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ON THE ROBUSTNESS OF BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF RESOLUTION $2l+1$ IN THE PRESENCE OF OUTLIERS*

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Summary

By use of the algebraic structure, we obtain a simplified expression for the outlier-insensitivity factor for balanced fractional 2^m factorial (2^m-BFF) designs of resolution $2l+1$ derived from simple arrays (S-arrays), whose measure has been introduced by Ghosh and Kipngeno (1985, *J. Statist. Plann. Inference*, **11**, 119–129). It is defined by use of the measure suggested by Box and Draper (1975, *Biometrika*, **62** (2), 347–352). As examples, we study the sensitivity of A-optimal 2^m-BFF designs of resolution VII (i.e., $l=3$) given by Shirakura (1976, *Ann. Statist.*, **4**, 515–531 ; 1977, *Hiroshima Math. J.*, **7**, 217–285). We observe that these designs are robust in the sense that they have low sensitivities.

1. Introduction

The concept of a balanced array (B-array) was introduced and first studied by Chakravarti [2]. A general connection between a B-array of strength $2l$ and a 2^m-BFF design of resolution $2l+1$ was established by Yamamoto, Shirakura and Kuwada [15]. Furthermore, these authors ([16]) obtained an explicit expression for the characteristic polynomial of the information matrix of a 2^m-BFF design of resolution $2l+1$ by utilizing the decomposition of the triangular multidimensional partially balanced (TMDPB) association algebra into its $l+1$ two-sided ideals. This polynomial includes the results obtained by Srivastava and Chopra [12] as a special case. A- and/or D-optimal 2^m-BFF designs of resolution V and VII have been obtained by several authors (e.g., Shirakura [9] and [10] and Srivastava and/or Chopra [3]–[7], [13] and [14]).

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As a measure of sensitivity in the sense that the design should be insensitive to wild observations, Box and Draper [1] introduced the sum of squares of diagonal elements of the projection matrix, when the number of observations and the number of unknown effects in the model assumed are both fixed. Recently, using the measure suggested by Box and Draper [1], Ghosh and Kipngeno [8] have defined a new measure of robustness of a design with respect to outliers, which is called the "outlier-insensitivity factor", and they have found these values for A-optimal 2^m-BFF designs of resolution V given by Srivastava and/or Chopra [3]–[7], [13] and [14].

In this paper, we obtain a simplified expression for the outlier-insensitivity factor for a 2^m-BFF design of resolution $2l+1$ derived from an S-array, by use of the properties of the TMDPB association scheme and its algebra. As examples, we find its value for A-optimal 2^m-BFF designs of resolution VII given by Shirakura [9] and [10], when $6 \leqq m \leqq 9$.

2. Measures of sensitivity

Consider a fractional experiment with m factors each at two levels (0 and 1, say). Then the ordinary linear model associated with a fraction T with N assemblies is

$$(2.1) \qquad \mathcal{E}[\boldsymbol{y}(T)] = E_T \boldsymbol{\Theta}, \qquad \mathrm{Var}\,[\boldsymbol{y}(T)] = \sigma^2 I_N, \qquad \mathrm{Rank}\,(E_T) = \nu_l,$$

where $\mathcal{E}[\boldsymbol{y}]$ stands for the expected value of $\boldsymbol{y}$, $\boldsymbol{y}(T)$ is a vector of N observations, E_T is the $N \times \nu_l$ design matrix, $\boldsymbol{\Theta}$ is a vector of unknown effects up to the l-factor interactions, σ^2 is a constant which may or may not be known, and $\nu_l = 1 + \binom{m}{1} + \cdots + \binom{m}{l}$. Here $l \leqq [m/2]$, where $[x]$ denotes the largest integer not exceeding x. The predicted value of $\boldsymbol{y}(T)$ is $\hat{\boldsymbol{y}}(T) = R\boldsymbol{y}(T)$, where $R = E_T(E_T' E_T)^{-1} E_T'$ which is known as the projection matrix. Here A' denotes the transpose of a matrix A. Suppose that the u-th observation in $\boldsymbol{y}(T)$ is an outlier in the sense that an unknown aberration c, a fixed constant, is added to it. And we denote the resulting observation vector as $\boldsymbol{y}^*(T)$ and the corresponding predicted value as $\hat{\boldsymbol{y}}^*(T) = R\boldsymbol{y}^*(T)$. Then the quantity $d_u = \{\hat{\boldsymbol{y}}^*(T) - \hat{\boldsymbol{y}}(T)\}' \{\hat{\boldsymbol{y}}^*(T) - \hat{\boldsymbol{y}}(T)\}$ is a measure of overall discrepancy caused by the effect of c on the u-th observation, and it is equal to $c^2 r_{uu}$, where r_{uu} is the u-th diagonal element of R. Clearly

