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SOME CHARACTERIZATION OF LOCALLY RESISTANT BIB DESIGNS OF DEGREE ONE*

SANPEI KAGEYAMA

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Summary

This paper investigates locally resistant balanced incomplete block (LRBIB) designs of degree one. A new necessary condition for the existence of such an LRBIB design is presented. This condition yields a complete characterization of affine α-resolvable LRBIB designs of degree one. Furthermore, regarding construction methods of LRBIB designs of degree one, it is shown that Shah and Gujarathi's method (1977, *Sankhyā*, B**39**, 406–408) yields the same parameters as Hedayat and John's method (1974, *Ann. Statist.*, **2**, 148–158), but their block structures are different and interesting.

1. Introduction

Let T be a set of v treatments. Let D be a block design with the $v\times b$ incidence matrix $N=((n_{ij}))$ consisting of b blocks of size k_j $(j=1,2,\cdots,b)$ such that the i-th treatment occurs r_i times $(i=1,2,\cdots,v)$ and the i-th treatment occurs in the j-th block n_{ij} times, where $n_{ij}=0$ or 1. Let $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$. Under the standard homoscedastic linear additive model, it is known that the coefficient matrix of the least square normal equations for treatment effects adjusted for block effects is $C=R-NK^{-1}N'$, where $R=\mathrm{diag}\{r_1, r_2,\cdots, r_v\}$ and $K=\mathrm{diag}\{k_1, k_2,\cdots, k_b\}$. We shall deal only with connected designs (i.e., rank of C is $v-1$) throughout this paper.

We shall mention definitions of several designs used here. A design D is said to be variance-balanced (Rao [11]) if C is of the form $C=\rho\{I-(1/v)J\}$ where $\rho=(n-b)/(v-1)$, I the identity matrix and J a matrix of all ones. Let D be a block design on a set of v treatments, T. Let L be a subset of T consisting of m $(\leqq v-2)$ treatments. We

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denote by $\bar{D}$ the remaining design upon the loss of all experimental units in D assigned to the treatments in L. In this case,

(i) D is said to be globally resistant of degree m (GR(m)) if $\bar{D}$ is variance-balanced with respect to the loss of any subset L of cardinality m;

(ii) D is said to be locally resistant of degree m (LR(m)) if $\bar{D}$ is variance-balanced with respect to the loss of some subsets (but not all) L of cardinality m.

Thus, resistant designs may be useful in a robustness problem when some or all of the experimental units assigned to one or more treatments are lost. As other applications, see Hedayat and John [3].

An incomplete block design D with parameters v, b, r $(=r_1=r_2=\cdots=r_v)$, k $(=k_1=k_2=\cdots=k_b)$ $(<v)$ is called an s-(v, k, λ_s) design if every s-subset of T is contained in exactly λ_s blocks of D. It is well known (cf. Hedayat and Kageyama [4]) that for each $0\leqq u\leqq s$ every s-(v, k, λ_s) design is a u-(v, k, λ_u) design with $\lambda_u=\lambda_s\binom{v-u}{s-u}\Big/\binom{k-u}{s-u}$. Note that a 2-$(v, k, \lambda_2)$ design is well known as a balanced incomplete block (BIB) design with parameters v, b $(=\lambda_0)$, r $(=\lambda_1)$, k and λ $(=\lambda_2)$, denoted by BIBD(v, b, r, k, λ). A BIB design is said to be α-resolvable if the blocks can be separated into t sets of β blocks each $(b=\beta t)$ such that each set contains every treatment exactly α times $(r=\alpha t)$. An α-resolvable BIB design is further said to be affine α-resolvable if any pair of blocks belonging to the same set contain q_1 treatments in common, whereas any pair of blocks belonging to different sets contain q_2 treatments in common. In this case, we have (cf. Kageyama [6]) $q_1=(\alpha-1)k/(\beta-1)=k+\lambda-r$, $q_2=\alpha k/\beta=k^2/v$ and

$$b=v+t-1\ . \tag{1.1}$$

An (affine) 1-resolvable design is simply called (affine) resolvable in the usual sense.

