Ann. Inst. Statist. Math.
Vol. 40, No. 1, 101-110 (1988)

AN EIGENVALUE APPROACH TO THE LIMITING BEHAVIOR OF TIME SERIES AGGREGATES

WILLIAM W. S. WEI[1] AND DANIEL O. STRAM[2]

[1]*Department of Statistics, Temple University, Philadelphia, Pennsylvania 19122, U.S.A.*
[2]*Radiation Effects Research Foundation, Minami-ku, Hiroshima 732, Japan*

(Received January 6, 1987; revised July 7, 1987)

Abstract. Many time series variables such as rainfall, industrial production, and sales exist only in some aggregated forms. To see the implication of time series aggregation it is important to know the limiting behavior of the time series aggregates. From the relationship of autocovariances between the underlying time series variable and its aggregates, we show that the limiting behavior of time series aggregates is closely related to the eigenvalues and the eigenvectors of the aggregation operator. Specifically, the vector of admissible autocorrelations of the limiting model for the time series aggregates is the eigenvector associated with the largest eigenvalue of the aggregation transformation. This provides an interesting and simple method for deriving the limiting model for time series aggregates. Systematic sampling of time series can be treated similarly. The method is illustrated with an empirical example.

Key words and phrases: Time series, aggregation, systematic sampling, ARIMA process, autocovariance, limiting model.

1. Introduction

Some time series variables such as the price of a given commodity and the temperature at a given place exist at every time point. Other variables such as industrial production and sales exist only through aggregation over a time interval. Unless the time series analyst is also the data collector, he or she often cannot choose the time unit or interval for which observations are provided. For example, in the stock market the often reported series is the daily closing price obtained by a systematic sampling, and imports and exports are normally aggregated and published by a government agency in terms of a month or a quarter but not for shorter intervals. Thus, the generating time unit for an underlying variable may not be the same as the observation time unit for which data are available. To draw a proper conclusion based on the analysis of time series data, it is of importance to examine the consequence of data sampling and aggregation. Recent studies on this topic include Tiao (1972), Harvey (1981), Abraham (1982), Wei (1982), Weiss (1984), Lutkepohl (1986), Stram and Wei (1986) and many others.

Suppose the underlying generating process $\{X_t\}$ follows an autoregressive integrated moving average ARIMA (p, d, q) process

$$\phi_p(B)(1 - B)^d X_t = \theta_q(B) a_t \, , \tag{1.1}$$

where B is the backshift operator such that $BX_t = X_{t-1}$, $\phi_p(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$, $\theta_q(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$, and $\{a_t\}$ is a white noise process of zero mean and constant variance σ_a^2. We also assume that the autoregressive polynomial $\phi_p(B)$ satisfies the stationarity condition with its roots being outside the unit circle. The roots of the moving average polynomial $\theta_q(B)$ can be outside or on the unit circle. If the roots of the moving average polynomial $\theta_q(B)$ are also outside the unit circle, then the model is said to be invertible. We call a model admissible if both its autoregressive and moving average polynomials satisfy the conditions of stationarity and invertibility. A set of autocorrelations is said to be admissible if it corresponds to an admissible model.

Let the observed time series $\{Y_T\}$ be the m-period nonoverlapping aggregates of $\{X_t\}$ defined as

$$\begin{aligned} Y_T &= \sum_{t=m(T-1)+1}^{mT} X_t \\ &= (1 + B + \cdots + B^{m-1}) X_{mT} \, , \end{aligned} \tag{1.2}$$

where $(1-B)^d X_t$ for some $d \geq 0$ is stationary but not necessarily invertible. Let $W_t = (1-B)^d X_t$ and $U_T = (1 - \mathscr{B})^d Y_T$ where $\mathscr{B}$ is the backshift operator on the aggregate time unit T such that $\mathscr{B} Y_T = Y_{T-1}$. Stram and Wei (1986) show that the autocovariance function $\gamma_W(k)$ for $\{W_t\}$ and the autocovariance function $\gamma_U(k)$ for $\{U_T\}$ are related as follows.

