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ERROR BOUNDS FOR ASYMPTOTIC EXPANSION OF THE SCALE MIXTURES OF THE NORMAL DISTRIBUTION

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Summary

Let X be a standard normal random variable and let σ be a positive random variable independent of X. The distribution of $\eta=\sigma X$ is expanded around that of $N(0,1)$ and its error bounds are obtained. Bounds are given in terms of $\mathrm{E}(\sigma^2 \vee \sigma^{-2}-1)^k$, where $\sigma^2 \vee \sigma^{-2}$ denotes the maximum of the two quantities σ^2 and σ^{-2}, and k is a positive integer, and of $\mathrm{E}(\sigma^2-1)^k$, if k is even.

1. Introduction

Let X and ε be mutually independent random variables and let ξ be their sum; $\xi=X+\varepsilon$. If ε is small in some sense, then we would expect that the distribution of ξ is close to that of X's. Assuming the smoothness of the distribution function (d.f.) G of X, Fujikoshi [3] gave an expansion of the d.f. F of ξ around G and its error bound. In particular, when G is the standard normal distribution function Φ and when the conditional distribution of ε given a random matrix V is the normal distribution $N(0, h(V))$, where h is a positive function, Fujikoshi showed that the error bound is given in terms of a higher order moment of $h(V)$.

In view of the particular property of the normal distribution, ξ has the identical distribution with that of the random variable $\eta=\sqrt{1+h(V)}X$ in this case. This is a special case of the scale mixtures of the normal distribution. A distribution F is said to be a scale mixture of that of X's if there exists a positive random variable σ independent of X such that the variable $\eta=\sigma X$, follows F.

The problem of approximating the scale mixtures of the normal or of other distributions has received recent interest. Keilson and Steutel [7] investigated the properties of the mixtures and established

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some moment measures of the distance of a scale mixture from its parent distribution. In particular, they showed that when X follows the standard normal distribution $N(0,1)$, the kurtosis $\gamma(\sigma X)$ $(=3\,\mathrm{E}\,(\sigma^2-1)^2$, if $\mathrm{E}\,(\sigma^2)=1)$ gives a convenient distance between the distribution F of σX and $N(0,1)$. Heyde [5] and Heyde and Leslie [6] related $\gamma(\sigma X)$ with the usual uniform measure of distance showing that the inequality

$$\sup|\Pr\{\sigma X\leqq x\}-\Phi(x)|\leqq 0.85\gamma(\sigma X)$$

holds provided $\mathrm{E}\,(\sigma^2)=1$. Hall [4] noticed that 0.85 can be replaced by 0.648. Similar inequalities for classes of distributions which include scale mixtures of the exponential distribution were obtained by Azlarov and Volodin [1], Brown [2], Heyde and Leslie [6] and Shimizu [8]. Hall [4] proved some inequalities assuming that the distribution of X has a probability density $p(x)$ such that $xp(x)$ is bounded.

The present article is concerned with the scale mixtures of the normal distribution. Fujikoshi's results ([3]) essentially imply that if $\sigma^2\geqq 1$ with probability 1, then the following inequality holds.

$$\left|\Pr\{\sigma X\leqq x\}-\Phi(x)-\sum_{l=1}^{k-1}\frac{1}{2^l l!}\,\mathrm{E}\,(\sigma^2-1)^l H_{2l-1}(x)\varphi(x)\right|$$
$$\leqq\frac{1}{2^k k!}\,\mathrm{E}\,(\sigma^2-1)^k\sup|H_{2k-1}(x)\varphi(x)|\,,$$

where φ and H's are the probability density of $N(0,1)$ and the Hermite polynomials, respectively. We shall give a similar inequalities without assuming $\sigma^2\geqq 1$. Our result is generalized to the non-central case, i.e., the situation where X is $N(\mu,1)$ with $\mu\neq 0$. We also generalize to the expansion of $\Pr\{\eta\in A\}$ for any Borel set A.