Chandak [1], Hedayat and John [3], Most [10], Shah and Gujarathi [12] have discussed several problems on resistant BIB designs. It is known that the property of being resistant depends not only on the parameters of the design, but also depends on the way the design has been constructed. Hedayat and John [3] gave some necessary conditions on parameters for a BIB design to be LR(1). The purpose of this paper is further to investigate LRBIB designs of degree one. By introducing a new necessary condition, we shall characterize affine α-resolvable LRBIB designs of degree one. Finally, we make a useful comment on a construction method of LRBIB designs of degree one which are not 3-designs, with illustrations.

2. Necessary condition and characterization

At first, we present a trivial result worth writing. From the method of construction by taking as blocks all possible combinations of k out of v treatments, the following is immediately given.

LEMMA 2.1. *Any* BIBD $\left(v, b=\binom{v}{k}, r=\binom{v-1}{k-1}, k, \lambda=\binom{v-2}{k-2}\right)$ *of the unreduced type (i.e., all combination type) is* GR(m) *with* $m \leqq k$.

Note that a GRBIB design is also LR. From Lemma 2.1 and the present purpose of investigation of LRBIB designs of degree one, we shall deal only with BIB designs satisfying $k \geqq 3$ throughout this paper.

Let D be a BIBD $(v, b, r, k\,(\geqq 3), \lambda)$ on T and let $L=\{x\} \subset T$. Divide D into two parts, D_1 and D_2^*. D_1 consists of all the blocks which do not contain the treatment x; D_2^* is the set of blocks which contain x. Next, let D_2 be the design obtained by deleting x from the blocks of D_2^*. Hedayat and John [3] proved the following.

PROPOSITION 2.1. *D is* LR(1) *if and only if D_1 is a BIB design (if and only if D_2 is a BIB design).*

This proposition yields

COROLLARY 2.1. *If D is* LR(1), *then its parameters satisfy the following conditions:*

(i) $r \geqq v-1$ (*if and only if* $\lambda \geqq k-1$),

(ii) $\lambda(k-2)/(v-2)=$ *positive integer*,

(iii) $b \geqq v+r-1$.

PROOF. (i) and (ii) are already given by Hedayat and John ([3], Corollary 4.2). From Proposition 2.1 and an assumption, D_1 is a BIBD $(v_1=v-1, b_1=b-r, r_1=r-\lambda, k_1=k, \lambda_1=\lambda-\lambda(k-2)/(v-2))$. The Fisher inequality for D_1 requires $b_1 \geqq v_1$, i.e., $b \geqq v+r-1$ which completes the proof.

Remark 2.1. Hedayat and John [3] gave another necessary condition, $\lambda > 1$. However, by using the Fisher inequality for parameters of D_2, we obtain $\lambda \geqq k-1$ (which is equivalent to (i)). Therefore, their condition $\lambda > 1$ is superfluous. The condition (iii) is new. It is known (cf. Kageyama [5]) that each of the following conditions is sufficient for the validity of $b \geqq v+r-1$; (a) D is resolvable, (b) $k \mid v$, (c) D has disjoint blocks.

In the light of the sufficient condition (a) presented in Remark 2.1, we next consider a BIB design with the property of affine α-resolvability to characterize a family of affine α-resolvable LRBIB designs of degree one. In this case as an example of validating a condition (iii) in Corollary 2.1, we have the following.

THEOREM 2.1. *Any affine α-resolvable BIB design is for $\alpha \geqq 2$ not* LR (1).

Proof is obvious from (1.1) and Corollary 2.1 (iii) with $r=\alpha t$ if $\alpha \geqq 2$.

From now on, we shall characterize affine resolvable BIB designs as the remaining case.