LEMMA 1.1. *$\gamma_U(k) = (1 + B + \cdots + B^{m-1})^{2(d+1)} \gamma_W(mk + (d+1)(m-1))$ where B now operates on the index of $\gamma_W(j)$ such that $B\gamma_W(j) = \gamma_W(j-1)$.*

For a fixed m and when $(1-B)^d X_t$ is also invertible, Stram and Wei (1986) derive exact autoregressive and moving average orders of the aggregate model for $\{Y_T\}$. In a special case when the underlying series $\{X_t\}$ follows an ARIMA(p, d, q) model with $p=0$, i.e., an IMA(d, q) model, they show that the aggregate series $\{Y_T\}$ follows an IMA(d, Q) process with $Q \leq [d + 1 + (q-d-1)/m]$ where $[z]$ denotes the integer part of z. For a limiting case, Tiao (1972) obtains the following interesting result in terms of the binomial coefficients.

LEMMA 1.2. *Suppose that* $\{X_t\}$ *follows an admissible* ARIMA(p, d, q) *model in* (1.1). *When* $m \to \infty$, *the admissible limiting model for* $\{Y_T\}$ *exists and equals the* IMA(d, d) *process whose d-th difference sequence is a stationary process with the following autocorrelation function*

$$\rho_d^{(\infty)}(k) = \frac{\sum_{j=0}^{d-k} \binom{2d+2}{j}(-1)^j(d+1-k-j)^{2d+1}}{\sum_{j=0}^{d} \binom{2d+2}{j}(-1)^j(d+1-j)^{2d+1}}, \tag{1.3}$$

which is independent of p and q.

The result of Lemma 1.2 implies that the limiting process of the time series aggregate from a stationary model is white noise. It follows that the limiting aggregate process from an ARIMA(p, d, q) model in (1,1) with its autoregressive polynomial being stationary but its moving average polynomial not necessarily being invertible is still an IMA(d, d) model. However, unless the underlying moving average polynomial is invertible, the limiting autocorrelation function may not be invertible. Consider the following non-invertible MA(1) process with $\theta_1(B)=(1-B)$:

$$X_t = (1 - B)a_t \,. \tag{1.4}$$

Since

$$\begin{aligned} Y_T &= (1 + B + \cdots + B^{m-1})X_{mT} \\ &= (1 + B + \cdots + B^{m-1})(1 - B)a_{mT} \\ &= (1 - B^m)a_{mT}\,, \end{aligned}$$

it follows that

$$Y_T = (1 - \mathscr{B})e_T\,, \tag{1.5}$$

where the e_T is white noise. That is, the aggregate model is also non-invertible with the moving average polynomial being $(1-\mathscr{B})$. In summary, we have the following lemma.

LEMMA 1.3. *Let* $\{X_t\}$ *be an* ARIMA(p, d, q) *model in* (1.1) *with a stationary autoregressive polynomial* $\phi_p(B)$. *Then the limiting model for* $\{Y_T\}$ *is an* IMA(d, d) *process. Furthermore, if* 1 *is a root of the moving average polynomial for the* $\{X_t\}$ *process, then* 1 *is also a root of the moving average polynomial for* $\{Y_T\}$.

Using the results of Lemmas 1.1, 1.2 and 1.3, we develop in this paper an alternative method to calculate the autocorrelations and hence the parameters of the admissible limiting model of the time series aggregates. We start with the transformation matrix for aggregation in the next section.

