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STOCHASTIC NEURODYNAMICS

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Abstract. Stochastic dynamics of relative membrane potential in the neural network is investigated. It is called stochastic neurodynamics. The least action principle for stochastic neurodynamics is assumed, and used to derive the fundamental equation. It is called a neural wave equation. A solution of the neural wave equation is called a neural wave function and describes stochastic neurodynamics completely. Linear superposition of neural wave functions provides us with a mathematical model of associative memory process. As a simple application of stochastic neurodynamics, a mathematical representation of static neurodynamics in terms of equilibrium statistical mechanics of spin system is derived.

Key words and phrases: Neurodynamics, neural holography, neural wave equation, neural network, associative memory.

1. Kinematics of nerve system

We are now looking for a faithful mathematical model of neurodynamics. Consider a large scale integration of neurons up to the order 10^{10} in a typical cell assembly of the cerebrum. Each neuron is connected with many other ones via about 10^4 chemical synapses, electrical ephapses and tight junctions and the totality of neurons can be considered as a huge dynamical system. We call it a nerve system or a neural network. A nerve system is said to be closed if no neurons in it are connected with external neurons. If a nerve system is not closed, it is said to be open. Among neurons of any open nerve system, those connected directly with external ones are called visible neurons. Those remaining are called hidden neurons.

Let us consider, here, the degrees of freedom belonging to a neuron in the nerve system. Neurological action of the neuron can be well described by illustrating the temporal changes of cell membrane potential and firing threshold. Let t be a time parameter in milli-second unit taking continuous values between 0 and ∞. Let $u(t)$ and $\theta(t)$ be values in milli-volt unit of membrane potential and firing threshold of the neuron at each time t. Firing mechanism of neuron is well known:

The neuron fires and generates impulse if and only if $u(t)>\theta(t)$.

The threshold $\theta(t)$ is slowly-varying compared with the membrane potential $u(t)$. Therefore, it is convenient to introduce a reduced degree of freedom $x(t)$ by

$$x(t) = u(t) - \theta(t) , \tag{1.1}$$

which will be called a relative membrane potential of the neuron. Then the neuron is active if $x(t)>0$, and not if $x(t)<0$. We assume the fundamental equation for kinematics of the relative membrane potential $x(t)$ of the neuron,

$$dx(t) = A(t)dt + dw(t) . \tag{1.2}$$

This is a stochastic differential equation (Nelson (1967)). The dynamical variable $A(t)$ describing the total electric current flowing into the membrane is called a drift. It represents the purely electric interaction between the neuron in question and others via neural connections. Therefore, $A(t)$ can be thought to depend on relative membrane potentials of all neurons in the nerve system. The second term in the right-hand side of equation (1.2) represents the fluctuating contribution to the membrane potential due to the spontaneous fluctuation (Abeles (1982), Buhmann and Schulten (1986)). It is given by a stochastic differential of a Gaussian stochastic process $w=\{w(t)|0\leq t<\infty\}$ called a Wiener process such that

$$E[dw(t)] = 0 , \tag{1.3}$$

$$E[|dw(t)|^2] = \nu dt , \tag{1.4}$$

where $E[\cdot]$ means to take the mathematical expectation and the diffusion constant ν is of the order of 10 (milli-volt)^2/(milli-second) (Buhmann and Schulten (1986)).

It is a standard result of the probability theory that the stochastic differential equation (1.2) determines a stochastic process $x=\{x(t)|0\leq t<\infty\}$ called semi-martingale if the drift $A(t)$ and the initial condition $x(0)$ are given (Nelson (1967)). Furthermore, the drift $A(t)$ is a mean forward derivative of the semi-martingale x defined by

$$\begin{aligned} Dx(t) &\equiv \lim_{h\to 0+} E_t\left[\frac{x(t+h) - x(t)}{h} \right] \\ &= A(t) . \end{aligned} \tag{1.5}$$

Here, $E_t[\cdot]$ means to take the conditional expectation given the present value $x(t)$. We call the drift $A(t)$ a mean forward current of the relative membrane potential $x(t)$.

From the mathematical point of view, a mean backward derivative of the semi-martingale x,

$$D_* x(t) \equiv \lim_{h \to 0+} E_t\left[\frac{x(t) - x(t-h)}{h} \right], \tag{1.6}$$

can be considered also as the total electric current flowing into the membrane. We call it a mean backward current of the relative membrane potential $x(t)$.

Kinematics of a neuron can be illustrated in those three "coordinates", one for the relative membrane potential, one for the mean forward current and one for the mean backward current. Now we proceed further to kinematics of a nerve system made up of N neurons, where N is of the order $O(10^{10})$. Since the activity of each neuron can be manifested in the coordinate of relative membrane potential, we use those of all neurons as a fundamental dynamical variable of the nerve system. Let $x_i(t)$ be relative membrane potential of the i-th neuron at each time t. Considering an ordered set of relative membrane potentials $\boldsymbol{x}(t) \equiv (x_1(t), x_2(t), \ldots, x_N(t))$, we can illustrate the dynamical variable of the nerve system by a vector $\boldsymbol{x}(t)$ in an N-dimensional coordinate space $\boldsymbol{R}^N$. This extremely higher dimensional space is called a configuration space of the nerve system and denoted by $\mathscr{X}$. Then, total action of the nerve system can be described by a stochastic process $\boldsymbol{x} = \{\boldsymbol{x}(t) \mid 0 \le t < \infty\}$ in $\mathscr{X}$ subject to a stochastic differential equation

$$d\boldsymbol{x}(t) = \boldsymbol{A}(t)dt + d\boldsymbol{w}(t) . \tag{1.7}$$

Here, $\boldsymbol{A}(t) = (A_1(t), A_2(t), \ldots, A_N(t))$, and $A_i(t)$ denotes the mean forward current of the i-th neuron for $i = 1, 2, \ldots, N$. The stochastic process $\boldsymbol{w} = \{\boldsymbol{w}(t) = (w_1(t), w_2(t), \ldots, w_N(t)) \mid 0 \le t < \infty\}$ is an N-dimensional Wiener process such that

$$E[dw_i(t)] = 0 , \tag{1.8}$$

$$E[dw_i(t) dw_j(t)] = v dt \delta_{ij} , \tag{1.9}$$

where δ_{ij} is a Kronecker's delta symbol.

