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SOME CONSTRUCTIONS OF TWO-ASSOCIATE CLASS PBIB DESIGNS

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Summary

Two-associate class PBIB designs, having association schemes of GD or L_2 types, are constructed by using patterned matrices and by methods of taking unions of sets of blocks.

1. Introduction

A large number of methods of construction of two-associate partially balanced incomplete block (PBIB) designs are available in literature (cf. Clatworthy [1] and Raghavarao [3]). We here present further methods of construction of semi-regular group divisible (GD) designs from self-complementary balanced incomplete block (BIB) designs. The method of block extensions applied to incidence matrices of certain BIB designs gives SRGD designs and L_2 designs. Series of SRGD and L_2 designs are constructed by a method of taking unions of blocks in affine resolvable block designs. Some methods are explained in Section 2, while our constructions are presented in Sections 3 and 4.

2. Definitions and methods

For most of definitions of incidence structures discussed here, refer to Raghavarao [3]. We describe only some definitions to avoid confusion of available notations here.

An incomplete block design with parameters v (number of treatments), b (number of blocks) and r (replication number of each treatment) is said to be α-resolvable, if the blocks can be partitioned into t resolution classes, $S_1, S_2, \cdots, S_t$, each of β blocks ($b=\beta t$) such that every treatment is repeated α times in each class S_i ($r=\alpha t$). An α-resolvable block design is said to be affine α-resolvable, if any pair of blocks in the same class S_i intersect in x treatments (say) and any pair

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of blocks from different classes S_i and S_j ($i\neq j$) intersect in y treatments (say). It is well-known that if $\alpha=1$ (and then $t=r$), the design is affine resolvable in the usual sense with $x=0$.

If N is the incidence matrix of a binary block design D, then $\bar{N}=J_{v,b}-N$ is the incidence matrix of a design $\bar{D}$ called the complement of D where $J_{v,b}$ is a $v\times b$ matrix with unit elements everywhere. A design D is said to be self-complementary, if its complement $\bar{D}$ has precisely the same parameters as D. If D is self-complementary and if D and $\bar{D}$ are also isomorphic, then D is said to be truly self-complementary.

A BIB design is an incomplete block design with parameters v, b, r in which every block is of size k and every pair of distinct treatments occurs exactly in λ blocks. We shall denote such a design as a BIBD (v, b, r, k, λ). If $v=b$, such a BIB design is said to be symmetric, and in that case $r=k$. Therefore, a symmetric BIB design can be written as an SBIBD (v, k, λ). It is well-known that any two distinct blocks of an SBIBD (v, k, λ) have λ treatments in common. The complement of a BIBD (v, b, r, k, λ) is a BIBD $(v, b, b-r, v-k, b-2r+\lambda)$. In order that a BIB design is self-complementary, it is necessary and sufficient that $v=2k$. It is known (cf. Shrikhande and Raghavarao [4]) that for an affine α-resolvable BIBD (v, b, r, k, λ), $x=k-r+\lambda$ and $y=k^2/v$.

Let $N=[N_1, N_2,\cdots, N_b]$ be the $v\times b$ incidence matrix of an incomplete block design D, where N_i, $i=1, 2,\cdots, b$, are the i-th column of N. A method of block extensions used in this paper for some s consists of choosing s new blocks each of two blocks out of the original b blocks of an incomplete block design with equal block size k and keeping one block over the other, thus making s new blocks of size $2k$, i.e., the incidence matrix N^* of the new design is of the form $N^*=[N_1^*, N_2^*,\cdots, N_s^*]$, where $N_i^*=\begin{bmatrix} N_l \\ N_{l'} \end{bmatrix}$ for $i=1, 2,\cdots, s$; $1\leqq l, l'\leqq b$.

A block design derivable from D by a method of block unions is a design in which any block is of the form $B_i\cup B_j$ for some $i\neq j$; B_i, B_j being blocks in D.

3. Construction based on patterned matrices

Let $O_{i,j}$ be an $i\times j$ matrix whose all elements are zero.