2. Aggregation matrix for autocovariances

Let the first $(d+1)$ autocovariances for the aggregate series $\{U_T=(1-\mathscr{B})^d Y_T\}$ be $\gamma_U(0), \gamma_U(1), \ldots, \gamma_U(d)$. Lemma 1.1 implies that

$$\text{(2.1)} \qquad \begin{bmatrix} \gamma_U(0) \\ \gamma_U(1) \\ \vdots \\ \gamma_U(d) \end{bmatrix} = A \begin{bmatrix} \gamma_W(-(d+1)(m-1)) \\ \gamma_W(-(d+1)(m-1)+1) \\ \vdots \\ \gamma_W(0) \\ \vdots \\ \gamma_W(md+(d+1)(m-1)) \end{bmatrix}.$$

The A in (2.1) is a $(d+1)\times N_0$ aggregation matrix, where $N_0=[md+2(d+1)(m-1)+1]$, equal to

$$\begin{bmatrix} \mathbf{c}' & \mathbf{0}_{N_0-N_1} & \\ \mathbf{0}'_m & \mathbf{c}' & \mathbf{0}_{N_0-m-N_1} \\ \mathbf{0}'_{2m} & \mathbf{c}' & \mathbf{0}_{N_0-2m-N_1} \\ \vdots & \vdots & \\ \mathbf{0}'_{dm} & \mathbf{c}' & \end{bmatrix},$$

where $\mathbf{c}'$ is a $1\times N_1$ row vector of c_i, where $N_1=[2(d+1)(m-1)+1]$. The c_i's are the coefficients of B^i in the polynomial $(1+B+\cdots+B^{m-1})^{2(d+1)}$, and $\mathbf{0}'_n$ is a $1\times n$ row vector of zeroes. Since $\gamma_W(-k)=\gamma_W(k)$ for all k, we can delete the first $(d+1)(m-1)$ columns in A corresponding to $\gamma_W(-(d+1)(m-1)), \ldots, \gamma_W(-1)$ by adding them to the columns corresponding to $\gamma_W((d+1)(m-1)), \ldots, \gamma_W(1)$, respectively. This gives

$$\text{(2.2)} \qquad \begin{bmatrix} \gamma_U(0) \\ \gamma_U(1) \\ \vdots \\ \gamma_U(d) \end{bmatrix} = A_m^d \begin{bmatrix} \gamma_W(0) \\ \gamma_W(1) \\ \vdots \\ \gamma_W(md+(d+1)(m-1)) \end{bmatrix},$$

where A_m^d is the corresponding $(d+1)\times(md+(d+1)(m-1)+1)$ matrix resulting from this deletion and addition.

To study the limiting model, by Lemmas 1.2 and 1.3, we need only consider the case where the underlying process X_t follows an IMA(d, d) model. Hence $\gamma_W(j)=0$ for $j>d$, and we can further eliminate these $\gamma_W(j)$'s and the unnecessary columns in A corresponding to $\gamma_W(j)$ for $j>d$. The resulting A_m^d becomes a $(d+1)\times(d+1)$ square matrix. We denote this matrix by $A_m^d(d)$ to emphasize the fact that this is the aggregation matrix of the linear transformation which maps the autocovariances, $\gamma_W(i)$'s, of the underlying IMA(d, d) model for $\{X_t\}$ to the autocovariances, $\gamma_U(j)$'s, of the IMA(d, d) aggregate model for $\{Y_T\}$, when Y_T is the m-period nonoverlapping aggregate of X_t's. From the construction, it is easily seen that this aggregation matrix $A_m^d(d)$ is nonsingular.

3. Eigenstructure of the aggregation matrix and the limiting aggregate model

For a given autocorrelation vector $\boldsymbol{\rho}'_W=(\rho_W(0), \rho_W(1),\ldots, \rho_W(d))$ where $\rho_W(i)=\gamma_W(i)/\gamma_W(0)$, let $\boldsymbol{\rho}'_U=(\rho_U(0), \rho_U(1),\ldots, \rho_U(d))$ be the corresponding autocorrelation vector for the aggregate series where $\rho_U(i)=\gamma_U(i)/\gamma_U(0)$. It is easy to see that