$$(2.2) \qquad \sum_{u=0}^{N} d_u = c^2 \sum_{u=1}^{N} r_{uu} = c^2 \nu_l,$$

because of the idempotency of R. If it is equally likely that c occurs with any of N observations, giving rise to $d_1, \cdots, d_N$, the average discrepancy is $\bar{d}=c^2 \sum_{u=0}^{N} r_{uu}/N=c^2\nu_l/N$. If the d_u $(u=1,\cdots,N)$ are as small as possible, then a design is said to be insensitive with respect to outliers. Because $\sum d_u$ is fixed as in (2.2), this means that the d_u are as uniformly as possible for a design insensitive to outliers. As one convenient measure of uniformity, Box and Draper [1] introduced the following:

$$V(d)=\sum_{u=1}^{N}(d_u-\bar{d})^2/N=c^4\{r-(\nu_l)^2/N\}/N ,$$

where

$$r=\sum_{u=1}^{N}(r_{uu})^2 .$$

Thus to ensure insensitivity to outliers, $V(d)$ should be made small. Minimization of $V(d)$ is equivalent to minimization of r, when N and ν_l are both fixed. Recently, Ghosh and Kipngeno [8] have defined the outlier-insensitivity factor, E, say, by

$$E=100\times(\nu_l)^2/(Nr) ,$$

because $r\geqq(\nu_l)^2/N$.

Note that under the model (2.1), for an orthogonal design or $N=\nu_l$, i.e., saturated design, we have $E=100$.

3. Outlier-insensitivity factors

Under the model (2.1), the expected value of an observation associated with an assembly $(\varepsilon_1,\cdots,\varepsilon_m)$ with $\varepsilon_k=0$ or 1 is given by

$$\mathcal{E}[y(\varepsilon_1,\cdots,\varepsilon_m)]=\sum_{\eta_1,\cdots,\eta_m} d_{\varepsilon_1}(\eta_1)\cdots d_{\varepsilon_m}(\eta_m)\theta(\eta_1,\cdots,\eta_m) , \tag{3.1}$$

where the summation is over all binary numbers $(\eta_1,\cdots,\eta_m)$ with $\eta_k=0$ or 1 such that $0\leqq\eta_1+\cdots+\eta_m\leqq l$, and

$$d_0(0)=d_1(0)=d_1(1)=1 \quad\text{and}\quad d_0(1)=-1 . \tag{3.2}$$

Note that when $\eta_1+\cdots+\eta_m=j$ $(j=0,1,\cdots,l)$, $\theta(\boldsymbol{\eta}_j)$ is called the j-factor interaction, where $\boldsymbol{\eta}_j'=(\eta_1,\cdots,\eta_m)$.

When T is a B-array of strength m, size N, m constraints, 2 levels and index set $\{\lambda_0,\lambda_1,\cdots,\lambda_m\}$, T is called an S-array, written SA$(m;\lambda_0,\lambda_1,\cdots,\lambda_m)$ for brevity (see [9]). For T being an SA$(m;\lambda_0,\lambda_1,\cdots,\lambda_m)$, T can be expressed as $T=\|\boldsymbol{j}_{\lambda_i}\otimes T_i\|$ if $\lambda_i\geqq1$ $(i=0,1,\cdots,m)$, where T_i are the (0, 1) matrices of size $\binom{m}{i}\times m$ whose rows denote all distinct vectors with weight i. Here $\boldsymbol{j}_p$ and $A\otimes B$ denote, respectively, the $p\times 1$ vector with all unity and the Kronecker product of two matrices A and B, and the weight of a (0, 1) vector means the number of ones in the vector.