Kinematics of the nerve system described by equation (1.7) is a direct result of that of each neuron given by equation (1.2). Recall that the mean forward current of a neuron depends on relative membrane potentials of all the neurons by the electric interaction via neural connections. Therefore, we may write down

$$\text{(1.10)} \qquad A_i(t) = a_i(x_1(t), x_2(t), \ldots, x_N(t), t) = a_i(\boldsymbol{x}(t), t),$$

for $i=1, 2, \ldots, N$, where $a_i = a_i(\boldsymbol{x}, t)$'s are certain functions of coordinates $\boldsymbol{x}=(x_1, x_2, \ldots, x_N)$ in the configuration space $\mathscr{X}$ and time t in the interval $[0, \infty)$. It seems worthwhile to notice here that there is no retardation in the electric interaction between neurons. Propagation speed of such an electric interaction is much higher than any other ones taking place in the brain. This is the reason why we can assume the validity of equation (1.10). Putting $\boldsymbol{a}(\boldsymbol{x}, t) \equiv (a_1(\boldsymbol{x}, t), a_2(\boldsymbol{x}, t), \ldots, a_N(\boldsymbol{x}, t))$, we obtain the fundamental equation for kinematics of the nerve system,

$$\text{(1.11)} \qquad d\boldsymbol{x}(t) = \boldsymbol{a}(\boldsymbol{x}(t), t)dt + d\boldsymbol{w}(t).$$

This is a stochastic differential equation of Itô type in the N-dimensional configuration space $\mathscr{X}$. It is known that the stochastic differential equation of Itô type determines a specific stochastic process called a diffusion process (Nelson (1967)). The time development of relative membrane potentials $x_i(t)$'s is now represented by a diffusion process $\boldsymbol{x}=\{\boldsymbol{x}(t) | 0 \leq t < \infty\}$ in $\mathscr{X}$. It is an N-dimensional Markov process, though each component process $x_i = \{x_i(t) | 0 \leq t < \infty\}$ is not.

Thus, we arrived at a mathematical model of the nerve system in which the total action of relative membrane potentials manifests an N-dimensional Markov process $\boldsymbol{x}=\{\boldsymbol{x}(t) | 0 \leq t < \infty\}$ in the configuration space $\mathscr{X}$. The ordered sets of relative membrane potentials $x_i(t)$'s, mean forward currents $Dx_i(t) = a_i(\boldsymbol{x}(t), t)$'s and mean backward currents $D_* x_i(t)$'s form three basic dynamical variables of the nerve system, $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t))$, $D\boldsymbol{x}(t) \equiv (Dx_1(t), Dx_2(t), \ldots, Dx_N(t))$ and $D_*\boldsymbol{x}(t) \equiv (D_*x_1(t), D_*x_2(t), \ldots, D_*x_N(t))$.

2. Dynamics of nerve system

We proceed further to dynamics of the relative membrane potentials which represents neurodynamics in the present mathematical model of the nerve system. If the total action of neural network manifests a systematic order, there might be a certain dynamical law of neurodynamics. In our model, the relative membrane potentials $\boldsymbol{x}(t)$, the mean forward currents $D\boldsymbol{x}(t)$ and the mean backward currents $D_*\boldsymbol{x}(t)$ are the basic dynamical variables. Kinematics of the nerve system is described by the N-dimensional Markov process $\boldsymbol{x}=\{\boldsymbol{x}(t) | 0 \leq t < \infty\}$ governed by the stochastic differential equation of Itô type (1.11). Therefore, it seems natural to suppose the fundamental dynamical law of neurodynamics in a form of variational (i.e., optimal) principle with respect to a certain universal quantity corresponding to the action account, the energy account or the entropy production (i.e., the negentropy account). We call the universal quantity to be minimized an action functional of the nerve system, and denote it by $J[x]$. The variational principle with respect to the action functional J will be called a least action principle. The action functional J may take different values for different stochastic processes. The least action principle claims that the real action of the nerve system is described by a Markovian diffusion process $x=\{x(t)|0\leq t<\infty\}$ of the relative membrane potentials which achieves the minimum or minimal value of the action functional J. From the point of view of optimal control theory, such a variational dynamical law as the least action principle can be thought of as a feedback control mechanism. Namely the mean forward and backward currents, $Dx(t)=a(x(t), t)$ and $D_*x(t)$, control the Markov process $x=\{x(t)|0\leq t<\infty\}$ so that it achieves the minimum value of the action functional (i.e., the cost function).

