm{Cov}\,(\hat{\xi}_1) & + E[\phi(\hat{\xi}_2-\hat{\xi}_1)'(\hat{\xi}_2-\hat{\xi}_1)\phi'] \\
& + E[(\hat{\xi}_1-\xi)'(\hat{\xi}_2-\hat{\xi}_1)\phi'] \\
& + E[\phi(\hat{\xi}_2-\hat{\xi}_1)'(\hat{\xi}_1-\xi)]\,.
\end{aligned}
$$

Since conditional distribution of $\mathrm{vec}\,\hat{\xi}_1$ given S_1 is $N(\mathrm{vec}\,\xi, (A_1A_1')^{-1}\otimes (B_1'S_1^{-1}B_1)^{-1}B_1'S_1^{-1}\Sigma_1S_1^{-1}B_1(B_1'S_1^{-1}B_1)^{-1})$, we can see that conditional distribution of $\hat{\xi}_1$ and $\hat{\xi}_2-\hat{\xi}_1$ given S_1 and S_2, is normal with mean $(\xi, 0)$ and covariance matrix

$$\begin{pmatrix} T_1, & -T_1 \\ -T_1, & T_1 + T_2 \end{pmatrix},$$

where $T_i = (A_i A_i')^{-1} B_i' S_i^{-1} \Sigma_i S_i^{-1} B_i / (B_i' S_i^{-1} B_i)^2$. This gives $E[(\hat{\xi}_1 - \xi)' | \hat{\xi}_2 - \hat{\xi}_1] = -T_1(T_1 + T_2)^{-1}(\hat{\xi}_2 - \hat{\xi}_1)'$ and

$$E[(\hat{\xi}_1 - \xi)'(\hat{\xi}_2 - \hat{\xi}_1)\phi'] = -E[T_1(T_1 + T_2)^{-1}(\hat{\xi}_2 - \hat{\xi}_1)'(\hat{\xi}_2 - \hat{\xi}_1)\phi'] .$$

Combined with (3.21), a necessary and sufficient condition for $\mathrm{Cov}(\hat{\xi}) \le \mathrm{Cov}(\hat{\xi}_1)$ is given by

$$\begin{aligned} \text{(3.22)} \qquad & E[\phi(\hat{\xi}_2 - \hat{\xi}_1)'(\hat{\xi}_2 - \hat{\xi}_1)\phi'] \\ & \le E[T_1(T_1 + T_2)^{-1}(\hat{\xi}_2 - \hat{\xi}_1)'(\hat{\xi}_2 - \hat{\xi}_1)\phi'] \\ & \quad + E[\phi(\hat{\xi}_2 - \hat{\xi}_1)'(\hat{\xi}_2 - \hat{\xi}_1)(T_1 + T_2)^{-1} T_1] . \end{aligned}$$

Put

$$\text{(3.23)} \qquad Z_i = (B_i'\Sigma_i^{-1}B_i)(B_i'S_i^{-1}\Sigma_i S_i^{-1}B_i)/(B_i'S_i^{-1}B_i)^2 ,$$

and $\Sigma_i^* = 1/(B_i'\Sigma_i^{-1}B_i)$. Then $T_i = Z_i\Sigma_i^*(A_iA_i')^{-1}$. Note that $Z_i \ge 1$ and the distribution of Z_i is free from Σ_1 and Σ_2. Let Q be a square root of $T_1 + T_2$ defined by $Q = (D_1^{-1}Z_1\Sigma_1^* + D_2^{-1}Z_2\Sigma_2^*)^{1/2}H$ where $D_i = \mathrm{diag}(d_1^{(i)}, \dots, d_m^{(i)})$. Then for given S_1 and S_2, $V = (\hat{\xi}_2 - \hat{\xi}_1)Q^{-1}$ has $N(0, I)$ which is independent of S_1, S_2. We can rewrite the condition (3.22) by V, Z_i, Σ_i^* as