THEOREM 3.1. *A truly self-complementary SRGD design D^* with incidence matrix N^* and parameters $v^*=4k=b^*$, $r^*=2k=k^*$; $\lambda_1^*=0$, $\lambda_2^*=k$; $m^*=2k$, $n^*=2$ exists if and only if*

$$N^*=\begin{bmatrix} N & J_{2k,1} & O_{2k,1} \\ \bar{N} & O_{2k,1} & J_{2k,1} \end{bmatrix}$$

where N is the incidence matrix of a self-complementary BIBD $(2k, 4k-2, 2k-1, k, k-1)$ *and $\bar{N}$, its complement.*

PROOF. Suppose that a self-complementary BIBD $(2k, 4k-2, 2k-1, k, k-1)$ exists. Let treatments corresponding to the i-th row and $(2k+i)$-th row in N^* constitute the i-th group of a GD association scheme for $1 \leqq i \leqq 2k$. Then it follows that N^* is the incidence matrix of a truly self-complementary SRGD design with the required parameters. Conversely, starting from a truly self-complementary symmetric SRGD design with parameters $v^*=4k$, $k^*=2k$, $\lambda_1^*=0$, $\lambda_2^*=k$, $m^*=2k$, $n^*=2$, it is easy to see that we can get a pair of disjoint blocks, to form at least one resolution class each of size $2k$ such that one treatment of one block paired with the treatment from the other block forms a group. Then the incidence matrix of the SRGD design admits the following natural matrix decomposition on writing treatments of the i-th group as the i-th and $(2k+i)$-th treatments:

$$\begin{bmatrix} J_{2k,1} & O_{2k,1} & N \\ O_{2k,1} & J_{2k,1} & M \end{bmatrix}.$$

Now, it is obvious to show the values of parameters as $v'=2k$, $b'=4k-2$, $r'=2k-1$ and $\lambda'=k-1$ in N. Let b_i be the number of unity in the i-th block of N. Then it follows that

$$\sum_{i=1}^{4k-2} b_i = 2k(k-1), \qquad \sum_{i=1}^{4k-2} b_i(b_i-1) = 2k(2k-1)(k-1).$$

In this case, letting $\bar{b}=\{1/(4k-2)\}\sum_{i=1}^{4k-2} b_i = k$, we can show $\sum_{i=1}^{4k-2}(b_i-\bar{b})^2=0$ after some calculation. Then $b_i=\bar{b}=k$ for all i. Hence, N is the incidence matrix of a self-complementary BIBD $(2k, 4k-2, 2k-1, k, k-1)$, and M is the incidence matrix of its complement, because of $\lambda_1^*=0$ and the grouping of treatments. Thus, the proof is completed.

Theorem 3.1 can be generalized by replacing $J_{2k,1}$ and $O_{2k,1}$ by $J_{2k,l}$ and $O_{2k,l}$, respectively, for some positive integer l. This process yields some repeated blocks.

A GD design with parameters $v=mn$, b, r, k, λ_1, λ_2, m, n is denoted by a GD design $(v, b, r, k; \lambda_1, \lambda_2; m, n)$.

THEOREM 3.2. *If N is the incidence matrix of a self-complementary* BIBD $(2k, l(4k-2), l(2k-1), k, l(k-1))$ *for a positive integer l, then*

$$N^*=\begin{bmatrix} N & \bar{N} & J_{2k,2l} & O_{2k,2l} \\ J_{1,l(4k-2)} & O_{1,l(4k-2)} & [J_{1,l}:O_{1,l}] & [J_{1,l}:O_{1,l}] \\ \bar{N} & N & O_{2k,2l} & J_{2k,2l} \\ O_{1,l(4k-2)} & J_{1,l(4k-2)} & [O_{1,l}:J_{1,l}] & [O_{1,l}:J_{1,l}] \end{bmatrix}$$

is the incidence matrix of a truly self-complementary SRGD design $(4k+2, 8kl, 4kl, 2k+1; 0, 2kl; 2k+1, 2)$.

PROOF. Let the treatments corresponding to the i-th and $(2k+1+i)$-th rows define new groups for $1\leqq i\leqq 2k+1$. Clearly $\lambda_1^*=0$ and in N^* any inner product of any two rows corresponding to different groups has value $2kl$. Therefore, $\lambda_2^*=2kl$. The other relations for v^*, b^*, r^*, k^* and the semi-regularity condition of $r^*k^*=v^*\lambda_2^*$ can easily be verified, along with the truly self-complement.

We now mention some combinatorial meaning of Theorems 3.1 and 3.2. Preece [2] gave various non-isomorphic solutions of self-complementary BIB designs with $v=2k$, and hence each of such solutions would give a solution to the symmetric SRGD design, and therefore our theorems are powerful enough to produce as many non-isomorphic solutions to the SRGD design as there are non-isomorphic solutions to the BIB design. For example, Preece [2] presented 4 non-isomorphic solutions of a BIBD (8, 14, 7, 4, 3) which, from Theorem 3.1, yield 4 non-isomorphic solutions for an SRGD design with parameters $v=b=16$, $r=k=8$, $\lambda_1=0$, $\lambda_2=4$, $m=8$, $n=2$, though Clatworthy [1] listed only one solution as SR 92. Similarly, 3 non-isomorphic solutions of a BIBD (10, 18, 9, 5, 4) yield 3 non-isomorphic solutions for an SRGD design with parameters $v=b=20$, $r=k=10$, $\lambda_1=0$, $\lambda_2=5$, $m=10$, $n=2$, one of them is known as SR 108.

