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SOME PROPERTIES OF MULTIVARIATE EXTREME VALUE DISTRIBUTIONS AND MULTIVARIATE TAIL EQUIVALENCE

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Summary

Denote by H a k-dimensional extreme value distribution with marginal distribution $H_i(x)=\Lambda(x)=\exp(-e^{-x})$, $x\in \boldsymbol{R}^1$. Then it is proved that $H(\boldsymbol{x})=\Lambda(x_1)\cdots\Lambda(x_k)$ for any $\boldsymbol{x}=(x_1,\cdots,x_k)\in\boldsymbol{R}^k$, if and only if the equation holds for $\boldsymbol{x}=(0,\cdots,0)$. Next some multivariate extensions of the results by Resnick (1971, *J. Appl. Probab.*, 8, 136–156) on tail equivalence and asymptotic distributions of extremes are established.

1. Introduction

Multivariate extreme order statistics have been studied by many authors, and their results have been summarized by Galambos (see [3], Chapter 5). In this paper, we establish some properties of multivariate extreme value distributions, and by using the results of Marshall and Olkin [4] we extend some of the results given in Resnick [5] to the multivariate case. We may use the same notations as in Marshall and Olkin [4].

For $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}\in\boldsymbol{R}^k$, write $\boldsymbol{ax}+\boldsymbol{b}$ to denote the vector

$$(a_1x_1+b_1,\cdots,a_kx_k+b_k)\,.$$

Basic arithmetical operations are always meant componentwise. Let $\boldsymbol{X}^{(1)},\boldsymbol{X}^{(2)},\cdots$ be a sequence of independent k-dimensional random vectors with common distribution function F and let

$$Z_j^{(n)}=\max_{1\le i\le n} X_j^{(i)}\,,\qquad j=1,\cdots,k\,.$$

If there exist $\boldsymbol{a}^{(n)}>0$, $\boldsymbol{b}^{(n)}\in\boldsymbol{R}^k$, $n=1,2,\cdots$ ($\boldsymbol{a}^{(n)}>0$ means $a_j^{(n)}>0$, $j=1,\cdots,k$) such that $(\boldsymbol{Z}^{(n)}-\boldsymbol{b}^{(n)})/\boldsymbol{a}^{(n)}$ converges in distribution to a random vector $\boldsymbol{U}$ with nondegenerate distribution function H (i.e., all univari-

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ate marginals of H are nondegenerate), then F is said to be in the domain of attraction of H with the notation $F \in \boldsymbol{D}(H)$ and H is said to be a multivariate extreme value distribution. The convergence in distribution is equivalent to the condition

$$\lim_{n\to\infty} F^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})=H(\boldsymbol{x}) \tag{1.1}$$

for all $\boldsymbol{x}$, because multivariate extreme value distributions are continuous (see Theorem 5.2.2 of Galambos [3]).

If $(\boldsymbol{Z}^{(n)}-\boldsymbol{b}^{(n)})/\boldsymbol{a}^{(n)}$ converges in distribution to $\boldsymbol{U}$, then the j-th component of $(\boldsymbol{Z}^{(n)}-\boldsymbol{b}^{(n)})/\boldsymbol{a}^{(n)}$ converges to the j-th component of $\boldsymbol{U}$ and thus normalizing constants $\{a_j^{(n)}\}$, $\{b_j^{(n)}\}$ can be determined from well-known univariate considerations, $j=1,\cdots,k$.

We make extensive use of the following result (see Marshall and Olkin [4] and Theorem 5.3.1 of Galambos [3]).

LEMMA 1.1. *Equation* (1.1) *is equivalent to*

$$\lim_{n\to\infty} n\{1-F(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\}=-\log H(\boldsymbol{x}) \tag{1.2}$$

for all $\boldsymbol{x}$ *such that* $0<H(\boldsymbol{x})<1$.