Let $E_T=\|\boldsymbol{j}_{\lambda_i}\otimes E_i(j)\|$ if $\lambda_i\geqq 1$ ($i=0,1,\cdots,m$; $j=0,1,\cdots,l$), where $E_i(j)$ denote the submatrices of E_T corresponding, respectively, to T_i and $\theta(\boldsymbol{\eta}_j)$. Then from (3.1), (3.2) and Appendix, we have

$$E_i(j)=\sum_{\alpha=0}^{\min(i,j,m-i)}(-1)^{j-\min(i,j)+\alpha}A_\alpha^{(i,j)}$$
$$\text{for}\quad i=0,1,\cdots,m\quad\text{and}\quad j=0,1,\cdots,l,$$

where $\min(a_1,\cdots,a_n)$ denotes the minimum value of integers $a_1,\cdots,a_n$, and $A_\alpha^{(i,j)}$ are the local association matrices which are given in Appendix. Hence, it follows from Appendix that

$$(3.3)\quad E_i(j)=\begin{cases}\displaystyle\sum_{\alpha=0}^{\min(i,j)}(-1)^{j-\min(i,j)+\alpha}A_\alpha^{(i,j)} & \text{for}\quad 0\leqq i\leqq[m/2],\\ \displaystyle\sum_{\alpha=0}^{\min(m-i,j)}(-1)^{\min(m-i,j)-\alpha}A_\alpha^{(m-i,j)} & \text{for}\quad [m/2]<i\leqq m.\end{cases}$$

Thus, from (3.3) and Appendix, the following is immediate.

LEMMA 3.1. *The submatrices $E_i(j)$ of E_T can be expressed as*

$$E_i(j)=\begin{cases}\displaystyle\sum_{\beta=0}^{\min(i,j)}h_\beta^{(i,j)}A_\beta^{*(i,j)} & \\ \qquad\qquad\textit{for}\quad 0\leqq i\leqq[m/2]\quad\textit{and}\quad 0\leqq j\leqq l,\\ \displaystyle\sum_{\beta=0}^{\min(m-i,j)}h_\beta^{*(m-i,j)}A_\beta^{*(m-i,j)} & \\ \qquad\qquad\textit{for}\quad [m/2]<i\leqq m\quad\textit{and}\quad 0\leqq j\leqq l,\end{cases}$$

where

$$h_\beta^{(i,j)}=\begin{cases}(-1)^j(-2)^i\left\{\binom{m-i-\beta}{j-i}\Big/\binom{j-\beta}{j-i}\right\}^{1/2}\\ \qquad\times\left\{\displaystyle\sum_{b=0}^{i-\beta}(-1/2)^b\binom{j-\beta}{i-\beta-b}\binom{m-i-\beta+b}{b}\right\}\\ \qquad\qquad\qquad\textit{if}\quad 0\leqq i\leqq j\leqq[m/2],\\ 2^j\left\{\binom{m-j-\beta}{i-j}\Big/\binom{i-\beta}{i-j}\right\}^{1/2}\\ \qquad\times\left\{\displaystyle\sum_{b=0}^{j-\beta}(-1/2)^b\binom{i-\beta}{j-\beta-b}\binom{m-j-\beta+b}{b}\right\}\\ \qquad\qquad\qquad\textit{if}\quad 0\leqq j\leqq i\leqq[m/2],\end{cases}$$

and $h_\beta^{*(m-i,j)}$ are given by replacing i in $h_\beta^{(i,j)}$ by $m-i$.