An old method of juxtaposing incidence matrices of some available designs will yield other designs. This is fundamental and sometimes useful. For example, if N is the incidence matrix of a BIBD (lk, b, r, k, λ) for a positive integer l, then $N^*=[N: I_l\otimes J_{k,p}]$ is the incidence matrix of an RGD design $(lk, b+lp, r+p, k; \lambda+p, \lambda; l, k)$ for any positive integer p. As an individual example, an existing BIBD (27, 39, 13, 9, 4) with $l=3$ and $p=2$ yields the RGD design (27, 45, 15, 9; 6, 4; 3, 9), which may be new.

4. Method of block extension

THEOREM 4.1. *The existence of a* BIBD (v, b, r, k, λ) *implies the existece of an SRGD design* $(lv, b^l, rb^{l-1}, lk; \lambda b^{l-1}, r^2b^{l-2}; l, v)$ *for any positive integer* $l\geqq 2$.

PROOF. Let N be the $v \times b$ incidence matrix of a BIBD (v, b, r, k, λ) and let N_i be the i-th column of N, as $N=[N_1, N_2, \cdots, N_b]$. Call

$$N^*_{i_1, i_2, \cdots, i_l} = \begin{bmatrix} N_{i_1} \\ N_{i_2} \\ \vdots \\ N_{i_l} \end{bmatrix},$$

where $(i_1, i_2, \cdots, i_l)$ is a permutation on blocks of order l. We consider all possible b^l vectors of the type $N^*_{i_1, i_2, \cdots, i_l}$ and let $N^* = [N^*_{i_1, i_2, \cdots, i_l}, N^*_{j_1, j_2, \cdots, j_l}, \cdots, N^*_{s_1, s_2, \cdots, s_l}]$ of size $lv \times b^l$. If we write the lv treatments as follows:

$$\begin{matrix} 1 & v+1 & \cdots & (l-1)v+1 \\ 2 & v+2 & \cdots & (l-1)v+2 \\ \vdots & \vdots & & \vdots \\ v & 2v & \cdots & lv \end{matrix},$$

then the i-th column forms the i-th group, $i=1, 2, \cdots, l$, i.e., $m^*=l$ and $n^*=v$. Now, consider a pair of first associates, which belongs to the same group. It was occurring originally in λ blocks, each of which is now replicated b^{l-1} times. Hence $\lambda_1^* = \lambda b^{l-1}$. A pair of second associates $\{\theta, \phi\}$, belonging to different groups, occurs in the resulting design as many times as the pair of columns N_i, N_j, say, containing θ and ϕ, respectively, occurs together as columns of N^*, i.e., $\lambda_2^* = r^2 b^{l-2}$. Other constants are easy to check, besides verification of the relation $r^*k^* - v^*\lambda_2^* = 0$ for the semi-regularity of the GD design, since $r^* - \lambda_1^* = b^{l-1}(r-\lambda) > 0$.

THEOREM 4.2. *The existence of an* SBIBD (v, k, λ) *implies the existence of an* $L_2(v)$ *design with parameters* $v^*=v^2$, $b^*=2v$, $r^*=2k$, $k^*=vk$, $\lambda_1^*=k+\lambda$, $\lambda_2^*=2\lambda$.

PROOF. In the incidence matrix N^* of Theorem 4.1 with $l=2$, when $v=b$, N^*_{ij} and $N^*_{ij'}$ for $j \neq j'$ will have $k+\lambda$ treatments in common. The same is true for N^*_{ij} and $N^*_{i'j}$ for $i \neq i'$; whereas N^*_{ij} and $N^*_{i'j'}$ for $i \neq i'$, $j \neq j'$ have 2λ treatments in common. If B^*_{ij} is a block corresponding to N^*_{ij} and if we arrange these blocks as follows:

$$B^* = \begin{bmatrix} B^*_{11} & B^*_{12} & B^*_{13} & \cdots & B^*_{1v} \\ B^*_{21} & B^*_{22} & B^*_{23} & \cdots & B^*_{2v} \\ \vdots & \vdots & \vdots & & \vdots \\ B^*_{v1} & B^*_{v2} & B^*_{v3} & \cdots & B^*_{vv} \end{bmatrix}.$$

Then it follows that the dual of the SRGD design N^* constructed in

Theorem 4.1 with $l=2$ gives an $L_2(v)$ design with the required parameters, where the usual association scheme of the $L_2(v)$ design is the dual to the block arrangement B^* given above. Thus, the proof is completed.