It is well-known that the univariate extreme value distributions can only be one of the following types

$$\Phi_\alpha(x)=\exp(-x^{-\alpha}), \qquad x>0 \ (\alpha>0),$$

$$\Psi_\alpha(x)=\exp(-(-x)^\alpha), \qquad x\leqq 0 \ (\alpha>0),$$

$$\Lambda(x)=\exp(-e^{-x}), \qquad -\infty<x<\infty.$$

For $k=1$, let

$$x^0=x_F^0=\sup\{x: F(x)<1\}\leqq\infty, \qquad \bar{F}(x)=1-F(x),$$

and

$$\bar{F}^{-1}(p)=F^{-1}(1-p), \qquad p\in(0,1),$$

where $F^{-1}(p)=\inf\{x: F(x)\geqq p\}$ denotes the generalized inverse of F.

If $k>1$ and H is the joint distribution of $(Y_1,\cdots,Y_k)$, then H_i and H_{ij} denote the marginal distributions of Y_i and (Y_i, Y_j), respectively, where $i, j=1,2,\cdots,k$ and $i<j$. For a k-dimensional distribution F, let

$$\boldsymbol{x}_F^0=(x_{F_1}^0,\cdots,x_{F_k}^0).$$

2. Multivariate extreme value distributions

In this section we establish some properties of multivariate extreme value distributions.

THEOREM 2.1. *Let H be a nondegenerate k-dimensional distribution function. Then a necessary and sufficient condition that H is an extreme value distribution is that for all $s>0$ there exist vectors $\boldsymbol{A}^{(s)}>\boldsymbol{0}$ and $\boldsymbol{B}^{(s)}$ such that*

$$H^s(\boldsymbol{A}^{(s)}\boldsymbol{x}+\boldsymbol{B}^{(s)})=H(\boldsymbol{x}) \tag{2.1}$$

for all $\boldsymbol{x}\in \boldsymbol{R}^k$.

PROOF. Sufficiency is obvious so that we shall prove necessity. If H is an extreme value distribution, then there exist a distribution function F and vectors $\boldsymbol{a}^{(n)}$ and $\boldsymbol{b}^{(n)}$ such that

$$\lim_{n\to\infty} F^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})=H(\boldsymbol{x}) . \tag{2.2}$$

It follows from Lemma 1.1 that

$$\lim_{n\to\infty} n\{1-F(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\}=-\log H(\boldsymbol{x}) .$$

Hence for all $s>0$

$$\lim_{n\to\infty} [ns]\{1-F(\boldsymbol{a}^{([ns])}\boldsymbol{x}+\boldsymbol{b}^{([ns])})\}=-\log H(\boldsymbol{x}) ,$$

where $[ns]$ is the greatest integer less than or equal to ns. Then by Lemma 1.1

$$\lim_{n\to\infty} F^n(\boldsymbol{a}^{([ns])}\boldsymbol{x}+\boldsymbol{b}^{([ns])})=H^{1/s}(\boldsymbol{x}) . \tag{2.3}$$

Hence by (2.2), (2.3) and Lemma 2.2.3 of Galambos [3] (which can easily be extended to the multivariate case, see also proof of Theorem 5.2.1 of Galambos [3]), there exist vectors $\boldsymbol{A}^{(s)}>\boldsymbol{0}$ and $\boldsymbol{B}^{(s)}$ such that

$$H^s(\boldsymbol{A}^{(s)}\boldsymbol{x}+\boldsymbol{B}^{(s)})=H(\boldsymbol{x})$$

for all $\boldsymbol{x}\in \boldsymbol{R}^k$.

COROLLARY 2.1. *Let H be an extreme value distribution. Then for any $t>0$, H^t is an extreme value distribution.*

COROLLARY 2.2. *Let H be an extreme value distribution. Then for all $s>0$, there exist vectors $\boldsymbol{A}^{(s)}>\boldsymbol{0}$ and $\boldsymbol{B}^{(s)}$ such that* (2.1) *holds, and if*
(i) *$H_i=\Phi_{\alpha_i}$, $i=1,\cdots,k$, then $\boldsymbol{A}^{(s)}=(s^{1/\alpha_1},\cdots,s^{1/\alpha_k})$ and $\boldsymbol{B}^{(s)}=\boldsymbol{0}$;*
(ii) *$H_i=\Psi_{\alpha_i}$, $i=1,\cdots,k$, then $\boldsymbol{A}^{(s)}=(s^{-1/\alpha_1},\cdots,s^{-1/\alpha_k})$ and $\boldsymbol{B}^{(s)}=\boldsymbol{0}$;*
(iii) *$H_i=\Lambda$, $i=1,\cdots,k$, then $\boldsymbol{A}^{(s)}=\boldsymbol{1}=(1,\cdots,1)$ and $\boldsymbol{B}^{(s)}=(\log s,\cdots,\log s)$, where $\alpha_i>0$, $i=1,\cdots,k$.*