For T being an SA$(m;\lambda_0,\lambda_1,\cdots,\lambda_m)$, the submatrices $M_{u,v}$ of $E_T'E_T$ corresponding, respectively, to $\{\theta(\eta_u)\}$ and $\{\theta(\eta_v)\}$ can be expressed as

$$M_{u,v}=\sum_{\alpha=0}^{\min(u,v)}\gamma_{|v-u|+2\alpha}A_\alpha^{(u,v)}\qquad\text{for}\quad 0\leqq u,\,v\leqq l\,,$$

where a connection between γ's and λ's is given by

$$\gamma_i=\sum_{j=0}^{m}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{m-i}{j-i+p}\lambda_j$$

(see [15]). It follows from Appendix that $M_{u,v}$ can be expressed as

$$M_{u,v}=\sum_{\beta=0}^{\min(u,v)}\kappa_\beta^{u-\beta,v-\beta}A_\beta^{\sharp(u,v)}\qquad\text{for}\quad 0\leqq u,\,v\leqq l\,,$$

where

$$\kappa_\beta^{u,v}=\sum_{\alpha=0}^{\beta+u}\gamma_{v-u+2\alpha}z_{\beta\alpha}^{(\beta+u,\beta+v)}\qquad\text{for}\quad 0\leqq u\leqq v\leqq l-\beta$$

and $\kappa_\beta^{v,u}=\kappa_\beta^{u,v}$ for $u\leqq v$ (see [16]). If $E_T'E_T$ is non-singular, then the submatrices $M_{u,v}^*$ of $(E_T'E_T)^{-1}$ corresponding, respectively, to $\{\theta(\eta_u)\}$ and $\{\theta(\eta_v)\}$ is

$$M_{u,v}^*=\sum_{\beta=0}^{\min(u,v)}\kappa_{u-\beta,v-\beta}^{\beta}A_\beta^{\sharp(u,v)}\qquad\text{for}\quad 0\leqq u,\,v\leqq l\,,\tag{3.4}$$

where $\|\kappa_{u,v}^\beta\|=\|\kappa_\beta^{u,v}\|^{-1}$. Note that the order of $\|\kappa_{u,v}^\beta\|$ is $l+1-\beta$ ($\beta=0,1,\cdots,l$) and does not depend on m, while the order of $E_T'E_T$ is ν_l. Thus from Lemma 3.1, (3.4) and Appendix, the following is immediate.

LEMMA 3.2. *The diagonal submatrices R_{ii} of R corresponding to T_i can be expressed as*

$$R_{ii}=\begin{cases}\displaystyle\sum_{j=0}^{l}\sum_{k=0}^{l}\left\{\sum_{\beta=0}^{\min(i,j,k)}h_\beta^{(i,j)}h_\beta^{(i,k)}\kappa_{j-\beta,k-\beta}^{\beta}A_\beta^{\sharp(i,i)}\right\} & \\ \qquad\qquad\qquad\qquad\qquad\text{if}\quad 0\leqq i\leqq[m/2]\,, & \\ \displaystyle\sum_{j=0}^{l}\sum_{k=0}^{l}\left\{\sum_{\beta=0}^{\min(m-i,j,k)}h_\beta^{*(m-i,j)}h_\beta^{*(m-i,k)}\kappa_{j-\beta,k-\beta}^{\beta}A_\beta^{\sharp(m-i,m-i)}\right\} & \\ \qquad\qquad\qquad\qquad\qquad\text{if}\quad [m/2]<i\leqq m\,. & \end{cases}$$

Since $A_0^{(i,i)}=I_{\binom{m}{i}}$ for $i=0,1,\cdots,m$, it follows from Appendix that the coefficients of $A_0^{(i,i)}$ in $A_\beta^{\sharp(i,i)}$ are given by

$$z_{(i,i)}^{\beta0}=\phi_\beta z_{\beta0}^{(i,i)}\Big/\left\{\binom{m}{i}\binom{i}{0}\binom{m-i}{0}\right\}=\phi_\beta\Big/\binom{m}{i}\,,\tag{3.5}$$

where $z_{\beta\alpha}^{(u,v)}$, $z_{(u,v)}^{\beta\alpha}$ and ϕ_β are given in Appendix. Hence, from Lemma 3.2 and (3.5), the diagonal elements r_{ii} of R_{ii} are given by