2. The results

For a positive random variable σ, let $k\geqq 1$ be an integer and write

$$\alpha_l=\mathrm{E}\,(\sigma^2-1)^l\,,\qquad l=0,1,2,\cdots,k\,,\qquad\text{and}\qquad\bar{\alpha}_k=\mathrm{E}\,(\sigma^{-2}-1)^k\,,$$

if they exist. Note that α_l is expressible as a linear combination of the moments of even order of $\eta=\sigma X$: As

$$\mathrm{E}\,(\eta^{2p})=\mathrm{E}\,(\sigma^{2p}\cdot X^{2p})=\mathrm{E}\,(\sigma^{2p})\cdot\mathrm{E}\,(X^{2p})=\frac{(2p)!}{2^p p!}\,\mathrm{E}\,(\sigma^{2p})\,,$$

we have

$$\alpha_l=\sum_{p=0}^{l}\binom{l}{p}\mathrm{E}\,(\sigma^{2p})(-1)^{l-p}=\sum_{p=0}^{l}\binom{l}{p}\frac{(-1)^{l-p}2^p p!}{(2p)!}\,\mathrm{E}\,(\eta^{2p})\,.$$

Put

$$G_k(x)=\Phi(x)-\sum_{l=1}^{k-1}\frac{1}{2^l l!}\alpha_l H_{2l-1}(x)\varphi(x)\,. \tag{2.1}$$

The following theorems state that if σ is close to one, then the d.f. $F(x)$ of the variable $\eta=\sigma X$ is approximated by $G_k(x)$.

THEOREM 2.1. *If* $\mathrm{E}(\sigma^{2k})$ *and* $\mathrm{E}(\sigma^{-2k})$ *exist, the following inequality holds.*

$$\sup_x|F(x)-G_k(x)|\leqq\frac{1}{2\pi k}\mathrm{E}(\sigma^2\vee\sigma^{-2}-1)^k \tag{2.2}$$

$$\left(\leqq\frac{1}{2\pi k}(\alpha_k+\bar{\alpha}_k)\,,\ \textit{if } k \textit{ is even}\right).$$

THEOREM 2.2. *If* $\mathrm{E}(\sigma^{2k})$ *exists for some even integer* $k\geqq 2$, *and if* $\mathrm{E}(\sigma^2)=1$, *then*

$$\sup_x|F(x)-G_k(x)|\leqq A(k)\,\mathrm{E}(\sigma^2-1)^k\,, \tag{2.3}$$

where

$$A(k)=\inf_{0<c<1}\sup_x\sup_{0<\tau<c}\left|\frac{|\Delta_\tau(x)|}{(1-c^2)^k}+\frac{1}{2\pi kc^{2k}}\right|,$$

with

$$\Delta_\tau(x)=\Phi(x/\tau)-\Phi(x)+\sum_{l=1}^{k-1}\frac{(\tau^2-1)^l}{2^l l!}H_{2l-1}(x)\varphi(x)\,.$$

Numerical computation shows that $A(2)\leqq 2.48$, $A(4)\leqq 11.26$ *and* $A(6)\leqq 51.24$.

The result can be extended to the non-central case. Let μ be a real constant and consider the variable

$$\eta=\sigma(X+\mu)-\mu\,.$$

Writing

$$G_k(x)=\Phi(x)-\sum_{\substack{l+m\geqq 1\\ l<k,m<k}}\frac{\mu^l}{2^m l!\,m!}\mathrm{E}\left((\sigma-1)^l(\sigma^2-1)^m\right)H_{l+2m-1}(x)\varphi(x)\,, \tag{2.4}$$

we have

THEOREM 2.3. *If* $\mathrm{E}(\sigma^{3k-2})$ *and* $\mathrm{E}(\sigma^{-3k+2})$ *exist, then*

$$\sup_x|F(x)-G_k(x)|\leqq\beta_k+\gamma_k\,, \tag{2.5}$$

where

$$\beta_k = \frac{1}{2\pi k} \mathrm{E}\,(\sigma^2 \vee \sigma^{-2} - 1)^k, \qquad \textit{and}$$

$$\gamma_k = \frac{2^{k/2}|\mu|^k}{2\pi k!} \sum_{m=0}^{k-1} \frac{\Gamma(m+k/2)}{m!} \mathrm{E}\,((\sigma \vee \sigma^{-1} - 1)^k (\sigma^2 \vee \sigma^{-2} - 1)^m).$$