Example. (See Galambos [3], p. 254.) The distribution function

$$H(x_1, x_2, \cdots, x_k) = \exp\{-\exp[-\min(x_1, x_2, \cdots, x_k)]\}$$

is an extreme value distribution (with $H_i = \Lambda$, $i = 1, \cdots, k$), since for any $s > 0$

$$H^s(x_1 + \log s, x_2 + \log s, \cdots, x_k + \log s) = H(x_1, x_2, \cdots, x_k) .$$

On the other hand, the distribution function

$$H(x_1, x_2) = \Lambda(x_1)\Lambda(x_2)[1 + (1 - \Lambda(x_1))(1 - \Lambda(x_2))/2]$$

is not an extreme value distribution, since

$$\begin{aligned} &H^s(x_1 + \log s, x_2 + \log s) \\ &\quad = \Lambda(x_1)\Lambda(x_2)[1 + (1 - \Lambda^{1/s}(x_1))(1 - \Lambda^{1/s}(x_2))/2]^s \neq H(x_1, x_2) . \end{aligned}$$

COROLLARY 2.3. (Lemma 5.4.1 of Galambos [3]) *Let H be an extreme value distribution and denote by $D_H(\boldsymbol{y}) = H(H_1^{-1}(y_1), \cdots, H_k^{-1}(y_k))$, $\boldsymbol{y} \in (0, 1)^k$, its dependence function. Then, D_H satisfies for all $s > 0$*

$$D_H^s(\boldsymbol{y}^{1/s}) = D_H(\boldsymbol{y}) .$$

LEMMA 2.1. *Let H be an extreme value distribution and D_H be the dependence function of H. If there exists a real number $c \in (0, 1)$ such that*

$$D_H(\boldsymbol{y}) = y_1 y_2 \cdots y_k \quad \text{for all} \quad \boldsymbol{y} \in (c, 1)^k , \tag{2.4}$$

then

$$D_H(\boldsymbol{y}) = y_1 y_2 \cdots y_k \quad \text{for all} \quad \boldsymbol{y} \in (0, 1)^k .$$

PROOF. For any $\boldsymbol{y} \in (0, 1)^k$, there exists an $s > 0$ such that $\boldsymbol{y}^{1/s} \in (c, 1)^k$. Hence by Corollary 2.3 and (2.4)

$$D_H(\boldsymbol{y}) = (D_H(\boldsymbol{y}^{1/s}))^s = (y_1^{1/s} y_2^{1/s} \cdots y_k^{1/s})^s = y_1 y_2 \cdots y_k$$

for all $\boldsymbol{y} \in (0, 1)^k$.

We now prove the following result which is concerned with the asymptotic independence of maxima.

THEOREM 2.2. *Let H be an extreme value distribution such that $H_i = \Lambda$, $i = 1, \cdots, k$. Then a necessary and sufficient condition that*

$$H(\boldsymbol{x}) = \Lambda(x_1) \cdots \Lambda(x_k) \tag{2.5}$$

for any $\boldsymbol{x} = (x_1, \cdots, x_k) \in \boldsymbol{R}^k$ is that

$$H(0, \cdots, 0) = \Lambda(0)^k . \tag{2.6}$$

PROOF. Necessity is obvious so that we shall prove sufficiency. Since H is an extreme value distribution, there exist a distribution function $F(x_1, \cdots, x_k)=P(X_1 \leqq x_1, \cdots, X_k \leqq x_k)$ and vectors $\boldsymbol{a}^{(n)}$ and $\boldsymbol{b}^{(n)}$ such that

$$\lim_{n\to\infty} F^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})=H(\boldsymbol{x}) .$$