Remark. A trivial SBIBD $(v, v-1, v-2)$ through Theorem 4.2 yields the complement of a well-known simple lattice with $v^*=v^2$, $b^*=2v$, $r^*=2$, $k^*=v$, $\lambda_1^*=1$, $\lambda_2^*=0$.

THEOREM 4.3. *The existence of an affine α-resolvable* BIBD $(v, b=\beta t, r=\alpha t, k, \lambda)$ *implies the existence of an $\alpha\beta$-resolvable SRGD design* $(2v, t\beta^2, t\alpha\beta, 2k; \beta\lambda, \alpha^2 t; 2, v)$.

PROOF. Let the incidence matrix N of the affine α-resolvable BIBD $(v, b=\beta t, r=\alpha t, k, \lambda)$ be written as:

$$N=[\overbrace{N_1^1, N_2^1, \cdots, N_\beta^1}^{S_1} : \overbrace{N_1^2, N_2^2, \cdots, N_\beta^2}^{S_2} : \cdots : \overbrace{N_1^t, N_2^t, \cdots, N_\beta^t}^{S_t}]$$

where S_i, $i=1, 2, \cdots, t$, is the i-th resolution class. Now consider

$$N^*=[\overbrace{N_{11}^1, N_{12}^1, \cdots, N_{\beta\beta}^1}^{S_1^*} : \overbrace{N_{11}^2, N_{12}^2, \cdots, N_{\beta\beta}^2}^{S_2^*} : \cdots : \overbrace{N_{11}^t, N_{12}^t, \cdots, N_{\beta\beta}^t}^{S_t^*}]$$

where $N_{ij}^l=\begin{bmatrix} N_i^l \\ N_j^l \end{bmatrix}$, $i, j=1, 2, \cdots, \beta$; $l=1, 2, \cdots, t$. Then N^* is the $2v\times \beta^2 t$ incidence matrix of a design with $v^*=2v$ treatments, t sets of β^2 blocks each of size $2k$. Every treatment occurs in each set S_i^* $\alpha\beta$ times and hence $\alpha\beta t$ times in N^*. Any pair of treatments, u, s, occurs together in $\beta\lambda$ blocks if $1\leqq u, s\leqq v$ or $v+1\leqq u, s\leqq 2v$. If $1\leqq u\leqq v$ and $v+1\leqq s\leqq 2v$, then the pair u, s occur together in $\alpha^2 t$ blocks. Considering the following arrangement of treatments

Group	Treatments			
1	1	2	$\cdots$	v
2	$v+1$	$v+2$	$\cdots$	$2v$

we obtain an SRGD design, since $r^*-\lambda_1^*=\alpha\beta t-\lambda\beta=\beta(r-\lambda)>0$ and $r^*k^*-v^*\lambda_2^*=2\alpha\beta tk-2v\alpha^2 t=2v\alpha(r-\alpha t)=0$. Thus, the proof is completed.

THEOREM 4.4. *The existence of an affine α-resolvable* BIBD $(v, b=\beta t, r=\alpha t, k, \lambda)$ *implies the existence of t $L_2(\beta)$ designs with parameters $v^*=\beta^2$, $b^*=2v$, $r^*=2k$, $k^*=\alpha\beta$, $\lambda_1^*=2k+\lambda-r$, $\lambda_2^*=2(k+\lambda-r)$.*

PROOF. With the same notations as in Theorem 4.3 and the procedure used in Theorem 4.2, take any set S_i^*, $i=1, 2, \cdots, t$, of N^*, say S_m^*. Then N_{ij}^m and $N_{ij'}^m$ for $j \neq j'$ have $k+x$ treatments in common. Note that $x=k+\lambda-r$ is a block intersection number in the affine α-resolvability. Similarly, N_{ij}^m and $N_{i'j}^m$ for $i \neq i'$ have $k+x$ treatments in common, while N_{ij}^m and $N_{i'j'}^m$ for $i \neq i'$, $j \neq j'$ intersect in $2x$ treatments. Let us arrange these blocks as follows:

$$B_m^* = \begin{bmatrix} B_{11}^m & B_{12}^m & \cdots & B_{1\beta}^m \\ B_{21}^m & B_{22}^m & \cdots & B_{2\beta}^m \\ \vdots & \vdots & & \vdots \\ B_{\beta 1}^m & B_{\beta 2}^m & \cdots & B_{\beta\beta}^m \end{bmatrix}.$$

Then the dual of B_m^*, for each of $m=1, 2, \cdots, t$, gives the required association scheme, i.e., B_{ij}^m and $B_{i'j'}^m$ are first associates iff they lie either in the same row or the same column of the above arrangement, otherwise, they are second associates. Thus, the proof is completed.