Therefore, the action functional J may have a typical universal form for any semi-martingales (not necessarily Markovian diffusion processes) $y=\{y(t)|0\leq t<\infty\}$'s in the N-dimensional configuration space $\mathscr{X}$,

$$J[y] = E\left[\int_a^b \mathscr{L}(y(t),\ Dy(t),\ D_*y(t))dt\right]. \tag{2.1}$$

Here, $\mathscr{L}$ denotes a given function of dynamical coordinates, that is, the relative membrane potentials, the mean forward currents and the mean backward currents. The upper and lower limits, a and b, are arbitrary provided that $0\leq a<b<\infty$. Explicit form of the function $\mathscr{L}$ thus characterizes the total action of a nerve system, that is, neurodynamics. We call it a Lagrange function of the nerve system.

$$\mathscr{L}(x(t),\ Dx(t),\ D_*x(t)) = \frac{1}{2}\left(\frac{1}{2}\,|Dx(t)|^2 + \frac{1}{2}\,|D_*x(t)|^2\right) - U(x(t))\,. \tag{2.2}$$

Here, $|Dx(t)|^2\equiv\sum_{i=1}^{N}(Dx_i(t))^2$ and $|D_*x(t)|^2\equiv\sum_{i=1}^{N}(D_*x_i(t))^2$, and U is a certain function of coordinates $x=(x_1, x_2,\ldots, x_N)$ in the configuration space $\mathscr{X}$. This function U may be understood to represent the total electrostatic energy contained in the configuration of relative membrane potentials of the nerve system. We call it a potential energy of the nerve system. Correspondingly, the first term in the right-hand side of equation (2.2) can be thought as the electrokinetic energy contained in the configuration of mean forward and backward currents. We call it a kinetic energy of the nerve system.

Fundamental theorem of stochastic calculus of variations (Yasue (1981*a*, 1981*b*, 1983), Zambrini (1985)) asserts that the Markovian diffusion process $x=\{x(t)|0\leq t<\infty\}$ in the configuration space $\mathscr{X}$ governed by the stochastic differential equation of Itô type (1.11) is a stationary or critical point of the action functional J if it is subject to the following equation.

$$D\frac{\partial \mathscr{L}}{\partial D_{*}\boldsymbol{x}(t)} + D_{*}\frac{\partial \mathscr{L}}{\partial D\boldsymbol{x}(t)} - \frac{\partial \mathscr{L}}{\partial \boldsymbol{x}(t)} = 0 . \tag{2.3}$$

Substituting the explicit form (2.2) of the Lagrange function $\mathscr{L}$ into equation (2.3), we obtain

$$\frac{1}{2}\left(DD_{*}\boldsymbol{x}(t) + D_{*}D\boldsymbol{x}(t)\right) = -\nabla U(\boldsymbol{x}(t)) , \tag{2.4}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_N)$ denotes the gradient operator in the N-dimensional configuration space $\mathscr{X}$. This is the fundamental equation which describes the dynamical law of the action of a nerve system. It will be called a neural equation of action of the nerve system.

3. Stochastic neurodynamics

It is known that equations (2.4) and (1.11) are equivalent to the following linear partial differential equation (Nelson (1984), Blanchard *et al.* (1987)).

$$i\nu\frac{\partial \psi(\boldsymbol{x}, t)}{\partial t} = -\frac{\nu^2}{2}\Delta\psi(\boldsymbol{x}, t) + U(\boldsymbol{x})\psi(\boldsymbol{x}, t) , \tag{3.1}$$

for a complex-valued function ψ of the coordinates $\boldsymbol{x}$ in the configuration space $\mathscr{X}$ and time t. Due to the imaginary factor in the time derivative, this equation becomes hyperbolic. This means that the dynamics of nerve systems is essentially undulatory. It seems natural, therefore, to call it a neural wave equation. The complex-valued function $\psi(\boldsymbol{x}, t)$ subject to the neural wave equation will be called a neural wave function of the nerve system.

We have obtained the fundamental equation of stochastic neurodynamics, that is, the neural wave equation (3.1). Since it is a linear partial differential equation of hyperbolic type, analysis of action of a nerve system can be performed within the realm of conventional mathematical analysis. Equation (3.1) is of the same form as the many particle wave equation in quantum theory. It is well-known that such a wave equation as (3.1) has proper solutions in a class of square integrable functions defined on the configuration space $\mathscr{X}$. Therefore, we can assume the existence of solutions to the fundamental equations (1.11) and (2.4). To get solutions we assume the neural wave function of the form

$$\psi(\boldsymbol{x}, t) = \exp\{R(\boldsymbol{x}, t) + iS(\boldsymbol{x}, t)\} , \tag{3.2}$$

compute

$$R(\boldsymbol{x}, t) = \log|\psi(\boldsymbol{x}, t)| , \tag{3.3}$$

and

$$S(\boldsymbol{x}, t) = \arg \psi(\boldsymbol{x}, t) , \tag{3.4}$$

then find

$$\boldsymbol{a}(\boldsymbol{x}, t) = \nu(\nabla S(\boldsymbol{x}, t) + \nabla R(\boldsymbol{x}, t)) , \tag{3.5}$$

$$\boldsymbol{a}_*(\boldsymbol{x}, t) = \nu(\nabla S(\boldsymbol{x}, t) - \nabla R(\boldsymbol{x}, t)) , \tag{3.6}$$

and

$$\begin{aligned} p(\boldsymbol{x}, t) &= e^{2R(\boldsymbol{x}, t)} \\ &= |\psi(\boldsymbol{x}, t)|^2 . \end{aligned} \tag{3.7}$$

These functions $\boldsymbol{a}$, $\boldsymbol{a}_*$ and p determine the Markovian diffusion process $\boldsymbol{x}=\{\boldsymbol{x}(t)|0\leq t<\infty\}$ subject to the least action principle such that $D\boldsymbol{x}(t)=\boldsymbol{a}(x(t), t)$, $D_*\boldsymbol{x}(t)= \boldsymbol{a}_*(\boldsymbol{x}(t), t)$ and $p(\boldsymbol{x}, t)$ is the probability distribution density of $\boldsymbol{x}(t)$.