It is well-known that the weak convergence of probability measures implies the weak convergence of any finite-dimensional marginal distribution (see Billingsley [1], p. 30). Thus by Lemma 1.1 we have

$$\lim_{n\to\infty} n\,(1-F_{ij}(b_i^{(n)}, b_j^{(n)}))=-\log H_{ij}(0, 0) \tag{2.7}$$

for any $i<j$. By Theorem 5.4.1 of Galambos [3] we have

$$1 \leqq -\log H_{ij}(0, 0) \leqq 2 .$$

Now we shall prove that

$$-\log H_{ij}(0, 0)=2 \tag{2.8}$$

for any $i<j$. Indeed, if (2.8) does not hold i.e. if for example for $i=k-1$, $j=k$

$$-\log H_{k-1\,k}(0, 0)=c , \qquad 1 \leqq c<2 ,$$

then we have

$$\begin{aligned} n\,(1-F(\boldsymbol{b}^{(n)})) &\leqq n\left\{\sum_{i=1}^{k-2} (1-F_i(b_i^{(n)}))+(1-F_{k-1\,k}(b_{k-1}^{(n)}, b_k^{(n)}))\right\} \\ &\to (k-2)+c<k \qquad \text{as } n\to\infty , \end{aligned}$$

using the relation $\lim_{n\to\infty} n\,(1-F_i(b_i^{(n)}))=1$. This contradicts (2.6), thus we have (2.8). Let $\bar{F}_{ij}(x_i, x_j)=P(X_i>x_i, X_j>x_j)$, then

$$\bar{F}_{ij}(x_i, x_j)=(1-F_i(x_i))+(1-F_j(x_j))-(1-F_{ij}(x_i, x_j)) .$$

Therefore, by (2.7) and (2.8) we have

$$\lim_{n\to\infty} n\bar{F}_{ij}(b_i^{(n)}, b_j^{(n)})=0$$

for any $i<j$. From the definition of $\bar{F}_{ij}$ and the inequalities $a_i^{(n)}, a_j^{(n)} >0$ we get

$$\bar{F}_{ij}(b_i^{(n)}, b_j^{(n)}) \geqq \bar{F}_{ij}(a_i^{(n)}x_i+b_i^{(n)}, a_j^{(n)}x_j+b_j^{(n)})$$

for any $x_i, x_j \geqq 0$, thus we have

$$\lim_{n\to\infty} n\bar{F}_{ij}(a_i^{(n)}x_i+b_i^{(n)}, a_j^{(n)}x_j+b_j^{(n)})=0 .$$

So by Theorem 5.3.1 of Galambos [3] we have

$$H(x_1, \cdots, x_k) = \Lambda(x_1) \cdots \Lambda(x_k)$$

for any $x_i \geqq 0$, $i=1, \cdots, k$. Therefore, by Lemma 2.1, (2.5) holds for any $\boldsymbol{x} \in \boldsymbol{R}^k$.

In a similar way one proves the following theorems.

THEOREM 2.3. *Let H be an extreme value distribution such that $H_i = \Phi_{\alpha_i}$, $\alpha_i > 0$, $i=1, \cdots, k$. Then a necessary and sufficient condition that*

$$H(\boldsymbol{x}) = \Phi_{\alpha_1}(x_1) \cdots \Phi_{\alpha_k}(x_k)$$

for any $\boldsymbol{x} \in \boldsymbol{R}^k$ is that

$$H(\boldsymbol{1}) = \Phi_{\alpha_1}(1) \cdots \Phi_{\alpha_k}(1) .$$

THEOREM 2.4. *Let H be an extreme value distribution such that $H_i = \Psi_{\alpha_i}$, $\alpha_i > 0$, $i=1, \cdots, k$. Then a necessary and sufficient condition that*

$$H(\boldsymbol{x}) = \Psi_{\alpha_1}(x_1) \cdots \Psi_{\alpha_k}(x_k)$$

for any $\boldsymbol{x} \in \boldsymbol{R}^k$ is that

$$H(-\boldsymbol{1}) = \Psi_{\alpha_1}(-1) \cdots \Psi_{\alpha_k}(-1) .$$

3. Multivariate tail equivalence

In this section, by using the results of Marshall and Olkin [4] we extend some of the results given in Resnick [5] to the multivariate case.

The following theorem is a k-dimensional version of Lemma 2.1 of Resnick [5].