$$(3.6)\qquad r_{ii}=\begin{cases}\left[\sum_{j=0}^{l}\sum_{k=0}^{l}\left\{\sum_{\beta=0}^{\min(i,j,k)}\phi_\beta h_\beta^{(i,j)}h_\beta^{(i,k)}\kappa_{j-\beta,k-\beta}^{\beta}\right\}\right]\Big/\binom{m}{i} & \text{if } 0\leqq i\leqq[m/2]\,,\\ \left[\sum_{j=0}^{l}\sum_{k=0}^{l}\left\{\sum_{\beta=0}^{\min(m-i,j,k)}\phi_\beta h_\beta^{*(m-i,j)}h_\beta^{*(m-i,k)}\kappa_{j-\beta,k-\beta}^{\beta}\right\}\right]\Big/\binom{m}{i} & \text{if } [m/2]<i\leqq m\,.\end{cases}$$

Thus the following is immediate.

LEMMA 3.3. *For T being an* SA $(m;\ \lambda_0, \lambda_1, \cdots, \lambda_m)$, *under the model* (2.1), *r is given by*

$$r=\sum_{i=0}^{m}\lambda_i\binom{m}{i}(r_{ii})^2\,,$$

where r_{ii} are given by (3.6).

From Lemma 3.3, we have the main results of this paper as follows.

THEOREM 3.1. *Let T be an* SA $(m;\ \lambda_0, \lambda_1, \cdots, \lambda_m)$. *Then under the model* (2.1), *the outlier-insensitivity factor E is given by*

$$E=100\times(\nu_l)^2/(Nr)\,,$$

where r is given in Lemma 3.3.

4. Calculation of E for Shirakura's designs

In this section, we study the sensitivity of A-optimal 2^m-BFF designs of resolution VII (i.e., $l=3$) given by Shirakura [9] and [10]. It follows from Lemma 3.1 that

$$h_0^{(i,0)}=\left\{\binom{m}{i}\right\}^{1/2} \qquad \text{for} \quad 0\leqq i\leqq[m/2]\,,$$

$$h_0^{(0,j)}=(-1)^j\left\{\binom{m}{j}\right\}^{1/2} \qquad \text{for} \quad j=1, 2, 3\,,$$

$$h_0^{(i,1)}=-(m-2i)\left\{\binom{m-1}{i-1}\Big/i\right\}^{1/2} \qquad \text{for} \quad 1\leqq i\leqq[m/2]\,,$$

$$h_1^{(i,1)}=2\left\{\binom{m-2}{i-1}\right\}^{1/2} \qquad \text{for} \quad 1\leqq i\leqq[m/2]\,,$$

$$h_\beta^{(1,j)}=(-1)^{j-1}h_\beta^{(j,1)} \qquad \text{for} \quad j=2, 3 \quad \text{and} \quad \beta=0, 1\,,$$

$$h_0^{(i,2)}=\left\{4\binom{i}{2}-2(m-1)i+\binom{m}{2}\right\}\left\{\binom{m-2}{i-2}\Big/\binom{i}{2}\right\}^{1/2} \qquad \text{for} \quad 2\leqq i\leqq[m/2]\,,$$

$$h_1^{(i,2)}=-2(m-2i)\left\{\binom{m-3}{i-2}\Big/(i-1)\right\}^{1/2}\qquad \text{for}\quad 2\leqq i\leqq[m/2]\,,$$

$$h_2^{(i,2)}=4\left\{\binom{m-4}{i-2}\right\}^{1/2}\qquad \text{for}\quad 2\leqq i\leqq[m/2]\,,$$

$$h_\beta^{(2,3)}=-h_\beta^{(3,2)}\qquad \text{for}\quad \beta=0,1,2\,,$$

$$h_0^{(i,3)}=\left\{8\binom{i}{3}-4(m-2)\binom{i}{2}+2\binom{m-1}{2}i-\binom{m}{3}\right\}\left\{\binom{m-3}{i-3}\Big/\binom{i}{3}\right\}^{1/2}\qquad \text{for}\quad 3\leqq i\leqq[m/2]\,,$$