$$\boldsymbol{\rho}_U = A_m^d(d)\boldsymbol{\rho}_W/\boldsymbol{a}_1'\boldsymbol{\rho}_W \,, \tag{3.1}$$

where $\boldsymbol{a}_1'$ is the first row of $A_m^d(d)$. Note that

$$\boldsymbol{a}_1'\boldsymbol{\rho}_W = \boldsymbol{a}_1'\boldsymbol{\gamma}_W/\gamma_W(0) \,, \tag{3.2}$$

where $\boldsymbol{\gamma}_W=(\gamma_W(0), \gamma_W(1),\ldots, \gamma_W(d))'$. Now $\gamma_W(0)$ is the variance of $\{W_t\}$ and by (2.2), $\boldsymbol{a}_1'\boldsymbol{\gamma}_W=\gamma_U(0)$ which is the variance of $\{U_T\}$. Hence, $\boldsymbol{a}_1'\boldsymbol{\rho}_W>0$ and the aggregation transformation defined in (3.1) is continuous over the set of $\{\boldsymbol{\rho}_W\}$. This leads to the following result.

LEMMA 3.1. *Let f^m be the m-period aggregation transformation given in* (3.1). *If $\lim_{m\to\infty} f^m(\boldsymbol{\rho}_W)= \boldsymbol{\rho}$, then $f^m(\boldsymbol{\rho})=\boldsymbol{\rho}$.*

PROOF. $\lim_{m\to\infty} f^m(\boldsymbol{\rho}_W)=\boldsymbol{\rho}$ implies that $\lim_{k\to\infty} f^{m^k}(\boldsymbol{\rho}_W)=\boldsymbol{\rho}$. But $f^{m^k}=f^m(f^{m^{k-1}})$. Thus

$$\boldsymbol{\rho} = \lim_{k\to\infty} f^m(f^{m^{k-1}}(\boldsymbol{\rho}_W)) = f^m(\lim_{k\to\infty} f^{m^{k-1}}(\boldsymbol{\rho}_W)) = f^m(\boldsymbol{\rho}) \,. \qquad \square$$

Lemma 3.1 implies that the autocorrelation vector of a limiting model is unaffected by aggregation. Hence by (3.1) the limiting autocorrelation vector $\boldsymbol{\rho}$ is seen to be the eigenvector of the aggregation matrix $A_m^d(d)$ with the associated eigenvalue $\boldsymbol{a}_1'\boldsymbol{\rho}$. The problem is that, unless the model is properly specified, $\boldsymbol{\rho}$ may not be admissible. For example, by Lemma 1.3 if $\{X_t\}$ follows a non-invertible IMA(1,1) model, the limiting process for $\{Y_T\}$ is also a non-invertible IMA(1,1) model. In other words, the aggregate of a mis-specified non-invertible MA(1) model $W_t=(1-B)a_t$ where $W_t=(1-B)X_t$ is again a non-invertible MA(1) model $U_T=(1-\mathscr{B})e_T$ where $U_T=(1-\mathscr{B})Y_T$ and e_T

is white noise. Hence $\boldsymbol{\rho}'=[1,1/2]$ will be one of the eigenvectors for the aggregation matrix $A_m^1(1)$ relating the misspecified variables W_T and U_T. But this $\boldsymbol{\rho}$ is not admissible. More generally, let $\theta_i^{(\infty)}(\mathscr{B})$ be the i-th order MA polynomial corresponding to an admissible limiting model. Then for any given d,

$$U_T = (1 - \mathscr{B})^d Y_T = (1 - \mathscr{B})^i \theta_{d-i}^{(\infty)}(\mathscr{B}) e_T ,$$

for $i=0, 1,\ldots, d$ are $(d+1)$ limiting models whose autocorrelation vectors are unaffected by aggregation, although all these models other than $i=0$ are misspecified and hence can be simplified. These $(d+1)$ autocorrelation vectors are clearly linearly independent as their generating polynomials $(1-\mathscr{B})^i\theta_{d-i}^{(\infty)}(\mathscr{B})$ for $i=0, 1,\ldots, d$ are clearly not linearly related, as each of the polynomials for $i=0, 1,\ldots, d$ contains exactly i roots equal to 1.