$$\begin{aligned} \text{(3.24)} \qquad E[\phi Q'V'VQ\,\phi'] \le\ & E[Z_1\Sigma_1^*(A_1A_1')^{-1}Q^{-1}V'VQ\phi'] \\ & + E[\phi Q'V'VQ'^{-1}Z_1\Sigma_1^*(A_1A_1')^{-1}] , \end{aligned}$$

where

$$\begin{aligned} \text{(3.25)} \qquad \phi = aU_1\Sigma_1^*H'D_1^{-1} \Bigg\{ & \frac{bn_1}{n_2} U_2\Sigma_2^*D_2^{-1} + U_1\Sigma_1^*D_1^{-1} \\ & + cV(D_1^{-1}Z_1\Sigma_1^* + D_2^{-1}Z_2\Sigma_2^*)L_1^{-1}V'L_2 \Bigg\}^{-1} H'^{-1} , \end{aligned}$$

with $U_i = (B_i'\Sigma_i^{-1}B_i)/B_i'S_i^{-1}B_i$ having χ^2 distribution with $n_i - p_i + 1$ degrees of freedom.

Multiplying H^{-1} and H'^{-1} from right and left respectively in (3.24) and noting that $E[V'V | VDV']$ is diagonal for any diagonal matrix D, we can simplify the condition (3.24) to the inequality between two diagonal matrices, giving

$$\text{(3.26)} \qquad aE[\phi_i^2(Z_1 + \rho_i Z_2)V_i^2] \le 2E[Z_1\phi_i V_i^2] , \quad i = 1, 2, \dots, m ,$$

where

$$\phi_i = U_1 \left\{ \frac{bn_1}{n_2} \rho_i U_2 + U_1 + c \sum_{j=1}^{m} \frac{d_i^{(1)} l_i^{(2)}}{d_j^{(1)} l_j^{(1)}} (Z_1 + \rho_j Z_2) V_j^2 \right\}^{-1},$$

with $\rho_i = \Sigma_2^* d_i^{(1)} / (\Sigma_1^* d_i^{(2)})$ and $V=(V_1, \ldots, V_m)$. Here V_j^2, $j=1, \ldots, m$ have independently χ^2 distributions with one degree of freedom. Regarding V_i^2 as a χ^2 variate with three degrees of freedom in (3.26), we can delete V_i^2 from both sides in the expectations of (3.26) and get

$$a \leq 2 \inf_{\rho_i} \frac{E[Z_1 \phi_i]}{E[(Z_1 + \rho_i Z_2)\phi_i^2]} \quad \text{for} \quad i = 1, 2, \ldots, m, \tag{3.27}$$

which is necessary and sufficient for $\mathrm{Cov}(\hat{\xi}) \leq \mathrm{Cov}(\hat{\xi}_1)$. Since $Z_1 + \rho_i Z_2$ and ϕ_i^2 have negative covariance with respect to Z_2, we can get for $\alpha_2 = E(Z_2)$

$$\begin{aligned}
\frac{E(Z_1 \phi_i)}{E[(Z_1 + \rho_i Z_2)\phi_i^2]} &\geq \frac{E[Z_1 \phi_i]}{E[(Z_1 + \rho_i \alpha_2)\phi_i^2]} \\
&\geq \inf_{Z_1, Z_2} Z_1 \frac{E[(Z_1 + \rho_i \alpha_2)\phi_i \mid Z_1, Z_2]}{E[(Z_1 + \rho_i \alpha_2)^2 \phi_i^2 \mid Z_1, Z_2]} \\
&\geq \inf_{Z_1, Z_2} \min \left\{ 1, \frac{1}{\alpha_2} \frac{E(U_1)}{E(U_1^2)} \frac{E(1/\psi_i \mid Z_1, Z_2)}{E(1/\psi_i^2 \mid Z_1, Z_2)} \right\},
\end{aligned}$$

where

$$\psi_i = \frac{bn_1}{n_2} U_2 + \frac{c}{\rho_i} \sum_{j=1}^{m} \frac{d_i^{(1)} l_i^{(2)}}{d_j^{(1)} l_j^{(1)}} (Z_1 + \rho_j Z_2) V_j^2 .$$