We have found the following guiding principles in stochastic neurodynamics:

1) Action of the nerve system is described by a Markovian diffusion process of relative membrane potentials of neurons.

2) The Markovian diffusion process is completely determined by the neural wave equation.

The neural wave equation (3.1), therefore, becomes a key concept of stochastic neurodynamics and plays an important role in the investigation of fundamental thought processes. It seems surprising that the action of a nerve system is described by a wave equation similar to that of a many particle system in quantum theory. It is merely a formal similarity. Stochastic neurodynamics has nothing in common with conceptual aspect of quantum theory. However, the formal similarity may arise naturally in the following sense:

In quantum theory, the uncertainty principle gives intrinsic disorder in the motion of many particles. A systemized mechanism in such disordered motion of many particles could be found as a statistical dynamical law described by the wave equation.

In stochastic neurodynamics, spontaneous release of calcium ion gives disorder or fluctuation in the action of relative membrane potentials. A systemized function of fundamental thought processes in such a fluctuating action of the nerve system could be found as a statistical time evolution of relative membrane potentials described by the neural wave equation (3.1).

Such a formal similarity to quantum theory may arise frequently when one looks for a systemized mechanism in high disorder of huge elements. Indeed, Nagasawa (1980, 1981) applied the "wave equation" formulation of stochastic processes to population growing in biology. Albeverio *et al.* (1983, 1986) applied it to pattern formation problems in both earth and space sciences. Among them are confinement of winds in zones on the surface of planets, formation of jet-streams in the protosolar nebura, morphology of galaxies, and stratification of galaxy distribution in the universe.

4. Geometric representation of association

In stochastic neurodynamics, fundamental perceptual processes will be described in terms of Markov processes of relative membrane potentials or equivalently neural wave functions of the nerve system. They are governed by the neural wave equation (3.1). (Recall the guiding principles 1 and 2.)

Following the guiding principles, we can investigate fundamental perceptual processes by means of neural wave functions. We look for mechanisms of memory and perceptual processes characteristic to the mathematical structure of neural wave equation (3.1).

First, we consider a mechanism of memory processes in the nerve system.

Let us start with the neural wave equation (3.1) of the nerve system. As it is well known equation (3.1) can be reduced to the stationary neural wave equation. It defines an eigen value problem in the Hilbert space $\mathscr{H} \equiv L^2(\boldsymbol{R}^N)$,

$$Hu = \lambda u \,, \tag{4.1}$$

where H is a self-adjoint operator on $\mathscr{H}$ defined by

$$H = -\frac{\nu^2}{2}\Delta + U(\boldsymbol{x}) \,. \tag{4.2}$$

We call H an action operator of the nerve system. It is known that for a wider class of potential energy functions the eigen value problem (4.1) has infinitely many solutions h_n's (Kato (1966)). Let λ_n, $n=1, 2, \ldots$, be the n-th eigen value corresponding to the n-th eigen function $h_n = h_n(\boldsymbol{x})$ belonging to $\mathscr{H}$,

$$Hh_n = \lambda_n h_n \,. \tag{4.3}$$

Neural wave functions ψ_n's given by

$$\psi_n(\boldsymbol{x}, t) = h_n(\boldsymbol{x}) \exp\left(-i \frac{\lambda_n}{\nu} t\right) , \tag{4.4}$$

$n=1, 2, \ldots$, are all subject to the neural wave equation (3.1). Therefore, they determine infinitely many N-dimensional stationary Markov processes of relative membrane potentials $\boldsymbol{x}_n = \{\boldsymbol{x}_n(t) \mid 0 \le t < \infty\}$'s with equilibrium probability distributions

$$p_n(\boldsymbol{x}) \equiv |h_n(\boldsymbol{x})|^2 . \tag{4.5}$$

Each stationary Markov process $\boldsymbol{x}_n$ is governed by a stochastic differential equation of Itô type

$$d\boldsymbol{x}_n(t) = \boldsymbol{a}_n(\boldsymbol{x}_n(t))dt + d\boldsymbol{w}(t) \tag{4.6}$$

with

$$\begin{aligned} \boldsymbol{a}_n(\boldsymbol{x}) &= \nu \nabla \log h_n(\boldsymbol{x}) \\ &= \nu \frac{\nabla h_n(\boldsymbol{x})}{h_n(\boldsymbol{x})} . \end{aligned} \tag{4.7}$$

The eigen values λ_n's are assumed to be increasingly ordered, that is,

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots . \tag{4.8}$$

Those eigen values are nothing else but the possible mean total energy of relative membrane potentials of the nerve system,

$$\begin{aligned} \lambda_n &= E\left[\frac{1}{2} \left(\frac{1}{2} \, |D\boldsymbol{x}_n(t)|^2 + \frac{1}{2} \, |D_* \boldsymbol{x}_n(t)|^2 \right) + U(\boldsymbol{x}_n(t)) \right] \\ &= E\left[\frac{1}{2} \, |\boldsymbol{a}_n(\boldsymbol{x}_n(t))|^2 + U(\boldsymbol{x}_n(t)) \right] \\ &= \int_{\mathscr{X}} \left\{ \frac{1}{2} \, |\boldsymbol{a}_n(\boldsymbol{x})|^2 + U(\boldsymbol{x}) \right\} p_n(\boldsymbol{x}) d^N x \\ &= \int_{\mathscr{X}} \left\{ \frac{1}{2} \, |\nabla h_n(\boldsymbol{x})|^2 + U(\boldsymbol{x}) \, |h_n(\boldsymbol{x})|^2 \right\} d^N x \\ &= \langle h_n, H h_n \rangle , \end{aligned} \tag{4.9}$$

for $n=1, 2, \ldots$.