THEOREM 3.1. *Let F and G be k-dimensional distribution functions. Suppose for normalizing vectors $\boldsymbol{a}^{(n)} > \boldsymbol{0}$, $\boldsymbol{b}^{(n)}$, $n \geqq 1$, $F^n(\boldsymbol{a}^{(n)}\boldsymbol{x} + \boldsymbol{b}^{(n)}) \to H(\boldsymbol{x})$, where $H_i = \Phi_{\alpha_i}$, $\alpha_i > 0$, $i=1, \cdots, k$. Then a necessary and sufficient condition that*

$$G^n(\boldsymbol{a}^{(n)}\boldsymbol{x} + \boldsymbol{b}^{(n)}) \to H(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}) , \tag{3.1}$$

where $\boldsymbol{A} = (c^{1/\alpha_1}, \cdots, c^{1/\alpha_k})$, $c > 0$, is that $\boldsymbol{B} = \boldsymbol{0}$ and

$$\lim_{t \to \infty} \frac{1 - F(tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k)}{1 - G(tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k)} = c \tag{3.2}$$

for all $\boldsymbol{x} = (x_1, x_2, \cdots, x_k)$ such that $0 < H(\boldsymbol{x}) < 1$, where $\phi_i(t) = \bar{F}_i^{-1} \bar{F}_1(t)$, $i = 2, \cdots, k$.

PROOF. If $F \in \boldsymbol{D}(H)$, then it is known that we can take $\boldsymbol{b}^{(n)}=\boldsymbol{0}$, $a_1^{(n)}=\bar{F}_1^{-1}(1/n)$ and $a_i^{(n)}=\bar{F}_i^{-1}\bar{F}_1(a_1^{(n)})=\phi_i(a_1^{(n)})$, $i=2,\cdots,k$, $n\geqq 1$ (see the proof of Proposition 3.1 of Marshall and Olkin [4] and Appendix I).

Sufficiency. Since $a_1^{(n)}\to\infty$ as $n\to\infty$ and (3.2), we have that for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$,

$$c=\lim_{n\to\infty}\frac{1-F(a_1^{(n)}x_1,\phi_2(a_1^{(n)})x_2,\cdots,\phi_k(a_1^{(n)})x_k)}{1-G(a_1^{(n)}x_1,\phi_2(a_1^{(n)})x_2,\cdots,\phi_k(a_1^{(n)})x_k)}=\lim_{n\to\infty}\frac{n(1-F(\boldsymbol{a}^{(n)}\boldsymbol{x}))}{n(1-G(\boldsymbol{a}^{(n)}\boldsymbol{x}))}\ .$$

Then by Lemma 1.1 and Corollary 2.2 we have

$$G^n(\boldsymbol{a}^{(n)}\boldsymbol{x})\to H^{1/c}(\boldsymbol{x})=H(c^{1/\alpha_1}x_1,\cdots,c^{1/\alpha_k}x_k)=H(\boldsymbol{A}\boldsymbol{x})\ .$$

Necessity. From the univariate result (Lemma 2.1 of Resnick [5]), we see that $B_i=0$, $i=1,\cdots,k$. So we have $\boldsymbol{B}=\boldsymbol{0}$. Since $a_1^{(n)}\leqq a_1^{(n+1)}\to\infty$, for any sufficiently large t there exists an $n\in\boldsymbol{N}$ such that $a_1^{(n)}\leqq t\leqq a_1^{(n+1)}$. For any $\boldsymbol{x}=(x_1,\cdots,x_k)$ such that $0<H(\boldsymbol{x})<1$, we have $\boldsymbol{0}<\boldsymbol{x}\neq\boldsymbol{\infty}$. Moreover, ϕ_i is non-decreasing, $i=2,\cdots,k$. Therefore we have

$$\frac{1-F(\boldsymbol{a}^{(n+1)}\boldsymbol{x})}{1-G(\boldsymbol{a}^{(n)}\boldsymbol{x})}\leqq\frac{1-F(tx_1,\phi_2(t)x_2,\cdots,\phi_k(t)x_k)}{1-G(tx_1,\phi_2(t)x_2,\cdots,\phi_k(t)x_k)}\leqq\frac{1-F(\boldsymbol{a}^{(n)}\boldsymbol{x})}{1-G(\boldsymbol{a}^{(n+1)}\boldsymbol{x})}$$

for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$. Taking the limits of above inequalities, we have (3.2).