$$h_1^{(i,3)}=\left\{8\binom{i-1}{2}-4(m-3)(i-1)+2\binom{m-2}{2}\right\}\left\{\binom{m-4}{i-3}\Big/\binom{i-1}{2}\right\}^{1/2}\qquad \text{for}\quad 3\leqq i\leqq[m/2]\,,$$

$$h_2^{(i,3)}=-4(m-2i)\left\{\binom{m-5}{i-3}\Big/(i-2)\right\}^{1/2}\qquad \text{for}\quad 3\leqq i\leqq[m/2]\,,$$

$$h_3^{(i,3)}=8\left\{\binom{m-6}{i-3}\right\}^{1/2}\qquad \text{for}\quad 3\leqq i\leqq[m/2]\,,$$

and $h_\beta^{*(m-i,j)}$ are given by replacing i in $h_\beta^{(i,j)}$ by $m-i$. Thus for T being an SA $(m;\lambda_0,\lambda_1,\cdots,\lambda_m)$, we have

$$E=100\times(\nu_3)^2/(Nr)\,,$$

where

$$r=\sum_{i=0}^{m}\lambda_i\binom{m}{i}(r_{ii})^2$$

and

$$r_{ii}=\begin{cases}\left[\sum_{j=0}^{3}\sum_{k=0}^{3}\left\{\sum_{\beta=0}^{\min(i,j,k)}\phi_\beta h_\beta^{(i,j)}h_\beta^{(i,k)}\kappa_{j-\beta,k-\beta}^{\beta}\right\}\right]\Big/\binom{m}{i} & \text{if}\quad 0\leqq i\leqq[m/2]\,,\\ \left[\sum_{j=0}^{3}\sum_{k=0}^{3}\left\{\sum_{\beta=0}^{\min(m-i,j,k)}\phi_\beta h_\beta^{*(m-i,j)}h_\beta^{*(m-i,k)}\kappa_{j-\beta,k-\beta}^{\beta}\right\}\right]\Big/\binom{m}{i} & \text{if}\quad [m/2]<i\leqq m\,.\end{cases}$$

For $6\leqq m\leqq 9$, all A-optimal 2^m-BFF designs of resolution VII given by Shirakura [9] and [10] except for the designs corresponding to $m=8$ and $N=127$b, 128b are S-arrays. In Tables 1, 2, 3 and 4, the values of the outlier-insensitivity factor E of A-optimal 2^m-BFF designs of resolution VII are presented for $m=6$ and $42\leqq N\leqq 64$, $m=7$ and $64\leqq N\leqq 90$, $m=8$ and $93\leqq N\leqq 128$, and $m=9$ and $130\leqq N\leqq 150$, respectively.

These values are greater than 93, 87, 92 and 91 for $m=6$, 7, 8 and 9, respectively. Therefore we conclude that the sensitivities of the Shirakura's designs to outliers are low.

Table 1.

N	E
42	100.00000
43	99.81403
44	98.68351
45	97.42432
46	95.70471
47	99.30583
48	99.28371
48	98.83269
49	98.80295
50	97.79658
50	97.70547
51	96.72022
52	95.31230
52	95.27303
53	93.92315
54	95.26166
55	95.05672
56	94.22506
57	98.92164
58	99.10209
59	98.91176
60	98.30057
61	97.22206
62	99.83649
63	99.89680
64	100.00000

Table 2.

N	E
64	100.00000
65a	99.23683
65b	99.59870
66	98.81355
67a	97.59471
67b	97.86155
68	96.67693
69a	95.40590
69b	95.56592
70	98.29494
71a	97.54619
71b	95.35560
72a	97.25246
72b	94.81481
73a	96.33422
73b	94.09180
74a	95.54002
74b	93.06986
75a	94.48201
75b	92.16546
76a	93.52295
76b	91.07877
77	94.42167
78	93.82685
79	93.24751
80	92.47121
81	91.66177
82	90.74376
83	89.85773
84	88.90973
85a	95.80994
85b	90.07147
86a	95.59750
86b	89.86398
87a	95.32026
87b	89.95140
88a	94.71190
88b	89.20927
89a	93.95288
89b	88.75775
90a	93.18285
90b	87.91516

Table 3.