No other functions different from the stationary neural wave functions h_n's can describe stable action of the nerve system. In other words, the nerve system with the action function (2.1) (and (2.2)) admits only selectively limited actions of relative membrane potentials represented by each of the stationary neural wave functions h_n's. Therefore, the admissible actions of the nerve system are completely characterized by those eigen functions h_n's which can be considered as vectors in the Hilbert space $\mathscr{H}$.

The neural wave equation (3.1) and the eigen value problem (4.1) provide us with both mechanism and geometric representation of perceptual processes in the nerve system. The perceptual process is realized in the light of Helmholtz' old idea of resonating strings (Pribram (1971)). The nerve system behaves as an assembly of weakly coupled N strings resonating to the selectively limited neural "waves" of relative membrane potentials ψ_n's given by equation (4.4).

We develop now a geometric representation of perceptual processes within the realm of Hilbert space geometry. From the point of view of functional analysis, the eigen functions $\{h_n\}_{n=1}^{\infty}$ usually form a complete normalized orthogonal system of vectors in the Hilbert space $\mathscr{H}$. Namely, each eigen function h_n is normalized in a sense that

$$(4.10) \qquad \begin{aligned} \|h_n\| &\equiv \left(\int_{\mathscr{X}} |h_n(\boldsymbol{x})|^2 d^N x\right)^{1/2} \\ &= (\langle h_n, h_n\rangle)^{1/2} \\ &= 1 , \end{aligned}$$

holds. The quantity $\|h_n\|$ is called a norm of h_n. Two different eigen functions, say h_n and h_m, are orthogonal in a sense that their inner product vanishes,

$$(4.11) \qquad \langle h_n, h_m\rangle = \int_{\mathscr{X}} \overline{h_n(\boldsymbol{x})} h_m(\boldsymbol{x}) d^N x = 0 .$$

The totality of eigen functions $\{h_n\}_{n=1}^{\infty}$ is said to be a complete system of vectors in $\mathscr{H}$ if any function f in $\mathscr{H}$ can be given by a linear combination of h_n's,

$$(4.12) \qquad f(\boldsymbol{x}) = \sum_{n=1}^{\infty} a_n h_n(\boldsymbol{x}) .$$

Here, each coefficient a_n is in general a complex number given by

$$(4.13) \qquad \begin{aligned} a_n &= \langle h_n, f\rangle \\ &= \int_{\mathscr{X}} \overline{h_n(\boldsymbol{x})} f(\boldsymbol{x}) d^N x . \end{aligned}$$

Let us look at the neural wave functions as infinite dimensional vectors in the Hilbert space $\mathscr{H}$. We may be allowed there to call a neural wave function a vector in $\mathscr{H}$. Since the neural wave function describes completely the action of relative membrane potentials, it may be called a state vector or a perceptual state of the nerve system. Among many state vectors, those corresponding to stationary neural wave functions are called eigen vectors or eigen states of the nerve system. They are of particular importance because they describe selectively limited stable actions of the nerve system.

A general solution of the neural wave equation (3.1) can be given in terms of a state vector. For each time t, the neural wave function $\psi(\boldsymbol{x}, t)$ can be seen as a vector $\psi(t)$ in $\mathscr{H}$. By the completeness of eigen vectors h_n's, it is decomposed in a form

$$\psi(t) = \sum_{n=1}^{\infty} a_n(t) h_n \, . \tag{4.14}$$

Substitution of this expression into the neural wave equation (3.1) yields

$$i\nu \frac{da_n(t)}{dt} = \lambda_n a_n(t) \, , \tag{4.15}$$

$n=1, 2, \ldots$. Those ordinary differential equations can be integrated to be

$$a_n(t) = a_n \exp\left(-i \frac{\lambda_n}{\nu} t\right) , \tag{4.16}$$

$n=1, 2, \ldots$, where a_n's are constants of integration. It is found therefore that a general state vector of the nerve system is

$$\psi(t) = \sum_{n=1}^{\infty} a_n \exp\left(-i \frac{\lambda_n}{\nu} t\right) h_n \, . \tag{4.17}$$

This describes the most general form of possible time evolution of the perceptual state of the nerve system.

For the better understanding of state vectors, we calculate the mean energy of relative membrane potentials when the nerve system is in the perceptual state described by the state vector (4.17). Let $\boldsymbol{x}=\{\boldsymbol{x}(t) \mid 0 \leq t < \infty\}$ be a Markovian diffusion process of relative membrane potentials associated with the neural wave function (4.17). Then, the mean total energy becomes

$$\begin{aligned} (4.18) \quad & E\left[\frac{1}{2}\left(\frac{1}{2}\,|D\boldsymbol{x}(t)|^2 + \frac{1}{2}\,|D_*\boldsymbol{x}(t)|^2\right) + U(\boldsymbol{x}(t))\right] \\ & = \int \left\{\frac{1}{2}\left(\frac{1}{2}\,|\boldsymbol{a}(\boldsymbol{x}, t)|^2 + \frac{1}{2}\,|\boldsymbol{a}_*(\boldsymbol{x}, t)|^2\right) + U(\boldsymbol{x})\right\} p(\boldsymbol{x}, t) d^N x \\ & = \int \overline{\psi(\boldsymbol{x}, t)} \left(-\frac{\nu^2}{2} \Delta + U(\boldsymbol{x})\right) \psi(\boldsymbol{x}, t) d^N x \\ & = \langle \psi(t), H\psi(t) \rangle \, , \end{aligned}$$

by equations (3.3)–(3.7). See also Nelson (1984). Substituting equation (4.17) into equation (4.18), we find the mean energy of the nerve system

$$\langle \psi(t),\ H\psi(t)\rangle = \sum_{n=1}^{\infty} \lambda_n |a_n|^2 \ . \tag{4.19}$$