Theorems 2.1 and 2.2 are not sufficient for the purpose of approximating $\Pr\{\eta \in A\}$ for various sets A's. If, for example, A is a finite interval $[a, b)$, then the probability $\Pr\{\eta \in A\} = F(b) - F(a)$ can be approximated by $G_k(b) - G_k(a)$ only within the error $2 \cdot \mathrm{E}\,(\sigma^2 \vee \sigma^{-2} - 1)^k / 2\pi k$. The following theorems are stronger versions of Theorems 2.1 and 2.2, respectively.

THEOREM 2.4. *Under the assumption of Theorem* 2.1, *the distribution F has a probability density $f(x)$, and for any Borel set A, the following inequalities hold.*

$$(2.6) \qquad \left|\Pr\{\eta \in A\} - \int_A dG_k(x)\right| \leqq \int_{-\infty}^{\infty} |f(x) - g_k(x)|\,dx \leqq \frac{2}{\pi} \mathrm{E}\,(\sigma^2 \vee \sigma^{-2} - 1)^k,$$

where

$$g_k(x) = G_k'(x) = \sum_{l=0}^{k-1} \frac{1}{2^l l!} \alpha_l H_{2l}(x) \varphi(x).$$

THEOREM 2.5. *Under the assumption of Theorem* 2.2, *we have*

$$(2.7) \qquad \left|\Pr\{\eta \in A\} - \int_A dG_k(x)\right| \leqq 4kA(k)\, \mathrm{E}\,(\sigma^2 - 1)^k.$$

3. Proofs

If X is a standard normal random variable and if σ is a positive random variable independent of X, then the d.f. $F(x)$ of the variable $\eta = \sigma X$ is given by

$$F(x) = \mathrm{E}\,(\Phi(x/\sigma)).$$

We start with approximating the d.f. $\Phi(x/\sigma)$ of $N(0, \sigma^2)$ for a fixed σ. The distribution $N(0, \sigma^2)$ has the density $\varphi(x/\sigma)/\sigma$ and its characteristic function is given by $e^{-\sigma^2 t^2/2}$. Therefore, the inversion formula gives

$$\frac{1}{\sigma}\varphi\left(\frac{x}{\sigma}\right) - \varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{-\sigma^2 t^2/2} - e^{-t^2/2}) e^{-itx}\,dt.$$

Integrating the both sides from 0 to x, and noting that $(e^{-\sigma^2 t^2/2} - e^{-t^2/2})/t$ is an integrable odd function, we obtain

$$\Phi(x/\sigma)-\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\sigma^2t^2/2}-e^{-t^2/2}}{-it}e^{-itx}dt\,. \tag{3.1}$$

On the other hand we have

$$\begin{aligned} e^{-\sigma^2t^2/2}&=e^{-(\sigma^2-1)t^2/2}e^{-t^2/2} \\ &=\sum_{l=0}^{k-1}\frac{1}{2^l l!}(\sigma^2-1)^l(it)^{2l}e^{-t^2/2}+\frac{1}{2^k k!}(\sigma^2-1)^k(it)^{2k}e^{-(1+\theta(\sigma^2-1))t^2/2}\,, \end{aligned} \tag{3.2}$$

where $0\leqq\theta\leqq 1$, and differentiations of the both sides of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t^2/2}e^{-itx}dt=\varphi(x)$$

lead to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(it)^l e^{-t^2/2}e^{-itx}dt=H_l(x)\varphi(x)\,,\qquad l=0,1,\cdots\,.$$