The last inequality is obtained by applying Lemma 1.1 to the expression

$$(Z_1 + \rho_i \alpha_2)\phi_i = \left\{ p \frac{1}{Z_1} + (1 - p) \frac{\psi_i}{\alpha_2 U_1} \right\}^{-1},$$

with $p = Z_1 / (Z_1 + \rho_i \alpha_2)$. Since $U_2 + \sum_{j=1}^{m} V_j^2$ has χ^2 distribution with $n_2 - p_2 + m + 3$ degrees of freedom and is independent of $(U_2, V_1^2, \ldots, V_m^2) / (U_2 + \sum_{j=1}^{m} V_j^2)$, we can simplify the lower bound further to get

$$\frac{E[1/\psi_i \mid Z_1, Z_2]}{E[1/\psi_i^2 \mid Z_1, Z_2]}$$

$$= \frac{E\left[\left(U_2 + \sum_{j=1}^m V_j^2\right)^{-1}\right]}{E\left[\left(U_2 + \sum_{j=1}^m V_j^2\right)^{-2}\right]} \frac{E\left[\left(U_2 + \sum_{j=1}^m V_j^2\right)\Big/\psi_i \Big| Z_1, Z_2\right]}{E\left[\left(U_2 + \sum_{j=1}^m V_j^2\right)^2\Big/\psi_i^2 \Big| Z_1, Z_2\right]}$$

$$\geq (n_2 - p_2 + m - 1) \inf_{U_2, V_j^2} \left\{ \frac{bn_1}{n_2} \frac{U_2}{U_2 + \sum_{j=1}^m V_j^2} + \frac{c}{\rho_i} \sum_{j=1}^m \frac{d_i^{(1)} l_i^{(2)}}{d_j^{(1)} l_j^{(1)}} (Z_1 + \rho_j Z_2) \frac{V_j^2}{U_2 + \sum_{j=1}^m V_j^2} \right\}$$

$$\geq (n_2 - p_2 + m - 1) \min \left\{ \frac{bn_1}{n_2}, \min_{1 \leq j \leq m} \frac{c}{\rho_i} \frac{d_i^{(1)} l_i^{(2)}}{d_j^{(1)} l_j^{(1)}} \rho_j \right\}$$

$$\geq (n_2 - p_2 + m - 1) \min \left\{ \frac{bn_1}{n_2}, c \frac{(D_2 L_2)_{\min}}{(D_2 L_1)_{\max}} \right\}.$$

Hence a sufficient condition for (3.27) is given by

$$a \leq 2 \min \left\{ 1, \frac{1}{\alpha_2} \frac{E(U_1)}{E(U_1^2)} (n_2 - p_2 + m - 1) \min \left(\frac{bn_1}{n_2}, c \frac{(D_2 L_2)_{\min}}{(D_2 L_1)_{\max}} \right) \right\}.$$

Noting that $\alpha_2 = (n_2 - 1)/(n_2 - p_2)$ by Theorem 2.1, we get the desired result.

It is noted that the condition in Theorem 3.2 depends on $d_j^{(2)}$ but not on $d_j^{(1)}$ and that it is free from $A_i A_i'$ if we take $L_1 = L_2 = D_2^{-1}$. Put $p_1 = p_2 = m = 1$ and $L_1 = L_2$ in Theorem 3.2, and we get

$$a \leq 2 \min \left\{ 1, \frac{n_2 - 1}{n_1 + 2} \min \left(\frac{bn_1}{n_2}, c \right) \right\},$$

in which the case $bn_1/n_2 = c$ was obtained by Bhattacharya (1980).