The normalization condition claims

$$\| \psi(t) \|^2 = \sum_{n=1}^{\infty} |a_n|^2 = 1 \ . \tag{4.20}$$

Since λ_n is the mean energy of the nerve system in the perceptual state described by the eigen vector h_n, equation (4.19) implies that the mean energy of the nerve system in the perceptual state (4.17) coincides with the "mean" of all mean energies λ_n's of the eigen states h_n's with weights given by $|a_n|^2$'s. In other words, a general state vector of the form (4.17) can be thought to describe a composite action of membrane potentials which fluctuates between stable actions in the eigen states h_n's with probability $|a_n|^2$'s. Such a composite action coincides with one of the stable actions $\boldsymbol{x}_n$'s with probability $|a_n|^2$'s.

Validity of this statistical interpretation of the general state vector (4.17) may be understood also by computing the probability distribution density of the Markovian diffusion process of relative membrane potentials $\boldsymbol{x} = \{\boldsymbol{x}(t) | 0 \le t < \infty\}$. For simplicity of computation, we assume $a_n \neq 0$ only for $n=k$ and $n=m$, where k and m are certain given natural numbers. Then, equation (4.17) becomes

$$\psi(t) = a_k \exp\left(-i \frac{\lambda_k}{\nu} t\right) h_k + a_m \exp\left(-i \frac{\lambda_m}{\nu} t\right) h_m \ , \tag{4.21}$$

with the normalization condition

$$|a_k|^2 + |a_m|^2 = 1 \ . \tag{4.22}$$

The probability distribution density can be obtained immediately,

$$\begin{aligned} p(\boldsymbol{x}, t) &= |\psi(\boldsymbol{x}, t)|^2 \\ &= |a_k|^2 |h_k(\boldsymbol{x})|^2 + |a_m|^2 |h_m(\boldsymbol{x})|^2 \\ &\quad + 2|a_k a_m|\, |h_k(\boldsymbol{x}) h_m(\boldsymbol{x})| \cos\left\{ \frac{(\lambda_k - \lambda_m)}{2\nu} t \right\} \\ &= |a_k|^2 p_k(\boldsymbol{x}) + |a_m|^2 p_m(\boldsymbol{x}) \\ &\quad + 2|a_k a_m|\, |h_k(\boldsymbol{x}) h_m(\boldsymbol{x})| \cos\left\{ \frac{(\lambda_k - \lambda_m)}{2\nu} t \right\} . \end{aligned} \tag{4.23}$$

The last term in the right-hand side of equation (4.23) is rapidly oscillating and vanishes effectively. In this sense, we have

$$p(x, t) \simeq |a_k|^2 p_k(x) + |a_m|^2 p_m(x) , \tag{4.24}$$

which is the desired result.

Thus, we are now able to describe any admissible actions of relative membrane potentials, that is, any perceptual processes taking place in the nerve system, in terms of the Hilbert space geometry. This is because they are illustrated by Markovian diffusion processes in the N-dimensional configuration space $\mathscr{X}$ specified by neural wave functions subject to the neural wave equation. Those neural wave functions and the neural wave equation are respectively vectors and a unitary evolution equation in the Hilbert space of square integrable functions on $\mathscr{X}$. For any normalized vector u_0 in the Hilbert space $\mathscr{H}$ we apply a unitary operator

$$K_t \equiv \exp\left(-\frac{i}{\nu} Ht\right) , \tag{4.25}$$

where H is the action operator of the nerve system given by equation (4.2), obtaining another vector

$$u_t = K_t u_0 . \tag{4.26}$$

This vector u_t satisfies the neural wave equation (3.1) written in terms of Hilbert space geometry as

$$i\nu \frac{d}{dt} u_t = Hu_t . \tag{4.27}$$

Because u_0 is arbitrary, any vector in H can be thought to represent a possible initial state of perceptual process in the nerve system. Such an initial state changes in time according to the unitary operator K_t. Eigen vectors h_n's are only exceptions, that is, they are stable (i.e., unchanged) under the unitary operator K_t,

$$K_t h_n = e^{-i(\lambda_n/\nu)t} h_n . \tag{4.28}$$

Following are the abstracted scheme of geometrical representation of perceptual process:

Action of a nerve system with N neurons is represented in an infinite dimensional complex Hilbert space $\mathscr{H}$. We call it a neural state space.

Each vector in the neural state space $\mathscr{H}$ represents a possible form of action of the nerve system. We call it a perceptual state of the nerve system.

Time development of a perceptual state is given by applying the unitary operator (4.25). It represents a possible perceptual process in the nerve system characterized by the action operator (4.2).

A perceptual state which is stable under the application of unitary time development (4.25) represents a memory process in the nerve system. Eigen vectors of the action operator (4.2) manifests memory processes. Those specific perceptual states are called eigen states or memory states.