Then it follows from (3.1) and (3.2) that

$$\begin{aligned} \Delta_\sigma(x)&\equiv\Phi(x/\sigma)-\Phi(x)+\sum_{l=1}^{k-1}\frac{1}{2^l l!}(\sigma^2-1)^l H_{2l-1}(x)\varphi(x) \\ &=\frac{i}{2\pi k!}(-1/2)^k(\sigma^2-1)^k\int_{-\infty}^{\infty}t^{2k-1}e^{-(1+\theta(\sigma^2-1))t^2/2}e^{-itx}dt\,. \end{aligned} \tag{3.3}$$

PROOF OF THEOREMS 2.1 AND 2.2. As $0\leqq\theta\leqq 1$, the absolute value of the integral in (3.3) is bounded by

$$J(k,\sigma)\equiv 2\int_0^{\infty}t^{2k-1}e^{-(1\wedge\sigma^2)t^2/2}dt\,,$$

where $1\wedge\sigma^2$ is equal to 1 or σ^2 according as $\sigma^2\geqq 1$ or $\sigma^2<1$. It is easy to see that $J(k,\sigma)=2^k\Gamma(k)$ if $\sigma\geqq 1$, and $J(k,\sigma)=\sigma^{-2k}J(k,1)$ if $\sigma\leqq 1$. Therefore

$$\begin{aligned} |\Delta_\sigma(x)|&\leqq\frac{1}{2\pi k!}\,\frac{1}{2^k}|\sigma^2-1|^k J(k,\sigma) \\ &\leqq\frac{1}{2\pi k}(\sigma^2\vee\sigma^{-2}-1)^k\,. \end{aligned}$$

Taking the expectation of the both sides, we obtain

$$|F(x)-G_k(x)|=|\mathrm{E}\,(\Delta_\sigma(x))|\leqq\mathrm{E}\,(|\Delta_\sigma(x)|)\leqq\frac{1}{2\pi k}\,\mathrm{E}\,(\sigma^2\vee\sigma^{-2}-1)^k\,.$$

To prove (2.3), let $0<c<1$ and note that

$$\Pr\{\sigma\leqq c\}\leqq\Pr\{|\sigma^2-1|\geqq 1-c^2\}\leqq\mathrm{E}\,(\sigma^2-1)^k/(1-c^2)^k\,,$$

and that

$$J(k,\sigma)\leqq J(k,1)/c^{2k}\leqq 2^k\Gamma(k)/c^{2k}, \qquad \text{if} \quad \sigma\geqq c.$$

It follows that

$$\begin{aligned}\mathrm{E}\,(|\Delta_\sigma(x)|)&=\int_0^\infty |\Delta_\tau(x)|d\,\mathrm{Pr}\,\{\sigma\leqq\tau\}\\ &\leqq \sup_{\substack{x\\0<\tau<c}}|\Delta_\tau(x)|\cdot\mathrm{Pr}\,\{\sigma\leqq c\}+\frac{1}{2\pi kc^{2k}}\int_0^\infty(\tau^2-1)^k d\,\mathrm{Pr}\,\{\sigma\leqq\tau\}\\ &\leqq\{\sup_{\substack{x\\0<\tau<c}}|\Delta_\tau(x)|/(1-c^2)^k+1/(2\pi kc^{2k})\}\cdot\mathrm{E}\,(\sigma^2-1)^k.\end{aligned}$$

PROOF OF THEOREM 2.3. For a fixed σ, put $\nu=\mu(\sigma-1)$. Then, the key relation becomes

$$\Phi((x-\nu)/\sigma)-\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\nu t-\sigma^2t^2/2}-e^{-t^2/2}}{-it}e^{-itx}dt. \tag{3.4}$$