THEOREM 2.2. *Any affine resolvable BIB design is not* LR (1) *except for a series of parameters*

$$v=4t, \quad b=2(4t-1), \quad r=4t-1, \quad k=2t, \quad \lambda=2t-1, \quad t \geqq 1 . \tag{2.1}$$

PROOF. It is known (cf. Shrikhande [13]) that the parameters of an affine resolvable BIB design can be expressed in terms of only two integral parameters n and t as $v=n^2[(n-1)t+1]$, $b=n(n^2t+n+1)$, $r=n^2t+n+1$, $k=n[(n-1)t+1]$, $\lambda=nt+1$, $n \geqq 2$, $t \geqq 1$. In this case, from (i) in Corollary 2.1, it holds that $r \geqq v-1$ being equivalent to $0 \geqq (n-2)(n^2t+n+1)$ which implies that when $n \geqq 3$ there does not exist an affine resolvable LRBIB design of degree one. As the remaining case of possibility of LR (1), we have $n=2$ which yields that for $t \geqq 1$, $v=4t$, $b=2(4t-1)$, $r=4t-1$, $k=2t$, $\lambda=2t-1$. This completes the proof.

Remark 2.2. From Chandak [1], Hedayat and John [3], it follows that if a Hadamard matrix of order $4t$ exists, then an affine resolvable BIB design with (2.1) exists. Furthermore, this design is also GR (1) (i.e., the design is also a 3-design) and further is LR (k). Incidentally, it is conjectured (cf. Shrikhande [13]) that an affine resolvable BIB design with (2.1) exists for every positive integer t. Sprott [14] gave a difference set for an affine resolvable BIB design with (2.1) when $4t-1$ is a prime or a prime power. Finally, it is easily shown that a BIB design with $v=2k$ and $k \geqq 3$ satisfying $\lambda(k-2)/(v-2)$ being an integer is completely characterized by the following two series: For l, $l' \geqq 1$,

$$v=2k, \quad b=2l(2k-1), \quad r=l(2k-1), \quad k \text{ (being even)}, \quad \lambda=l(k-1) ;$$

$$v=2k, \quad b=4l'(2k-1), \quad r=2l'(2k-1), \quad k, \quad \lambda=2l'(k-1) .$$

Some practical examples in the above series will be seen in Section 3.

3. Comments on constructions

There are some construction methods of LR (or GR) BIB designs. Shah and Gujarathi [12] gave a construction theorem of an LRBIB design of degree one. But the parameters of designs constructed by their method are here characterized to be the same as those in a construction theorem (Theorem 5.1) by Hedayat and John [3] who presented the following.

PROPOSITION 3.1. *The existence of a* BIBD (v, b, r, k, λ) *satisfying* $b=3r-2\lambda$ *implies the existence of a* GRBIBD $(v+1, 2b, b, (v+1)/2, r)$ *of degree one.*

On the other hand, Shah and Gujarathi [12] presented the following.

PROPOSITION 3.2. *If a* BIBD (v, b, r, k, λ) *satisfying* $r=2\lambda$ *exists, then there exists an* LRBIBD $(v'=v+1, b'=2b, r'=b, k'=k, \lambda'=b-r)$ *of degree one with respect to two given treatments.*

We now show that Propositions 3.1 and 3.2 produce the same family of parameters each other, but their block structures are different.

We shall characterize a BIB design in Proposition 3.2 by using necessary conditions (i), (ii) and (iii) in Corollary 2.1. It is obvious from the Fisher inequality for the BIB design that $r' \geqq v'-1$ and $b' \geqq v'+r'-1$ hold. For (ii) we get

$$\frac{\lambda'(k'-2)}{v'-2}=\frac{\lambda(k-2)}{k}=\lambda-\frac{2\lambda}{k}$$

which should be an integer. Hence we can let $2\lambda=lk$ for a positive integer l. In this case, a BIB design satisfying $r=2\lambda$ can be expressed as

$$(3.1) \quad v=2k-1, \quad b=l(2k-1), \quad r=lk, \quad k, \quad \lambda=lk/2 \qquad \text{for} \quad l \geqq 1 .$$