Remark. The L_2 design with the same parameters as in Theorem 4.4 can be given by considering new blocks of the following type also:

$$N_{ij}^* = \begin{bmatrix} N_i^l \\ N_j^{l'} \end{bmatrix} \qquad \begin{array}{l} l \neq l'\,;\ l, l'=1, 2, \cdots, t\,; \\ l, l' \text{ fixed for each design}\,. \end{array}$$

5. Method of block unions

THEOREM 5.1. *The existence of an affine resolvable block design* (v, $b=\beta r$, r, k) *implies the existence of an SRGD design* (b/l, v, lk, r; 0, $l^2 y$; r, β/l) *with* $y=k^2/v$ *if* β/l *is a positive integer* ($=p$, *say*) *for any positive integer* l *satisfying* $lk<v$.

PROOF. Let the blocks of the block design partitioned into r resolution classes be written as follows:

$$[\overbrace{B_{11}, B_{12}, \cdots, B_{1\beta}}^{S_1} : \overbrace{B_{21}, B_{22}, \cdots, B_{2\beta}}^{S_2} : \cdots : \overbrace{B_{r1}, B_{r2}, \cdots, B_{r\beta}}^{S_r}]\,.$$

Here,

$$\begin{array}{ll} |B_{ij} \cap B_{ij'}| = 0\,, & j \neq j'\,; \\ |B_{ij} \cap B_{i'j'}| = y = k^2/v\,, & i \neq i'\,;\ i, i'=1, 2, \cdots, r\,, \\ & j, j'=1, 2, \cdots, \beta\,. \end{array}$$

Since $\beta=lp$, every set S_i can be divided into p distinct subsets of l blocks each. Form the unions of the l blocks in each of the p subsets for every set S_i. Then in the resulting design, there are v treatments and $pr=b/l$ blocks. Letting B^*_{ij} be the j-th block in the i-th class in the resulting design, $|B^*_{ij}\cap B^*_{ij'}|=0$ for $j\neq j'$ and $|B^*_{ij}\cap B^*_{i'j'}|=l^2y$, $i\neq i'$, since any two blocks from the same class S_i are disjoint, while any two blocks from different resolution classes intersect in y treatments. Every treatment is repeated once in every class of both the original design and the resulting design, and then the resulting block size is lk. In this case, the dual of the above design can be shown to be the required GD design with blocks in the i-th resolution class corresponding to the i-th group. Thus, $m=r$ and $n=\beta/l$. The proof is completed.

THEOREM 5.2. *The existence of an affine resolvable block design* $(v, b=\beta r, r, k)$ *for* $\beta\geqq 2$, *implies the existence of an* $L_2(\beta)$ *design with parameters* $v^*=\beta^2$, $b^*=v$, $r^*=2k-k^2/v$, $k^*=2\beta-1$, $\lambda_1^*=k$, $\lambda_2^*=2k^2/v$.

PROOF. Let S_1 and S_2 denote any two resolution classes of the affine resolvable block design, and let B_{1i}, $i=1, 2,\cdots,\beta$, and B_{2j}, $j=1, 2,\cdots,\beta$, be the β blocks in S_1 and S_2, respectively. Consider $B_{1i}\cup B_{2j}$, $i, j=1, 2,\cdots,\beta$, as new blocks. Then in the resulting β^2 blocks, each of the v treatments occurs $2\beta-1$ times. Since $|B_{1i}\cap B_{2j}|=k^2/v$, $|B_{1i}\cup B_{2j}|=2k-k^2/v$ for every i and j. If we write $B_{1i}\cup B_{2j}$ as ij and arrange the resulting blocks in the following $\beta\times\beta$ array:

$$\begin{array}{cccc} 11 & 12 & \cdots & 1\beta \\ 21 & 22 & \cdots & 2\beta \\ \vdots & \vdots & & \vdots \\ \beta 1 & \beta 2 & \cdots & \beta\beta \end{array}$$

and call any two blocks first associates if they lie either in the same row or in the same column of the array, otherwise call them second associates, then it is easy to see that any two blocks which are first associates intersect in k treatments and any two blocks which are second associates intersect in $2k^2/v$ treatments. Now, dualizing the above arrangement, we get the required result.

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