If we consider the particular case $F_1=\cdots=F_k$, then we have the following handy result.

COROLLARY 3.1. *Let F and G be k-dimensional distribution functions. Suppose $F_1=\cdots=F_k$ and that there exist $a^{(n)}>0$, $b^{(n)}$, $n\geqq 1$ such that $F^n(a^{(n)}\boldsymbol{x}+b^{(n)}\boldsymbol{1})\to H(\boldsymbol{x})$, where $H_i=\Phi_\alpha$, $\alpha>0$, $i=1,\cdots,k$. Then a necessary and sufficient condition that*

$$G^n(a^{(n)}\boldsymbol{x}+b^{(n)}\boldsymbol{1})\to H(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B})\ ,$$

where $\boldsymbol{A}=c^{1/\alpha}\boldsymbol{1}$, $c>0$, is that $\boldsymbol{B}=\boldsymbol{0}$ and

$$\lim_{t\to\infty}\frac{1-F(t\boldsymbol{x})}{1-G(t\boldsymbol{x})}=c$$

for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$.

Next we establish a k-dimensional version of Lemma 2.2 of Resnick [5], which can be proved similarly to Theorem 3.1.

THEOREM 3.2. *Let F and G be k-dimensional distribution functions and $\boldsymbol{a}^{(n)}>\boldsymbol{0}$, $\boldsymbol{b}^{(n)}$, $n\geqq 1$ are normalizing vectors such that $F^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H(\boldsymbol{x})$, where $H_i=\Psi_{\alpha_i}$, $\alpha_i>0$, $i=1,\cdots,k$. Then a necessary and suf-*

ficient condition that

$$G^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}),$$

where $\boldsymbol{A}=(c^{-1/\alpha_1},\cdots,c^{-1/\alpha_k})$, $c>0$, *is that* $\boldsymbol{B}=\boldsymbol{0}$, $\boldsymbol{x}_F^0=\boldsymbol{x}_G^0=\boldsymbol{x}^0\in\boldsymbol{R}^k$ *and*

$$\lim_{t\downarrow 0}\frac{1-F((tx_1,\phi_2(t)x_2,\cdots,\phi_k(t)x_k)+\boldsymbol{x}^0)}{1-G((tx_1,\phi_2(t)x_2,\cdots,\phi_k(t)x_k)+\boldsymbol{x}^0)}=c$$

for all $\boldsymbol{x}=(x_1,\cdots,x_k)$ *such that* $0<H(\boldsymbol{x})<1$, *where* $x_i^0=x_{F_i}^0$, $i=1,\cdots,k$ *and* $\phi_i(t)=x_i^0-\bar{F}_i^{-1}(\bar{F}_1(x_1^0-t))$, $i=2,\cdots,k$.

COROLLARY 3.2. *Let* F *and* G *be* k*-dimensional distribution functions. Suppose* $F_1=\cdots=F_k$ *and that there exist* $a^{(n)}>0$, $b^{(n)}$, $n\geqq 1$ *such that* $F^n(a^{(n)}\boldsymbol{x}+b^{(n)}\boldsymbol{1})\to H(\boldsymbol{x})$, *where* $H_i=\Psi_\alpha$, $\alpha>0$, $i=1,\cdots,k$. *Then a necessary and sufficient condition that*

$$G^n(a^{(n)}\boldsymbol{x}+b^{(n)}\boldsymbol{1})\to H(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}),$$

where $\boldsymbol{A}=c^{-1/\alpha}\boldsymbol{1}$, $c>0$, *is that* $\boldsymbol{B}=\boldsymbol{0}$, $x_{F_i}^0=x_{G_i}^0=x^0\in\boldsymbol{R}^1$, $i=1,\cdots,k$, *and*

$$\lim_{t\downarrow 0}\frac{1-F(t\boldsymbol{x}+x^0\boldsymbol{1})}{1-G(t\boldsymbol{x}+x^0\boldsymbol{1})}=c$$

for all $\boldsymbol{x}$ *such that* $0<H(\boldsymbol{x})<1$.

Finally, we establish a k-dimensional version of Lemma 2.5 of Resnick [5].