It suffices to prove

$$\mathrm{E}\left|\Phi((x-\nu)/\sigma)-\Phi(x)+\sum_{\substack{l+m\geqq1\\l<k,m<k}}\frac{\nu^l(\sigma^2-1)^m}{2^m l!\,m!}\int_{-\infty}^\infty (it)^{l+2m-1}e^{-t^2/2}e^{-itx}dt\right| \leqq\beta_k+\gamma_k. \tag{3.5}$$

We have

$$\begin{aligned}e^{i\nu t-\sigma^2t^2/2}-e^{-t^2/2}&=(e^{i\nu t+(\sigma^2-1)(it)^2/2}-1)e^{-t^2/2}\\ &=\sum_{\substack{l+m\geqq1\\l<k,m<k}}\frac{\nu^l(\sigma^2-1)^m}{2^m l!\,m!}(it)^{l+2m}e^{-t^2/2}\\ &\quad+\frac{1}{2^k k!}(\sigma^2-1)^k e^{i\nu t}(it)^{2k}e^{-(\sigma^2-1)t^2\theta/2}e^{-t^2/2}\\ &\quad+\frac{B}{k!}\nu^k(it)^k\sum_{m=0}^{k-1}\frac{(\sigma^2-1)^m}{2^m m!}(it)^{2m}e^{-t^2/2},\end{aligned}$$

where $0\leqq\theta\leqq1$ and $|B|\leqq1$. Substituting this expression into (3.4), we obtain

$$\begin{aligned}&\left|\Phi((x-\nu)/\sigma)-\Phi(x)+\sum_{\substack{l+m\geqq1\\l<k,m<k}}\frac{\nu^l(\sigma^2-1)^m}{2^m l!\,m!}\int_{-\infty}^\infty (it)^{l+2m-1}e^{-t^2/2}e^{-itx}dt\right|\\ &\leqq\frac{1}{\pi}\int_0^\infty|(\sigma^2-1)^k|t^{2k-1}e^{-t^2(1\wedge\sigma^2)/2}dt/2^k k!\\ &\quad+\frac{1}{\pi k!}\sum_{m=0}^{k-1}\frac{1}{2^m m!}|\nu^k(\sigma^2-1)^m|\int_0^\infty t^{2m+k-1}e^{-t^2(1\wedge\sigma^2)/2}dt\\ &\leqq\frac{1}{2\pi k}(\sigma^2\vee\sigma^{-2}-1)^k+\frac{2^{k/2}|\mu|^k}{2\pi k!}(\sigma\vee\sigma^{-1}-1)^k\sum_{m=0}^{k-1}\frac{\Gamma(m+k/2)}{m!}(\sigma^2\vee\sigma^{-2}-1)^m,\end{aligned}$$

as was to be proved.

PROOF OF THEOREMS 2.4 AND 2.5. The distribution F has the probability density

$$f(x)=\mathrm{E}\left(\frac{1}{\sigma}\varphi\left(\frac{x}{\sigma}\right)\right).$$

For a fixed $\sigma>0$, let $\Delta_\sigma(x)$ be as in (3.3). Then, differentiation gives

$$\delta_\sigma(x)\equiv\Delta'_\sigma(x)=\frac{1}{\sigma}\varphi\left(\frac{x}{\sigma}\right)-\xi_\sigma(x)\varphi(x)\,, \tag{3.6}$$

where

$$\xi_\sigma(x)=\sum_{l=0}^{k-1}\frac{(\sigma^2-1)^l}{2^l l!}H_{2l}(x)$$

is a polynomial of degree $k-1$ in x^2. As the ratio $\frac{1}{\sigma}\varphi\left(\frac{x}{\sigma}\right)\Big/\varphi(x)$ is an exponential function of x^2, it is easy to see that the $\delta_\sigma(x)$ changes its signs at most $2k$ times and is positive or negative for sufficiently large $|x|$ according as $\sigma>1$ or $\sigma<1$. If, for instance, $\sigma>1$, there exist at most $2k$ real numbers

$$c_0\ (\equiv-\infty)<c_1\leqq c_2\leqq\cdots\leqq c_{2k}<c_{2k+1}\ (\equiv\infty)\,,$$

such that

$$\delta_\sigma(x)\geqq 0 \quad \text{if} \quad x\in\bigcup_{j=0}^{k} C_{2j}\,, \qquad \text{and}$$

$$\delta_\sigma(x)\leqq 0 \quad \text{if} \quad x\in\bigcup_{j=1}^{k} C_{2j-1}\,,$$

where C_j denotes the interval $[c_j, c_{j+1})$.