Thus, Proposition 3.2 shows that the existence of a BIB design with parameters (3.1) implies the existence of an LRBIBD $(2k, 2l(2k-1), l(2k-1), k, l(k-1))$ of degree one. But, since $v<2k$ in (3.1), the complement of the design should be generally considered as

$$(3.2) \quad v'=2k-1, \quad b'=l(2k-1), \quad r'=l(k-1), \quad k'=k-1, \quad \lambda'=lk/2-l \qquad \text{for} \quad l \geqq 1 .$$

Therefore, Proposition 3.2 is equivalent to the following.

PROPOSITION 3.3. *If a* BIBD $(v'=2k-1, b'=l(2k-1), r'=l(k-1), k'$

$=k-1$, $\lambda'=lk/2-l$) *exists for* $l\geqq 1$, *then there exists an* LRBIBD $(v'+1, 2b', b', (v'+1)/2, r')$ *of degree one with respect to two given treatments.*

Since a BIB design satisfying $b=3r-2\lambda$ in Proposition 3.1 can easily be characterized as (3.2) (cf. Hedayat and John [3]), it follows that Propositions 3.1 and 3.2 produce the same family in the sense of constructing LRBIB designs of degree one. However, their methods produce designs having different remarkable block structures. This is explained as follows.

When N is the incidence matrix of a started BIB design, Proposition 3.1 yields the following structure, in an incidence form as

$$\begin{bmatrix} N & N^c \\ 1\cdots 1 & 0\cdots 0 \end{bmatrix} \tag{3.3}$$

where N^c is the complement of N. On the other hand, since N is generally partitioned as

$$N=\begin{bmatrix} 1\cdots 1 & 0\cdots 0 \\ N_1 & N_2 \end{bmatrix},$$

Proposition 3.2 shows the following structure, in an incidence form as

$$\begin{bmatrix} 1\cdots 1 & 0\cdots 0 & 0\cdots 0 & 1\cdots 1 \\ N_1 & N_2 & N_1 & N_2^c \\ 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & 1\cdots 1 \end{bmatrix}. \tag{3.4}$$

The structure (3.3) always yields a 3-design (i.e., the resultant design is GR), whereas the structure (3.4) does not yield a 3-design. It appears that most of available LRBIB designs of degree one have a block structure of a 3-design. Thus, Proposition 3.3 is very useful and interesting in the sense that we can get systematically an LRBIB design which is not a 3-design. Shah and Gujarathi [12] do not point out such a property.

Finally, we can present all the parameter sets (which are exhaustive) of existent LRBIB designs with $r\leqq 30$ of degree one, *constructed by their method*, as:

(1) $v=6$, $b=20$, $r=10$, $k=3$, $\lambda=4$;
(2) $v=6$, $b=40$, $r=20$, $k=3$, $\lambda=8$;
(3) $v=6$, $b=60$, $r=30$, $k=3$, $\lambda=12$;
(4) $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$;
(5) $v=8$, $b=28$, $r=14$, $k=4$, $\lambda=6$;
(6) $v=8$, $b=42$, $r=21$, $k=4$, $\lambda=9$;
(7) $v=8$, $b=56$, $r=28$, $k=4$, $\lambda=12$;
(8) $v=10$, $b=36$, $r=18$, $k=5$, $\lambda=8$;

(9) $v=12$, $b=22$, $r=11$, $k=6$, $\lambda=5$;
(10) $v=12$, $b=44$, $r=22$, $k=6$, $\lambda=10$;
(11) $v=14$, $b=52$, $r=26$, $k=7$, $\lambda=12$;
(12) $v=16$, $b=30$, $r=15$, $k=8$, $\lambda=7$;
(13) $v=16$, $b=60$, $r=30$, $k=8$, $\lambda=14$;
(14) $v=20$, $b=38$, $r=19$, $k=10$, $\lambda=9$;
(15) $v=24$, $b=46$, $r=23$, $k=12$, $\lambda=11$;
(16) $v=28$, $b=54$, $r=27$, $k=14$, $\lambda=13$.