4. k-sample problem

Let $X_i(p_i \times N_i)$ be observed random matrix having normal distribution $N_{p_i, N_i}(B_i \xi A_i; \Sigma_i, I)$ and assume that $X_1, \ldots, X_k$ are mutually independent, where $B_i(p_i \times q)$ and $A_i(m \times N_i)$ are known matrices of ranks q and m respectively. The problem is to estimate ξ when Σ_i are unknown. We shall consider the following class of unbiased estimators as an extension from two-sample problem:

$$(4.1)\qquad \operatorname{vec}\hat{\xi}_{[k]}=\left(\sum_{i=1}^{k}c_in_iA_iA_i'\otimes B_i'S_i^{-1}B_i\right)^{-1}\sum_{i=1}^{k}c_in_i(A_iA_i'\otimes B_i'S_i^{-1}B_i)\operatorname{vec}\hat{\xi}_i\,,$$

where c_i are positive constants and $\hat{\xi}_i$ is the MLE of ξ based on X_i only and is given by

$$(4.2)\qquad \operatorname{vec}\hat{\xi}_i=\{(A_iA_i')^{-1}A_i\otimes(B_i'S_i^{-1}B_i)^{-1}B_i'S_i^{-1}\}\operatorname{vec}X_i\,,$$

for

$$S_i=X_i(I-A_i'(A_iA_i')^{-1}A_i)X_i'\,.$$

The notation $[k]$ in (4.1) denotes the combined estimator from first up to k-th samples. When $p_i\equiv 1$, $q=1$ and $A_i=e'_{N_i}$, this class of estimators is discussed by Shinozaki (1978) and Bhattacharya (1978, 1984). Further special case of $c_i\equiv 1$ was discussed by Norwood and Hinkelmann (1977). The following theorem is an extension of Bhattacharya (1984).

THEOREM 4.1. *Let $q=1$, $n_i\geq p_i+1$ for $i=1,2,\ldots,k-1$ and $n_k\geq p_k+4$. Assume that $HA_iA_i'H'$ $(1\leq i\leq k)$ are simultaneously diagonal for some nonsingular matrix H. Then a necessary and sufficient condition for* $\operatorname{Cov}(\hat{\xi}_{[k]})\leq\operatorname{Cov}(\hat{\xi}_{[k-1]})$ *is given by*

$$(4.3)\qquad \frac{c_k}{c_i}\leq 2\,\frac{n_i(n_i-1)(n_k-p_k)(n_k-p_k-3)}{n_k(n_k-1)(n_i-p_i)(n_i-p_i+3)}\quad \textit{for}\quad i=1,2,\ldots,k-1\,.$$

Successively using Theorem 4.1 with respect to k, we get

COROLLARY 4.1. *Let $q=1$, $n_1\geq p_1+1$ and $n_i\geq p_i+4$ for $i=2,\ldots,k$. Assume that $HA_iA_i'H'$ are simultaneously diagonal for some nonsingular matrix H. Then a necessary and sufficient condition for* $\operatorname{Cov}(\hat{\xi}_{[k]})\leq\operatorname{Cov}(\hat{\xi}_1)$ *is given by*

$$(4.4)\qquad \frac{c_j}{c_i}\leq 2\,\frac{n_i(n_i-1)(n_j-p_j)(n_j-p_j-3)}{n_j(n_j-1)(n_i-p_i)(n_i-p_i+3)}\quad \textit{for}\quad 1\leq i<j\leq k\,.$$

COROLLARY 4.2. *Let $q=1$, $n_i\geq p_i+4$ for $i=1,2,\ldots,k$ and assume that $HA_iA_i'H'$ are simultaneously diagonal for some nonsingular matrix H. Then a necessary and sufficient condition for* $\operatorname{Cov}(\hat{\xi}_{[k]})\leq\operatorname{Cov}(\hat{\xi}_i)$ *for all i is given by*

$$(4.5)\qquad \frac{1}{2}\,\frac{n_j-p_j+3}{n_i-p_i-3}\leq\frac{c_jn_j(n_j-1)(n_i-p_i)}{c_in_i(n_i-1)(n_j-p_j)}\leq 2\,\frac{n_j-p_j-3}{n_i-p_i+3}\,,\quad 1\leq i<j\leq k\,.$$

It is easy to see that such c_j/c_i in Corollary 4.2 exist if and only if $(n_i-p_i-5)(n_j-p_j-5)\geq 16$ for all $i\neq j$, which is an extension of Shinozaki (1978), Bhattacharya (1978). If we put $p_i\equiv 1$, $c_i\equiv 1$ and $q=m=1$ in Corollary 4.2, the conditions become $(n_i-2)(n_j-8)\geq 16$ for all $i\neq j$ which was obtained by Norwood and Hinkelmann (1977).