THEOREM 3.3. *Let* F *and* G *be* k*-dimensional distribution functions, and* $\boldsymbol{a}^{(n)}>0$, $\boldsymbol{b}^{(n)}$, $n\geqq 1$ *are normalizing vectors such that* $F^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H(\boldsymbol{x})$, *where* $H_i=\Lambda$, $i=1,\cdots,k$. *Then a necessary and sufficient condition that*

$$G^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}), \tag{3.3}$$

where $\boldsymbol{A}>\boldsymbol{0}$, $\boldsymbol{B}=b\boldsymbol{1}$, *is that* $\boldsymbol{A}=\boldsymbol{1}$, $\boldsymbol{x}_F^0=\boldsymbol{x}_G^0=\boldsymbol{x}^0$ *and*

$$\lim_{t\uparrow x_1^0}\frac{1-F(\boldsymbol{a}(t)\boldsymbol{x}+\boldsymbol{b}(t))}{1-G(\boldsymbol{a}(t)\boldsymbol{x}+\boldsymbol{b}(t))}=e^b \tag{3.4}$$

for all $\boldsymbol{x}$ *such that* $0<H(\boldsymbol{x})<1$, *where* $x_1^0=x_{F_1}^0$, $a_i(t)=\bar{F}_i^{-1}(\bar{F}_1(t)/e)-\bar{F}_i^{-1}\bar{F}_1(t)$ *and* $b_i(t)=\bar{F}_i^{-1}\bar{F}_1(t)$, $i=1,\cdots,k$.

PROOF. If $F\in\boldsymbol{D}(H)$, then we can suppose without loss of generality that $b_i^{(n)}=\bar{F}_i^{-1}\bar{F}_1(b_1^{(n)})$ and $a_i^{(n)}=\bar{F}_i^{-1}(\bar{F}_1(b_1^{(n)})/e)-\bar{F}_i^{-1}\bar{F}_1(b_1^{(n)})$, $i=1,\cdots,k$. (See Proposition 3.3 of Marshall and Olkin [4] and Appendix II.)

Sufficiency. Since $\lim_{n\to\infty}b_1^{(n)}=x_1^0$ and $\{b_1^{(n)}\}$ is an increasing sequence, by (3.4) we have

$$e^b=\lim_{n\to\infty}\frac{1-F(\boldsymbol{a}(b_1^{(n)})\boldsymbol{x}+\boldsymbol{b}(b_1^{(n)}))}{1-G(\boldsymbol{a}(b_1^{(n)})\boldsymbol{x}+\boldsymbol{b}(b_1^{(n)}))}=\lim_{n\to\infty}\frac{n\{1-F(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\}}{n\{1-G(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\}}\ .$$

Hence from Lemma 1.1 and Corollary 2.2 we have

$$G^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H^{1/e^b}(\boldsymbol{x})=H(\boldsymbol{x}+\boldsymbol{B})\,,\qquad \text{where}\qquad \boldsymbol{B}=b\mathbf{1}\,.$$

Necessity. Consider the marginal distributions F_i, G_i and $H_i=\Lambda$, $i=1,\cdots,k$, then by Lemma 2.5 of Resnick [5], we have $\boldsymbol{A}=\mathbf{1}$ and $\boldsymbol{x}_F^0=\boldsymbol{x}_G^0=\boldsymbol{x}^0$. Proposition 3.3 of Marshall and Olkin [4] implies

$$\lim_{t\uparrow x_1^0}\frac{1-F(\boldsymbol{a}(t)\boldsymbol{x}+\boldsymbol{b}(t))}{1-F_1(t)}=-\log H(\boldsymbol{x}) \tag{3.5}$$

for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$. Now we shall prove that for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$

$$\lim_{t\uparrow x_1^0}\frac{1-G(\boldsymbol{a}(t)\boldsymbol{x}+\boldsymbol{b}(t))}{1-F_1(t)}=-e^{-b}\log H(\boldsymbol{x})\ . \tag{3.6}$$