Noting that $\int_{-\infty}^{\infty}\delta_\sigma(x)dx=\Delta_\sigma(\infty)-\Delta_\sigma(-\infty)=0$, we obtain,

$$\begin{aligned}\int_{-\infty}^{\infty}|\delta_\sigma(x)|dx&=\sum_{j=0}^{k}\int_{C_{2j}}d\Delta_\sigma(x)-\sum_{j=1}^{k}\int_{C_{2j-1}}d\Delta_\sigma(x)\\&=2\sum_{j=0}^{k}\int_{C_{2j}}d\Delta_\sigma(x)=2\sum_{j=0}^{k}(\Delta_\sigma(c_{2j+1})-\Delta_\sigma(c_{2j}))\\&\leqq 4k\frac{1}{2k\pi}(\sigma^2\vee\sigma^{-2}-1)^k\,.\end{aligned} \tag{3.7}$$

Similar argument can apply to the case when $\sigma<1$ to obtain the inequality (3.7). Therefore

$$\left|\Pr\{\eta\in A\}-\int_A dG_k(x)\right|\leqq\left|\int_A \mathrm{E}\,(\delta_\sigma(x))dx\right|$$
$$\leqq \mathrm{E}\int_{-\infty}^{\infty}|\delta_\sigma(x)|dx$$
$$\leqq\frac{2}{\pi}\,\mathrm{E}\,(\sigma^2\vee\sigma^{-2}-1)^k\,,$$

as was to be proved. The proof of the inequality (2.7) is quite similar and we omit the details.

4. Examples

Exaples 1. (t-distribution) Let X be a standard normal random variable and let χ_n^2 be independent of the X and have the chi-square distribution with n-degrees of freedom.

Put $\sigma=\sqrt{(n-2)/\chi_n^2}$ and $\eta=\sigma X$. Then the variable

$$\xi=\sqrt{n/(n-2)}\cdot\eta$$

has the t-distribution with n-degrees of freedom. As $\chi_n^2/2$ has the probability density $x^q e^{-x}/\Gamma(q+1)$, for $x\geqq 0$, where $q=\frac{n}{2}-1$, we have for any real c ($\leqq 2q$),

$$\mathrm{E}\,(\sigma^c)=\frac{1}{\Gamma(q+1)}\int_0^\infty (q/x)^{c/2}x^q e^{-x}dx=q^{c/2}\frac{\Gamma(q-c/2+1)}{\Gamma(q+1)}\,.$$

If, in particular, p is a positive number, then

$$(4.1)\qquad \begin{cases} \mathrm{E}\,(\sigma^{2p})=q^p/q^{(p)}\,, \\ \mathrm{E}\,(\sigma^{-2p})=(q+p)^{(p)}/q^p\,, \\ \mathrm{E}\,(\sigma^{2p-1})=\dfrac{q^{p-1}}{(q-1/2)^{(p-1)}}\,u(q)\,, \qquad \text{and} \\ \mathrm{E}\,(\sigma^{-(2p-1)})=\dfrac{(q+p-1/2)^{(p)}}{q^p}\,u(q)\,, \end{cases}$$

where $q^{(p)}=q(q-1)(q-2)\cdots(q-p+1)$, etc., and

$$(4.2)\qquad u(q)\equiv\frac{\sqrt{q}\cdot\Gamma(q+1/2)}{\Gamma(q+1)}$$
$$=1-\frac{1}{8q}+\frac{1}{128q^2}+\frac{5}{1024q^3}-\frac{21}{32768q^4}+O\left(\frac{1}{q^5}\right).$$