PROOF OF THEOREM 4.1. From Lemma 2.1 and Theorem 2.1 we get

$$\text{(4.6)} \qquad \mathrm{Cov}\,(\mathrm{vec}\,\hat{\xi}_{[k]}) = E\left[\sum_{i=1}^{k} W_i\theta_i W_i'\right],$$

where

$$\text{(4.7)} \qquad \begin{aligned} W_i &= \left(\sum_{i=1}^{k} c_in_iA_iA_i' \otimes B_i'S_i^{-1}B_i\right)^{-1} c_in_iA_iA_i' \otimes B_i'S_i^{-1}B_i\,,\\ \theta_i &= (A_iA_i')^{-1} \otimes l_i(B_i'\Sigma_i^{-1}B_i)^{-1}\,, \end{aligned}$$

for $l_i=(n_i-1)/(n_i-p_i)$. By the assumption of Theorem 4.1, we can find a nonsingular matrix H such that $HA_iA_i'H'=\mathrm{diag}(d_1^{(i)},\ldots, d_m^{(i)})$ simultaneously. Since $q=1$, we can write a necessary and sufficient condition for $\mathrm{Cov}(\hat{\xi}_{[k]})\leq \mathrm{Cov}(\hat{\xi}_{[k-1]})$ as

$$\text{(4.8)} \qquad E\left[\sum_{i=1}^{k} \theta_{ij}W_{ij}^2 - \sum_{i=1}^{k-1} \theta_{ij}W_{ij}^{*2}\right] \leq 0 \quad \text{for} \quad j=1, 2,\ldots, m\,,$$

where

$$\text{(4.9)} \qquad \begin{aligned} W_{ij} &= c_in_il_i\theta_{ij}^{-1}U_i^{-1}/\sum_{i=1}^{k} c_in_il_i\theta_{ij}^{-1}U_i^{-1}\,,\\ W_{ij}^* &= c_in_il_i\theta_{ij}^{-1}U_i^{-1}/\sum_{i=1}^{k-1} c_in_il_i\theta_{ij}^{-1}U_i^{-1}\,,\\ \theta_{ij} &= l_i/(B_i'\Sigma_i^{-1}B_id_j^{(i)})\,, \end{aligned}$$

and $U_i=B_i'\Sigma_i^{-1}B_i/B_i'S_i^{-1}B_i$ has χ^2 distribution with n_i-p_i+1 degrees of freedom. Put $\theta_{*j}=\left(\sum_{i=1}^{k-1}\theta_{ij}^{-1}\right)^{-1}$. Noting that $\theta_{*j}\leq\sum_{i=1}^{k-1} W_{ij}^{*2}\theta_{*j}$ and

$$\sum_{i=1}^{k} \theta_{ij}W_{ij}^2 - \sum_{i=1}^{k-1} \theta_{ij}W_{ij}^{*2} \leq W_{kj}^2\theta_{kj} + W_{kj}(W_{kj}-2)\theta_{*j}\,,$$

as in the proof of Theorem 3.1 in Bhattacharya (1984), we get a sufficient condition for (4.8)

$$\frac{1}{2} \leq \min_{1\leq j\leq m} \frac{1}{1+\tau_j} \frac{E(W_{kj})}{E(W_{kj}^2)}, \tag{4.10}$$

where $\tau_j = \theta_{kj}/\theta_{*j}$. By Lemma 1.1, a lower bound of RHS of (4.10) is given by