Linearity of the Hilbert space $\mathscr{H}$ admits to represent any perceptual state as a linear superposition (i.e., a sum) of other perceptual states. If a state vector u happens to be a linear superposition

$$u = u_1 + u_2 , \tag{4.29}$$

of the other two orthogonal state vectors such that $\langle u_1, u_2\rangle = 0$, we interpret the perceptual state u as a statistical composite of perceptual states u_1 and u_2 with probability $|u_1|^2 = |\langle u, u_1\rangle|^2$ and $|u_2|^2 = |\langle u, u_2\rangle|^2$. In other words, any perceptual state u in $\mathscr{H}$ can be thought in part as a statistical composite of any other one v in $\mathscr{H}$ with probability proportional to

$$|\langle u, v\rangle|^2 . \tag{4.30}$$

This fact opens the possibility to describe the mechanism of association in perceptual processes.

Two perceptual states are independent with each other if they are orthogonal, that is, their inner product vanishes. They cannot be a composite of each other.

Infinite dimensionality and completeness of the Hilbert space $\mathscr{H}$ claim the existence of infinitely many varieties of a set of infinitely many independent perceptual states of the nerve system. They are nothing else but complete normalized orthogonal systems in $\mathscr{H}$. Among them is the system of eigen states of the action operator, $\{h_n\}_{n=1}^{\infty}$. Any perceptual state of the nerve system u in $\mathscr{H}$ can be decomposed into a linear combination

$$u = \sum_{n=1}^{\infty} \alpha_n h_n , \tag{4.31}$$

where each complex coefficient α_n is given by the inner product

$$\alpha_n = \langle u, h_n\rangle . \tag{4.32}$$

Let $\{g_n\}_{n=1}^{\infty}$ be another complete normalized orthogonal system in $\mathscr{H}$. Then, the same perceptual state u can be decomposed into another linear combination

$$u = \sum_{n=1}^{\infty} \beta_n g_n , \tag{4.33}$$

with

$$\beta_n = \langle u, g_n \rangle \,. \tag{4.34}$$

Two ordered sets of infinitely many components $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ and $(\beta_1, \beta_2, \beta_3, \ldots)$ are coordinate representations of the perceptual state u in two different "coordinate" systems $\{h_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$. Thus any perceptual state of the nerve system has infinitely many varieties of coordinate representation. They provide us with a mechanism of multiple association characteristic to the perceptual process as follows.

Equation (4.31) asserts that the perceptual state u can be thought as a statistical composite of those given by h_n's with statistical weights $|\alpha_n|^2$'s. In other words, the perceptual state u resembles that of h_1 with probability $|\alpha_1|^2$, that of h_2 with $|\alpha_2|^2$, and so on. Each perceptual state h_n associates the perceptual state u with probability of association given by $|\alpha_n|^2$. Similarly, equation (4.33) asserts that each perceptual state g_n associates also the same perceptual state u with probability of association $|\beta_n|^2$. The choice of coordinate systems in the Hilbert space $\mathscr{H}$ is completely free, and the association manifests a wide variety.

5. A simple example

As an example of the analysis and interpretation of fundamental thought processes in the nerve system, we consider a simple model of nerve system arising from stochastic neurodynamics. We start with the neural wave equation (3.1) and look for a specific solution of the form

$$\psi(x, t) = u(x)h(t) \,. \tag{5.1}$$

Substituting equation (5.1) into equation (3.1), we obtain

$$i\nu \frac{dh(t)}{dt} = \lambda h(t) \tag{5.2}$$

and

$$\left(-\frac{\nu^2}{2}\Delta + U(x)\right)u(x) = \lambda u(x) \,, \tag{5.3}$$

where λ is a constant of separation. The former equation (5.2) can be integrated immediately, obtaining

$$h(t) = h_0 \exp\left(-i\frac{\lambda}{\nu}t\right) , \tag{5.4}$$

where h_0 is a constant of integration. The latter equation (5.3) becomes an eigen value problem, and the constant of separation λ is determined as the eigen value. We call equation (5.3) a stationary neural wave equation and its solution $u(\boldsymbol{x})$ a stationary neural wave function.

It is worthwhile to notice here that the normalization condition for the probability distribution density,

$$\int_{\mathscr{X}} p(\boldsymbol{x}, t) d^N x = \int_{\mathscr{X}} |\psi(\boldsymbol{x}, t)|^2 d^N x = 1 , \tag{5.5}$$

yields $h_0 = 1$ and

$$\int_{\mathscr{X}} |u(\boldsymbol{x})|^2 d^N x = 1 . \tag{5.6}$$

Given a solution $u(\boldsymbol{x})$ of the eigen value problem (5.3) with respect to the eigen value λ, we find a neural wave function

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x}) \exp\left(-i \frac{\lambda}{\nu} t\right) . \tag{5.7}$$

Equations (3.3) and (3.4) yield

$$R(\boldsymbol{x}, t) = \log |u(\boldsymbol{x})| , \tag{5.8}$$

and

$$S(\boldsymbol{x}, t) = -\frac{\lambda}{\nu} t . \tag{5.9}$$

We find by equation (3.5)

$$\boldsymbol{a}(\boldsymbol{x}, t) = \boldsymbol{a}(\boldsymbol{x}) \equiv \nu \frac{\nabla u(\boldsymbol{x})}{u(\boldsymbol{x})} . \tag{5.10}$$

The stochastic differential equation of Itô type (1.11), therefore, becomes

$$d\boldsymbol{x}(t) = \boldsymbol{a}(\boldsymbol{x}(t)) dt + d\boldsymbol{w}(t) . \tag{5.11}$$