From (3.3) and $\boldsymbol{A}=\mathbf{1}$, we have $G^n(\boldsymbol{a}^{(n)}\boldsymbol{x}+\boldsymbol{b}^{(n)})\to H^{e^{-b}}(\boldsymbol{x})$. So it holds

$$\lim_{s\to\infty}s\,\{1-G(\boldsymbol{\alpha}(s)\boldsymbol{x}+\boldsymbol{\beta}(s))\}=-e^{-b}\log H(\boldsymbol{x})\ ,$$

where $\alpha_i(s)=\bar{F}_i^{-1}(1/(es))-\bar{F}_i^{-1}(1/s)$ and $\beta_i(s)=\bar{F}_i^{-1}(1/s)$, $i=1,\cdots,k$. (This result can be proved similarly to Corollary 2.4.1 of de Haan [2].) Now, let $s(t)=1/(1-F_1(t))$, then $\boldsymbol{\alpha}(s(t))=\boldsymbol{a}(t)$ and $\boldsymbol{\beta}(s(t))=\boldsymbol{b}(t)$, thus we have (3.6). The relations (3.5) and (3.6) imply (3.4).

COROLLARY 3.3. *Let F and G be k-dimensional distribution functions. Suppose $F_1=\cdots=F_k$ and that there exist $a^{(n)}>0$, $b^{(n)}$, $n\geqq 1$ such that $F^n(a^{(n)}\boldsymbol{x}+b^{(n)}\mathbf{1})\to H(\boldsymbol{x})$, where $H_i=\Lambda$, $i=1,\cdots,k$. Then a necessary and sufficient condition that*

$$G^n(a^{(n)}\boldsymbol{x}+b^{(n)}\mathbf{1})\to H(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B})\,,$$

where $\boldsymbol{A}>\mathbf{0}$, $\boldsymbol{B}=b\mathbf{1}$, is that $\boldsymbol{A}=\mathbf{1}$, $x_{F_i}^0=x_{G_i}^0=x^0$, $i=1,\cdots,k$ and

$$\lim_{t\uparrow x^0}\frac{1-F(a(t)\boldsymbol{x}+t\mathbf{1})}{1-G(a(t)\boldsymbol{x}+t\mathbf{1})}=e^b$$

for all $\boldsymbol{x}$ such that $0<H(\boldsymbol{x})<1$, where $a(t)=\bar{F}_1^{-1}(\bar{F}_1(t)/e)-t$.

Finally, we remark that in general univariate tail equivalence does not imply multivariate tail equivalence of the joint distribution functions. Put, for example, $F(x_1, x_2)=H(x_1)H(x_2)$ and $G(x_1, x_2)=H(\min(x_1, x_2))$, where H is a univariate extreme value distribution. This counterexample shows in addition that k-dimensional versions of Corollaries 2.1 and 2.2 of Resnick [5] do not necessarily hold.

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Appendix I

By Lemma 2.2.3 of Galambos [3] it is sufficient to prove

$$\text{(A.1)} \qquad \lim_{n\to\infty} \frac{\bar{F}_i^{-1}(1/n)}{\bar{F}_i^{-1}(\bar{F}_1(a_1^{(n)}))}=1\,, \qquad i=2,\cdots,k\,.$$

The relation

$$\lim_{n\to\infty} \frac{n}{1/\bar{F}_1(a_1^{(n)})}=\lim_{n\to\infty} n\,(1-F_1(a_1^{(n)}))=1$$

holds. Hence by Theorem 2.3.1 and Corollary 1.2.1 of de Haan [2] we have (A.1).

Appendix II

It is sufficient to prove

$$\text{(A.2)} \qquad F_i^n(a_i^{(n)}x+b_i^{(n)})\to \Lambda(x)\,,$$

where $a_i^{(n)}=\bar{F}_i^{-1}(\bar{F}_1(b_1^{(n)})/e)-\bar{F}_i^{-1}\bar{F}_1(b_1^{(n)})$ and $b_i^{(n)}=\bar{F}_i^{-1}\bar{F}_1(b_1^{(n)})$, $i=2,\cdots,k$.

Since the relation $\lim_{n\to\infty} n\bar{F}_1(b_1^{(n)})=1$ holds, we obtain

$$\lim_{n\to\infty}(1-\bar{F}_1(b_1^{(n)}))^n=e^{-1},$$

$$\lim_{n\to\infty}(1-\bar{F}_1(b_1^{(n)})/e)^n=e^{-e^{-1}}.$$

Hence by Theorem 2.1.2* of de Haan [2] we have (A.2).