$$\min_{1\leq j\leq m} \left\{ 1, \frac{1}{\tau_j c_k n_k l_k \theta_{kj}^{-1} \theta_{*j}} \frac{E(U_k^{-1})}{E(U_k^{-2})} \frac{E(1/g_j)}{E(1/g_j^2)} \right\}, \tag{4.11}$$

where

$$g_j = \sum_{i=1}^{k-1} c_i n_i l_i (\theta_{*j}/\theta_{ij}) U_i^{-1} .$$

Applying Lemma 1.1 again, we get for all $j=1,\dots, m$

$$\frac{E(1/g_j)}{E(1/g_j^2)} \geq \min_{1\leq i\leq k-1} \left\{ c_i n_i l_i \frac{E(U_i)}{E(U_i^2)} \right\}.$$

Combined with (4.11), we finally get a sufficient condition for (4.8) as

$$\frac{1}{2} \leq \min_{1\leq i\leq k-1} \left\{ \frac{c_i n_i l_i}{c_k n_k l_k} \frac{n_k - p_k - 3}{n_i - p_i + 3} \right\},$$

which is equivalent to (4.3). The necessity of the condition (4.3) follows from nonpositiveness of the derivative of $E[\sum_{i=1}^{k} \theta_{ij} W_{ij}^2]$ with respect to $1/\theta_{kj}$ at $1/\theta_{kj}=0$ for all θ_{ij} $(i\neq k)$ in view of (4.8) and putting further $1/\theta_{ij}=0$ for $i\neq r$ $(r<k)$.

Assuming that $q=1$, we can consider another class of unbiased estimators given by

$$\hat{\eta}_j' = \hat{\xi}_1' + \sum_{i=1}^{j-1} \phi_i (\hat{\xi}_{i+1}' - \hat{\xi}_1') ,$$

where

$$\phi_i = a_i \left(b_i \frac{n_i}{n_{i+1}} (A_{i+1} A_{i+1}')^{-1} A_1 A_1' \frac{B_1' S_1^{-1} B_1}{B_{i+1}' S_{i+1}^{-1} B_{i+1}} + I_m \right)^{-1} ,$$

for positive constants a_i and b_i. This is an extension of (3.6) to k-sample problem. When $p_i \equiv 1$, $m=1$ and $A_i = e_{N_i}'$, this class of estimators is discussed by Brown and Cohen (1974) and Bhattacharya (1980). Noting that $\phi_i \leq a_i I$ and the same argument as Brown and Cohen (1974) yields the following theorem.

THEOREM 4.2. *Assume that* $q=1$, $n_1 \geq p_1+1$, $n_i \geq p_i+4$ *for* $i=2, \ldots, k$ *and that* $HA_iA_i'H'$ *are simultaneously diagonal for some nonsingular matrix* H. *Take positive constants* a_i *and* b_i *such that*

$$a_1 \leq \min\left\{1,\ 2b_1 \frac{n_1(n_1-1)(n_2-p_2)(n_2-p_2-3)}{n_2(n_2-1)(n_1-p_1)(n_1-p_1+3)}\right\},$$

$$\vdots$$

$$\frac{a_{k-2}}{1-a_1-\cdots-a_{k-3}} < \min\left\{1,\ 2b_{k-2} \frac{n_1(n_1-1)(n_{k-1}-p_{k-1})(n_{k-1}-p_{k-1}-3)}{n_{k-1}(n_{k-1}-1)(n_1-p_1)(n_1-p_1+3)}\right\},$$

$$\frac{a_{k-1}}{1-a_1-\cdots-a_{k-2}} < \min\left\{2,\ 2b_{k-1} \frac{n_1(n_1-1)(n_k-p_k)(n_k-p_k-3)}{n_k(n_k-1)(n_1-p_1)(n_1-p_1+3)}\right\}.$$

Then we have

$$\mathrm{Cov}(\hat{\eta}_k) \leq \mathrm{Cov}(\hat{\eta}_{k-1}) \leq \cdots \leq \mathrm{Cov}(\hat{\eta}_1) \leq \mathrm{Cov}(\hat{\xi}_1)$$

uniformly for all Σ_i.

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