N	E
93	100.00000
94	99.47096
95	99.05598
96	98.19897
97	97.51840
98	96.61199
99	95.83656
100	94.93192
101	96.21760
102	95.81256
103	95.06137
104	94.23622
105	93.58704
106	92.75629
107	98.81635
108	98.50349
109	98.27383
110	97.57208
111	97.05163
112	96.28676
113	95.65440
114	94.87960
115	96.66927
116	96.45581
117	96.00049
118	95.37221
119	94.79438
120	94.10029
121	93.47427
122	92.76839
123	95.96798
124	95.48760
125	95.07700
126	94.45884
127a	99.89345
128a	100.00000

Table 4.

N	E
130	100.00000
131	99.61979
132	98.99250
133	98.31154
134	97.96954
135	97.28181
136	96.78959
137	96.10858
138	95.55457
139	96.87867
140	96.57143
141	96.01454
142	95.40207
143	94.77291
144	94.13993
145	93.62473
146	93.00120
147	92.46373
148	92.08806
149	91.84034
150	91.35135

5. Concluding remarks

Let T^* be an SA (7; 3, 0, 0, 1, 1, 0, 1, 3), which is also a B-array of strength 6 and size 83. Then we have $\det(\|\kappa_0^{u,v}\|)=129466368$, $\det(\|\kappa_1^{w,s}\|)=73728$, $\det(\|\kappa_2^{p,q}\|)=3072$ and $\det(\|\kappa_3^{0,0}\|)=128$, where $\det(A)$ denotes the determinant of a matrix A. Hence, T^* is a 2^7-BFF design of resolution VII. It follows from Section 4 that $E=96.91502$ for T^*, which is the most insensitive design to outliers in the class of balanced designs derived from S-arrays, while, from Table 2, we have $E=89.85773$

for an A-optimal 2^7-BFF design T of resolution VII. On the other hand, it follows from Theorem 2.1 of Shirakura [9] that

$$\mathrm{tr}\{(E'_{T^*}E_{T^*})^{-1}\}=1.50710 \quad \text{for} \quad T^*$$

and

$$\mathrm{tr}\{(E'_T E_T)^{-1}\}=0.88119 \quad \text{for} \quad T.$$

This implies that in the restricted class mentioned above, the most insensitive design to outliers is not always good design in some sense. It, however, is worth to calculate the values of E for some optimal designs with respect to the popular criteria (e.g., A-, D- and E-optimal).

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Appendix

Let $(\varepsilon_1,\cdots,\varepsilon_m)$ $(=\boldsymbol{\varepsilon}_i'$, say) be a (0, 1) vector with weight i. Further let $S_i=\{\boldsymbol{\varepsilon}_i\}$ $(i=0,1,\cdots,m)$. Then $|S_i|=\binom{m}{i}$ $(=n_i$, say), where $|S|$ denotes the cardinality of a set S. Suppose a relation of association defined among the sets S_i in such a way that $\boldsymbol{\varepsilon}_i$ $(\in S_i)$ and $\boldsymbol{\varepsilon}_j$ $(\in S_j)$ are the α-th associates if

$$\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_j=\min(i,j)-\alpha\ , \tag{A.1}$$

where $\alpha=0,1,\cdots,\min(i,m-i,j,m-j)$. Let $\tilde{\boldsymbol{\varepsilon}}_k=\boldsymbol{j}_m-\boldsymbol{\varepsilon}_k$. Then if $\boldsymbol{\varepsilon}_i$ is the α-th associate of $\boldsymbol{\varepsilon}_j$, then (A.1) shows that $\boldsymbol{\varepsilon}_i$ is the α^*-th associate of $\tilde{\boldsymbol{\varepsilon}}_j$, $\tilde{\boldsymbol{\varepsilon}}_i$ is the α^{**}-th associate of $\boldsymbol{\varepsilon}_j$, and $\tilde{\boldsymbol{\varepsilon}}_i$ is the α^{***}-th associate of $\tilde{\boldsymbol{\varepsilon}}_j$, where $\alpha^*=\min(i,m-j)-i+\min(i,j)-\alpha$, $\alpha^{**}=\min(m-i,j)-j+\min(i,j)-\alpha$, and $\alpha^{***}=\min(m-i,m-j)-m+i+j-\min(i,j)+\alpha$.