Also,

$$(4.3)\quad \begin{cases} \alpha_p = \mathrm{E}\,(\sigma^2-1)^p = \sum_{l=0}^{p} \binom{p}{l}(-1)^{p-l} q^l / q^{(l)}\,, \qquad p=1,\cdots,k \quad \text{and} \\ \bar{\alpha}_k = \mathrm{E}\,(\sigma^{-2}-1)^k = \sum_{l=0}^{k} \binom{k}{l}(-1)^{k-l}(q+l)^{(l)}/q^l\,. \end{cases}$$

Therefore, setting

$$(4.4)\quad \begin{cases} G_2(x) = \Phi(x)\,, \\ G_4(x) = G_2(x) - \left(\dfrac{1}{8(q-1)} H_3(x) + \dfrac{1}{12(q-1)^{(2)}} H_5(x)\right)\varphi(x)\,, \quad \text{and} \\ G_6(x) = G_4(x) - \left(\dfrac{q+6}{128(q-1)^{(3)}} H_7(x) + \dfrac{3q^2+86q+120}{768(q-1)^{(4)}} H_9(x)\right)\varphi(x)\,, \end{cases}$$

we obtain the following inequalities for the error $\Delta_k \equiv \sup|\Pr\{\eta \leqq x\} - G_k(x)|$:

$$(4.5)\quad \begin{cases} \Delta_2 \leqq \beta_2 \equiv \dfrac{1}{4\pi}\left(\dfrac{1}{q-1} + \dfrac{q+2}{q^2}\right), \\ \Delta_4 \leqq \beta_4 \equiv \dfrac{1}{8\pi}\left(\dfrac{3(q+6)}{(q-1)^{(3)}} + \dfrac{3q^2+26q+24}{q^4}\right), \quad \text{and} \\ \Delta_6 \leqq \beta_6 \equiv \dfrac{1}{12\pi}\left(\dfrac{5(3q^2+86q+120)}{(q-1)^{(5)}} + \dfrac{15q^3+340q^2+1124q+720}{q^6}\right). \end{cases}$$

Table 1 shows the values of $\Delta_k = \sup_x |F_n(x) - G_k(x)|$ and their bounds β's for several values of k and n $(=2q+2)$.

Table 1. Approximation to t-distribution

n	Δ_2	β_2	Δ_4	β_4	Δ_6	β_6
6	0.0299	0.1592				
8	0.0208	0.0840				
10	0.0160	0.0564	0.0255	0.2263		
12	0.0130	0.0422	0.0121	0.0693		
16	0.0094	0.0279	0.0045	0.0188	0.0239	0.1670
20	0.0074	0.0208	0.0023	0.0045	0.0055	0.0249
30	0.0048	0.0126	0.0008	0.0024	0.0007	0.0021
40	0.0036	0.0091	0.0004	0.0011	0.0002	0.0005
50	0.0028	0.0071	0.0002	0.0006	0.0001	0.0002

Example 2. (M.L.E. for a multivariate regression model) Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a p-variate normal distribution $N_p(B'\beta, \Sigma)$, where B is a given $q\times p$ matrix of rank $q<p$, β is a q dimensional vector of unknown parameters and Σ is an unknown posi-

tive definite matrix. The maximum likelihood estimator of β is given by

$$\hat{\beta}'=(BS^{-1}B')^{-1}BS^{-1}\bar{Y},$$

where $\bar{Y}=\Sigma Y_j/n$ and $S=\Sigma(Y_j-\bar{Y})(Y_j-\bar{Y})'$. Fujikoshi [3] considered the distribution F of