Since in application, especially in forecasting, we often consider only an invertible model, we shall prove the following useful theorem.

THEOREM 3.1. *The admissible autocorrelation vector* $\boldsymbol{\rho}_d^{(\infty)}$ *of the limiting aggregate model*

$$(1 - \mathscr{B})^d Y_T = \theta_d^{(\infty)}(\mathscr{B}) e_T ,$$

is the eigenvector associated with the largest eigenvalue, $\lambda_{\max}$, *of* $A_m^d(d)$, *where an eigenvector is scaled so that its first element equals* 1.

PROOF. By Lemmas 1.2 and 3.1, we need consider only the case where the underlying process follows an admissible IMA(d, d) model. Let $\boldsymbol{\rho}_W'=(\rho_W(0), \rho_W(1),\ldots, \rho_W(d))$ be its admissible autocorrelation vector. We have

$$\lim_{k\to\infty} f^{m^k}(\boldsymbol{\rho}_W) = \boldsymbol{\rho}_d^{(\infty)} ,$$

where we take the limit through m^k. In terms of $A_m^d(d)$, we have

$$\lim_{k\to\infty} (A_m^d(d))^k \boldsymbol{\rho}_W / \boldsymbol{\alpha}_{1,mk}' \boldsymbol{\rho}_W = \boldsymbol{\rho}_d^{(\infty)} ,$$

where $\boldsymbol{\alpha}_{1,mk}'$ is the first row of the matrix $(A_m^d(d))^k$ which can be easily seen to be equal to $A_{mk}^d(d)$. Now, for given d let $\boldsymbol{\rho}_{d-i}^{(\infty)}$ be equal to the autocorrelation vector of the model

$$U_T = (1 - \mathscr{B})^d Y_T = (1 - \mathscr{B})^i \theta_{d-i}^{(\infty)}(\mathscr{B}) e_T ,$$

for $i=0, 1,\ldots, d$. Since these autocorrelation vectors are linearly independent, they form a basis for R^{d+1}, and we can write

(3.3) $$\boldsymbol{\rho}_W = \sum_{i=0}^{d} c_i \boldsymbol{\rho}_{d-i}^{(\infty)} ,$$

for some constants c_i which are not all equal to 0. Let $\boldsymbol{\rho}_{\max}^{(\infty)}$ be the eigenvector associated with $\lambda_{\max}$. We want to show that $\boldsymbol{\rho}_{\max}^{(\infty)} = \boldsymbol{\rho}_d^{(\infty)}$. To prove this, let $c_{\max}$ be the constant which multiplies $\boldsymbol{\rho}_{\max}^{(\infty)}$ in (3.3). From Stram and Wei (1986) we know that the set of admissible autocorrelations for an IMA(d, d) model is open. Thus we can choose an admissible $\boldsymbol{\rho}_W$ so that $c_{\max} \neq 0$. Now

$$(A_m^d(d))^k \boldsymbol{\rho}_W = \sum_{i=0}^{d} c_i \lambda_i^k \boldsymbol{\rho}_{d-i}^{(\infty)} ,$$

and $\boldsymbol{a}_{1,mk}' \boldsymbol{\rho}_W = \sum_{i=0}^{d} c_i \lambda_i^k$ where λ_i is the associated eigenvalue for the eigenvector $\boldsymbol{\rho}_{d-i}^{(\infty)}$ of $A_m^d(d)$. Hence

$$\begin{aligned} \boldsymbol{\rho}_d^{(\infty)} &= \lim_{k\to\infty} f^{m^k}(\boldsymbol{\rho}_W) = \lim_{k\to\infty} (A_m^d(d))^k \boldsymbol{\rho}_W / \boldsymbol{a}_{1,mk}' \boldsymbol{\rho}_W \\ &= \lim_{k\to\infty} \left\{ \left(\sum_{i=0}^{d} c_i \lambda_i^k \boldsymbol{\rho}_{d-i}^{(\infty)} \right) \Big/ \left(\sum_{i=0}^{d} c_i \lambda_i^k \right) \right\} = \boldsymbol{\rho}_{\max}^{(\infty)} . \end{aligned}$$ □