The Markovian diffusion process of relative membrane potentials $\boldsymbol{x} = \{\boldsymbol{x}(t) | 0 \leq t < \infty\}$ generated by equation (5.11) is a stationary process in the configuration space $\mathscr{X}$ with the (time-independent) equilibrium probability distribution density

$$p(x) = |u(x)|^2 . \tag{5.12}$$

The physical meaning of the eigen value λ is the mean total energy (i.e., kinetic energy plus potential energy) of relative membrane potentials $x_i(t)$'s, that is,

$$\begin{aligned} \lambda &= E\left[\frac{1}{2}\left(\frac{1}{2}|Dx(t)|^2 + \frac{1}{2}|D_*x(t)|^2\right) + U(x(t))\right] \\ &= \int_{\mathscr{X}}\left\{\frac{1}{2}\left(\frac{1}{2}|a(x,t)|^2 + \frac{1}{2}|a_*(x,t)|^2\right) + U(x)\right\}p(x,t)d^N x \\ &= \int_{\mathscr{X}}\left(\frac{1}{2}|a(x)|^2 + U(x)\right)p(x)d^N x . \end{aligned} \tag{5.13}$$

Let us suppose a restricted case of dynamics of the nerve system in which the relative membrane potentials $x_i(t)$'s take small values around 0. Then we can approximate the total electrostatic energy of the nerve system $U(x)$ by the second order expansion in the coordinates x_i's around 0. Namely, we may write down

$$\begin{aligned} U(x) &\simeq U_0 + \sum_{i=1}^{N}\frac{\partial U}{\partial x_i}(0)x_i + \frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2 U}{\partial x_i \partial x_j}(0)x_i x_j \\ &\equiv \frac{1}{2}\sum_{i,j=1}^{N}\sigma_{ij}(x_i - \xi_i)(x_j - \xi_j) , \end{aligned} \tag{5.14}$$

where σ_{ij}'s and ξ_i's are certain constants such that (σ_{ij}) is a positive definite symmetric N by N matrix. For such a quadratic function $U(x)$ as (5.14), the eigen value problem (5.3) can be solved explicitly, obtaining an infinite series of eigen values $\{\lambda_n\}_{n=0}^{\infty}$ and eigen functions $\{u_n(x)\}_{n=0}^{\infty}$. The eigen function $u_0(x)$ belonging to the lowest (i.e., smallest) eigen value λ_0 has a Gaussian form

$$u_0(x) = A\exp\left(-\frac{1}{4\nu}\sum_{i,j=1}^{N}\omega_{ij}(x_i - \xi_i)(x_j - \xi_j)\right) , \tag{5.15}$$

where (ω_{ij}) is a positive definite symmetric N by N matrix and A a normalization constant.

The neural wave function $\psi(x, t) = u_0(x)\exp(-i(\lambda_0/\nu)t)$ determines a stationary Markovian diffusion process $x_0 = \{x_0(t) | 0 \leq t < \infty\}$ with the equilibrium distribution density

$$p_0(x) \simeq \exp\left(-\frac{1}{\nu}\mathscr{H}_0(x)\right) , \tag{5.16}$$

where

$$\mathscr{H}_0(\boldsymbol{x}) \equiv \frac{1}{2} \sum_{i,j=1}^{N} \omega_{ij} x_i x_j + \sum_{i=1}^{N} \beta_i x_i \,, \tag{5.17}$$

for certain constants β_i's. This means that the total action of the nerve system in this restricted case can be approximated by relative membrane potentials x_i's manifesting a time-independent statistical configuration governed by the equilibrium distribution density (5.16). Such a statistical configuration of relative membrane potentials is formally equivalent to the well-known continuous model of Ising spin system in statistical mechanics. Namely, if we suppose each random configuration variable x_i to be the z-component of a spinning top, the ordered set $\boldsymbol{x}=(x_1, x_2,\ldots, x_N)$ may be considered as configuration variables of N classical spin system. The equilibrium distribution density (5.16) is nothing else but the Gibbs state (i.e., the Boltzmann distribution) of this classical spin system

$$p_G(\boldsymbol{x}) = \exp\left(-\frac{1}{k_B T} \mathscr{H}_0(\boldsymbol{x})\right) , \tag{5.18}$$

with temperature $T \equiv \nu / k_B$, where k_B denotes the Boltzmann constant.

Thus, classical statistical mechanics of spinning tops deserves to be a simple model of the action of a nerve system. The Hopfield model of neural network, that is, the so-called Boltzmann engine now becomes a simple case of stochastic neurodynamics (Hopfield (1982)). Let us introduce discrete random variables s_i's by

$$s_i \equiv \operatorname{sgn}(x_i) \,. \tag{5.19}$$

If the i-th neuron is active (i.e., firing) $s_i = 1$, and $s_i = -1$ otherwise. The random variables s_i's well characterize the total action of nerve system from the point of view of neuron activity, and called action variables. We rewrite equation (5.17) in terms of the action variables, obtaining

$$\mathscr{H}_0(\boldsymbol{s}) = \frac{1}{2} \sum_{i,j=1}^{N} T_{ij} s_i s_j + \sum_{i=1}^{N} I_i s_i \,, \tag{5.20}$$

for $\boldsymbol{s}=(s_1, s_2,\ldots, s_N)$. Here, T_{ij}'s and I_i's may take random values absorbing redundant random variables $|x_i x_j|$'s and $|x_i|$'s. Equation (5.20) can be thought to define the Hamiltonian (i.e., the energy function) of a spin glass system of N random bonding spins. The equilibrium distribution density (5.16) defines a Gibbs state of the spin glass system with temperature $T = \nu / k_B$. For further analysis of the Hopfield model, see Hopfield and Tank (1985).

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