Note that each parameter set has two designs with different block structures (i.e., one is a 3-design, but another is not a 3-design), and that (4), (9), (12) and (14) belong to series (2.1) of affine resolvable BIB designs (cf. Kageyama [8]). Further note that for a 3-(v, k, λ_3) design, $\lambda(k-2)/(v-2)$ $(=\lambda_3)$ is necessarily an integer. Furthermore, as examples of parameters of other existing LRBIB designs of degree one, we can present for the range of $r \leqq 20$,

(17) $v=10$, $b=30$, $r=12$, $k=4$, $\lambda=4$;
(18) $v=11$, $b=33$, $r=15$, $k=5$, $\lambda=6$;
(19) $v=17$, $b=68$, $r=20$, $k=5$, $\lambda=5$.

In fact, there exist 3-designs with parameters (17) to (19) (see Hedayat and Kageyama [4]).

There are not so many sets of parameters in a practical range for D. Hedayat and John [3] showed that D is GR(1) if and only if it is a 3-design, and that any t-design, $t \geqq 3$, is at least GR(1). Since there are a number of families of t-designs, there exist a number of families of LRBIB designs of degree one. However, these designs mostly have large values of the design parameters.

4. Additional remark

The concept of resistance for BIB designs can be similarly extended to a case of variance-balanced block (VBB) designs. The definition is still valid from replacing only a term "a BIB design" as a started design by "a VBB design". However, in this case it is difficult to characterize resistant VBB designs completely. This is due to the following grounds:

(i) Any equi-blocksized VBB design is a BIB design.

(ii) Any equi-replicated VBB design with $b=v$ is a symmetrical BIB design.

(iii) Any equi-replicated VBB design with $b=v+1$ does not exist.

Therefore, there is no need to consider further such a characterization problem for these three cases. Thus, it is in general sufficient to consider such a problem for unequal-blocksized VBB designs. In this case we have some observations from Kageyama [9]: For what forms of

N_α and $\boldsymbol{\alpha}$, can a VBB design N_α with v treatments and b blocks be extended to a VBB design $N=[\boldsymbol{\alpha}' : N_\alpha']'$ with $v+1$ treatments for a row vector $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \cdots, \alpha_b)$ with $\alpha_i=0$ or 1? For this problem, it is known that when $\boldsymbol{\alpha}=O_{1\times b}$, N_α is a disconnected VBB design with $k_1=\cdots=k_b=1$; when $\boldsymbol{\alpha}=J_{1\times b}$, $N_\alpha=J_{v\times b}$; when $\boldsymbol{\alpha}=[J_{1\times x} : O_{1\times(b-x)}]$, corresponding to the partition of $\boldsymbol{\alpha}$ the incidence matrix N_α can be decomposed into $[N_1 : N_2]$ as follows.

$$N=\begin{bmatrix} J_{1\times x} & O_{1\times(b-x)} \\ N_1 & N_2 \end{bmatrix}.$$

In this case, if the block sizes of N_1 are constant, then both N_1 and N_2 are VBB designs (in fact, N_1 is a BIB design). However, if the block sizes of N_1 are not necessarily constant, we have an example in which both $N_\alpha=[N_1 : N_2]$ and $N'=[\boldsymbol{\alpha}' : N_\alpha']$ are VBB designs and further N_1 and N_2 are not VBB designs. In this sense, it is not easy to characterize resistant VBB designs, similar to the characterization of LRBIB designs of degree one. In spite of such a situation, there are a number of families of LRVBB designs of degree one. For construction methods of those designs, refer to Hedayat and Federer [2] and Kageyama [7] and [9].

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HIROSHIMA UNIVERSITY

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