The parameter value θ_i in $\theta_d^{(\infty)}(\mathscr{B})$ can then be obtained through the relationship between $\rho_d^{(\infty)}(i)$'s and θ_i's of a moving average model.

4. An illustrative example with concluding remarks

To illustrate the above results, let us consider the monthly unemployed females per 1000 persons between the ages of 16 and 19 in the United States from January 1961 to December 1985. The data is from the "Economic Report of the President" published by the United States Government Printing Office, Washington, D.C. from 1961 to 1985. There are a total of 300 observations. Tables 1 and 2 show the sample autocorrelation functions of the original series and its first differences. It is evident that the original series $\{X_t\}$ is nonstationary and the series of its first differences $\{(1-B)X_t\}$ is stationary.

Table 1. Sample autocorrelations $\hat{\rho}_Z(k)$ of $\{X_t\}$.

k	1	2	3	4	5	6	7	8
$\hat{\rho}_Z(k)$	.97	.96	.95	.94	.93	.93	.92	.91

Table 2. Sample autocorrelations $\hat{\rho}_W(k)$ of $\{(1-B)X_t\} = \{W_t\}$.

k	1	2	3	4	5	6	7	8
$\hat{\rho}_W(k)$	−.41	.06	−.08	.06	−.09	.07	−.03	.07

To examine the limiting behavior of the autocorrelation function of the aggregates, we compute the sample autocorrelations of Y_T for $m=6$ and $m=12$ corresponding to semi-annual and annual total of the unemployed females. The autocorrelations for $\{Y_T\}$ show nonstationarity. Table 3 shows the autocorrelations of the differenced aggregates, i.e., $U_T=(1-\mathscr{B})Y_T$.

Is the above phenomenon expected? To answer the question, let us derive the admissible autocorrelations $\rho_1^{(\infty)}(0)$ and $\rho_1^{(\infty)}(1)$ of the limiting aggregate model

$$(1-\mathscr{B})Y_T=\theta_1^{(\infty)}(\mathscr{B})e_T\,.$$

Take $m=2$ and consider the limit of m^k as $k\to\infty$. For $m=2$ and $d=1$, Lemma 1.1 gives $\gamma_U(k)=(1+B)^4\,\gamma_W(2k+2)$, and (2.1) and (2.2) imply that

$$A=\begin{bmatrix}1&4&6&4&1&0&0\\0&0&1&4&6&4&1\end{bmatrix},$$
$$A_2^1=\begin{bmatrix}6&8&2&0&0\\1&4&6&4&1\end{bmatrix},$$

and hence

$$A_2^1(1)=\begin{bmatrix}6&8\\1&4\end{bmatrix}.$$

The characteristic equation, $\det(A_2^1(1)-\lambda I)=0$, is

$$\lambda^2-10\lambda+16=0\,,$$

and the eigenvalues are easily seen to be 2 and 8. Thus, $\lambda_{\max}=8$ and its corresponding eigenvector is the solution of the system of equations, $(A_2^1(1)-8I)\boldsymbol{x}=\mathbf{0}$, i.e.

Table 3. Sample autocorrelations $\hat{\rho}_U(k)$ of $\{(1-\mathscr{B})Y_t\}=\{U_t\}$.