$$\eta_1=\sqrt{n}\,\boldsymbol{a}'(\hat{\beta}-\beta)\boldsymbol{a}/\lambda,$$

where $\boldsymbol{a}$ is a given q dimensional vector and

$$\lambda=(\boldsymbol{a}'(B\Sigma^{-1}B')^{-1}\boldsymbol{a})^{1/2}.$$

The distribution F has the characteristic function

$$\mathrm{E}\,(e^{-t^2(1+V'W^{-1}V)/2}),$$

where V and W are mutually independent and follow $N_{p-q}(0, I)$ and $W_{p-q}(I, n-1)$, the Wishart distribution of dimensionality $r=p-q$ with $n-1$ degrees of freedom and covariance matrix I, respectively. This means that the distribution of η_1 is identical with that of $\eta=\sigma X$, where $\sigma=\sqrt{1+V'W^{-1}V}$. Fujikoshi approximated $F(x)$ by

$$G_k(x)=\Phi(x)-\sum_{l=1}^{k-1}\frac{1}{2^l l!}h_l H_{2l-1}(x)\varphi(x), \tag{4.6}$$

where

$$\begin{aligned} h_l&=\mathrm{E}\,(h(V))^l \qquad (=\mathrm{E}\,(\sigma^2-1)^l \text{ in our notation}) \\ &=\frac{r}{n-r-2}\cdot\frac{r+2}{n-r-4}\cdots\frac{r+2l-2}{n-r-2l}, \end{aligned} \tag{4.7}$$

and he obtained the following error bound.

$$|F(x)-G_k(x)|\leqq\frac{1}{2^k k!}l_{2k}h_k, \tag{4.8}$$

where

$$l_j=\sup|H_{j-1}(x)\varphi(x)|.$$

In particular, (4.8) gives

$$|F(x)-\Phi(x)|\leqq\frac{1}{2\sqrt{2\pi e}}h_1, \quad \text{and} \tag{4.9}$$

$$\left|F(x)-\Phi(x)+\frac{1}{2}h_1 x\varphi(x)\right|\leqq\frac{1.39}{8\sqrt{2\pi}}h_2. \tag{4.10}$$

In fact the inequality (4.9) can be improved:

Putting $c=\sqrt{(n-r-2)/(n-2)}$, consider the distribution $F_0(x)$ of the variable $c\eta_1$. In the above notation, $c\eta_1$ has the identical distribution with $\eta_0 \equiv c\sigma X$. But we have

$$\alpha_{01} \equiv \mathrm{E}\,((c\sigma)^2-1)=c^2\,\mathrm{E}\,(\sigma^2)-1=c^2(1+h_1)-1=0\,, \qquad \text{and}$$

$$\begin{aligned}\alpha_{02} &\equiv \mathrm{E}\,((c\sigma)^2-1)^2=\mathrm{E}\,(c\sigma)^4-1=c^4\,\mathrm{E}\,(1+h(V))^2-1\\ &=c^4(1+2h_1+h_2)-1=\frac{2r}{(n-2)(n-r-4)}\leqq h_2\,.\end{aligned}$$

Therefore

$$\begin{aligned}(4.11)\qquad |F_0(x)-\Phi(x)| &\leqq \frac{1}{4\pi}\,\mathrm{E}\,((c\sigma)^2\vee(c\sigma)^{-2}-1)^2\\ &\leqq \frac{1}{4\pi c^4}\,\mathrm{E}\,((c\sigma)^2-1)^2=\frac{1}{4\pi c^4}\,\alpha_{02}\\ &=\frac{r(n-2)}{2\pi(n-r-2)^2(n-r-4)}=\frac{1}{2\pi}\,\frac{n-2}{(n-r-2)(r+2)}h_2\,.\end{aligned}$$

This means that we can expect that $\Pr\{\sqrt{N}\boldsymbol{a}'(\hat{\beta}-\beta)\boldsymbol{a}/\lambda\leqq x\}$ can be better approximated by $\Phi\left(\sqrt{\frac{n-2}{n-r-2}}\,x\right)$ than by $\Phi(x)$ if n is large as compared to r.

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