Let $A_\alpha^{(u,v)}$ $(=A_\alpha^{(v,u)\prime})$ be the $n_u\times n_v$ local association matrix of the TMDPB association scheme, where $0\leqq u\leqq v\leqq[m/2]$ and $\alpha=0,1,\cdots,l$ (e.g., [15]). Further let $A_\beta^{\#(u,v)}$ $(=A_\beta^{\#(v,u)\prime})$ $(0\leqq u\leqq v\leqq[m/2]\,;\ \beta=0,1,\cdots,l)$ be the $n_u\times n_v$ matrices which are linearly linked with $A_\alpha^{(u,v)}$ as follows (e.g., [11] and [16]).

$$A_\alpha^{(u,v)}=\sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)}A_\beta^{\#(u,v)} \qquad \text{for}\quad 0\leqq\alpha\leqq u\leqq v\leqq[m/2] \tag{A.2}$$

and

$$A_\beta^{\#(u,v)}=\sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha}A_\alpha^{(u,v)} \qquad \text{for}\quad 0\leqq\beta\leqq u\leqq v\leqq[m/2]\ , \tag{A.3}$$

where

$$z_{\beta\alpha}^{(u,v)}=\sum_{b=0}^{\alpha}(-1)^{\alpha-b}\frac{\binom{u-\beta}{b}\binom{u-b}{u-\alpha}\binom{m-u-\beta+b}{b}\left\{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}\right\}^{1/2}}{\binom{v-u+b}{b}}$$

and

$$z_{(u,v)}^{\beta\alpha}=\phi_\beta z_{\beta\alpha}^{(u,v)}\Big/\left\{\binom{m}{u}\binom{u}{\alpha}\binom{m-u}{v-u+\alpha}\right\} \qquad \text{for}\quad 0\leqq u\leqq v\leqq[m/2]\ .$$

Here

$$\phi_\beta=\binom{m}{\beta}-\binom{m}{\beta-1} \quad \text{for} \quad \beta=0, 1, \cdots, \min(u, v) .$$

The matrices $A_\beta^{\#(u,v)}$ have the following properties,

$$\sum_{\beta=0}^{u} A_\beta^{\#(u,u)}=I_{n_u} ,$$

$$A_\beta^{\#(u,w)}A_\gamma^{\#(w,v)}=\delta_{\beta,\gamma}A_\beta^{\#(u,v)} ,$$

$$\text{Rank}\,(A_\beta^{\#(u,v)})=\phi_\beta ,$$

where $\delta_{a,b}$ denotes Kronecker's delta, i.e., $\delta_{a,b}=1$ or 0 according as $a=b$ or not. As mentioned above, we have

$$A_\alpha^{(i,j)}=\begin{cases} A_\alpha^{(i,j)} & \text{if} \quad 0\leqq i,\ j\leqq[m/2] , \\ A_{\alpha^*}^{(i,m-j)} & \text{if} \quad 0\leqq i\leqq[m/2]<j\leqq m , \\ A_{\alpha^{**}}^{(m-i,j)} & \text{if} \quad 0\leqq j\leqq[m/2]<i\leqq m , \\ A_{\alpha^{***}}^{(m-i,m-j)} & \text{if} \quad [m/2]<i,\ j\leqq m . \end{cases}$$

Thus $A_\alpha^{(i,j)}$ ($0\leqq i, j\leqq m$; $0\leqq\alpha\leqq\min(i, m-i, j, m-j)$) can be expressed as the linear combinations of $A_\beta^{\#(*,*)}$ as in (A.2) and (A.3).

It is to be noted that the importance of the TMDPB association algebra $\mathcal{A}$ generated by the ordered association matrices $D_\alpha^{(u,v)}$ and also generated by $D_\beta^{\#(u,v)}$ has been discussed in the works of Yamamoto, Shirakura and Kuwada [16], and others. A few references are given above; for further information the readers are requested to see the references therein.