$m=6$

k	1	2	3	4	5	6	7	8
$\hat{\rho}_U(k)$	.34	.03	.05	−.20	−.12	−.10	−.16	−.01

$m=12$

k	1	2	3	4	5	6	7	8
$\hat{\rho}_U(k)$	.25	−.21	−.22	−.03	.19	.08	.21	−.17

$$\begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\operatorname{rank}(A_2^1(1)-8I)=1$, there is one free variable and it can easily be shown that an associated eigenvector is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{4}x_1 \end{bmatrix}.$$

The admissible autocorrelation structure of the limiting aggregate model becomes

$$\boldsymbol{\rho}_1^{(\infty)} = \begin{bmatrix} \rho_1^{(\infty)}(0) \\ \rho_1^{(\infty)}(1) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}.$$

Thus, the result shown in Table 3 is expected from the limiting behavior of temporal aggregates.

This result may not be evident purely based on empirical values. For example, with 300 observations of X_t, Table 2 clearly implies an IMA(1,1) model for $\{X_t\}$. The result with $m=6$ as shown in Table 3 also implies an IMA(1,1) model for the aggregates $\{Y_t\}$. When $m=12$, the standard deviation of $\hat{\rho}_U(k)$ is approximately equal to .2 and hence as shown in Table 3, the sample autocorrelation for this case indicates a white noise phenomenon. However, while the value of $\hat{\rho}_U(k)$ is $-.41$ for $\{(1-B)X_t\}$, it changes to .34 for $\{(1-\mathscr{B})Y_t\}$ when $m=6$ and to .25 for $\{(1-\mathscr{B})Y_T\}$ when $m=12$. This reduction in $\hat{\rho}_U(k)$ is a direct effect of temporal aggregation. The limiting model IMA(1,1) is a correct one to be used for the aggregates and not the white noise model.

It is interesting and useful to note that in deriving the limiting aggregate model the m in $A_m^d(d)$ can be any integer larger than or equal to 2. The limit can then be taken through m^k as $k\to\infty$. Specifically, as shown in the above example, we can choose $m=2$ which greatly simplifies the calculation of the coefficients of B^i in the polynomial $(1+B+\cdots+B^{m-1})^{2(d+1)}$ and hence the construction of $A_m^d(d)$.

Systematic sampling of time series can be treated in much the same fashion as aggregation is here. This follows because the relationship between the autocovariances of an ARIMA(p, $d+1$, q) series X_t and those of a systematically sampled series, $Z_T=X_{mT}$, is exactly the same as between those of a basic ARIMA(p, d, q) X_t series and those of an aggregated Y_T series. Thus, let $S_m^{d+1}(d)$ be the systematic sampling operator. We have $S_m^{d+1}(d)=A_m^d(d)$, and as shown in Wei (1981) they have the same eigenstructures.

Acknowledgements

The authors would like to thank the referees for some helpful comments on an earlier version of this paper.

REFERENCES

Abraham, B. (1982). Temporal aggregation and time series, *Internat. Statist. Rev.*, **50**, 285-291.

Harvey, A. C. (1981). *Time Series Models*, Halstead Press, Oxford.

Lutkepohl, H. (1986). Forecasting temporaneously aggregated vector ARMA processes, *Journal of Forecasting*, **5**(2), 85-95.

Stram, D. and Wei, W. W. S. (1986). Temporal aggregation in the ARIMA process, *J. Time Ser. Anal.*, **7**(4), 279-292.

Tiao, G. C. (1972). Asymptotic behavior of time series aggregates, *Biometrika*, **59**, 523-531.

Wei, W. W. S. (1981). Effect of systematic sampling on ARIMA models, *Comm. Statist. A—Theory Methods*, **10**(23), 2389-2398.

Wei, W. W. S. (1982). The effects of systematic sampling and temporal aggregation on causality—a cautionary note on Geweke's paper, *J. Amer. Statist. Assoc.*, **77**(378), 316-319.

Weiss, A. A. (1984). Systematic sampling and temporal aggregation in time series models, *J. Econometrics*, **26**, 271-281.