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THE LOWER BOUND FOR THE VARIANCE OF UNBIASED ESTIMATORS FOR ONE-DIRECTIONAL FAMILY OF DISTRIBUTIONS

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Summary

In this paper we introduce the concept of one-directionality which includes both cases of location (and scale) parameter and selection parameter and also other cases, and establish some theorems for sharp lower bounds and for the existence of zero variance unbiased estimator for this class of non-regular distributions.

1. Introduction

For the lower bound for the variance of unbiased estimators, the most famous is the so-called Cramér-Rao bound. But the Cramér-Rao bound and its Bhattacharyya extension assume a set of regularity conditions. Chapman and Robbins [2], Kiefer [4] and Fraser and Guttman [3] obtained bounds with much less stringent assumptions, but they still require the independence of the support of the parameter θ or almost equivalently that the distribution with $\theta \neq \theta_0$ is absolutely continuous with respect to that with $\theta = \theta_0$ when θ_0 is the specified parameter value at which the variance is evaluated. Recently in the non-regular cases the Cramér-Rao bound has been discussed by Víncze [8], Móri [5] and others.

In the previous paper, Akahira, Puri and Takeuchi [1] get the Bhattacharyya type bound for the variance of unbiased estimators in non-regular cases.

In this paper we introduce the concept of one-directionality which includes both cases of location (and scale) parameter and selection parameter and also other cases, and show that the bound for the variance of unbiased estimators is sharp in the sense that the actual infimum of the variance of unbiased estimators is equal to the bound for a specified θ_0, for this class of non-regular distributions. We also estab-

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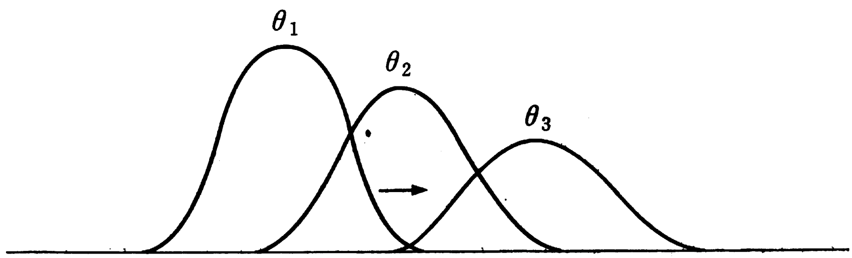
lish that for a wide class of the non-regular distributions the infimum of the variance of unbiased estimators can be zero when the sample size is not smaller than 2.

A simple but rather general case of the class of distributions of which none dominates another is that it is characterized by a real parameter θ and the distribution shifts monotonically as θ changes. For one dimensional random variables the case can be visualized by the following example.



$\theta_1 < \theta_2 < \theta_3$

Mathematical definition for such cases in a rather general set-up is given in Section 2 and is termed as one-directional family of distributions.

2. Definition of the one-directional family of distributions

We assume that we are given a model consisting of a sample space $(\chi, \mathcal{B})$ and a family $\mathcal{P}=\{P_\theta: \theta \in \Theta\}$ of probability measures, where a parameter space Θ is an open subset in a Euclidean 1-space R^1.

Throughout the subsequent discussion we shall assume the following:

(A.2.1) For each $\theta \in \Theta$, P_θ is absolutely continuous with respect to a σ-finite measure μ and the corresponding density w.r.t. μ is $f(x, \theta)$.

Let $A(\theta)$ be a support of $f(x, \theta)$, that is, $A(\theta)=\{x: f(x, \theta)>0\}$. The determination of $A(\theta)$ is not unique so far as any null set may be added to it, but in the sequel we take one and fixed determination of $A(\theta)$ for every $\theta \in \Theta$ which satisfies the following:

(A.2.2) For any disjoint points θ_1 and θ_2 in Θ, neither $A(\theta_1) \supset A(\theta_2)$ nor $A(\theta_1) \subset A(\theta_2)$.

(A.2.3) For $\theta_1 < \theta_2 < \theta_3$,

$$A(\theta_1) \cap A(\theta_3) \subset A(\theta_1) \cap A(\theta_2) ,$$

$$A(\theta_1) \cap A(\theta_3) \subset A(\theta_2) \cap A(\theta_3) .$$

(A.2.4) If θ_n tends to θ as $n \to \infty$, then

$$\mu\left(\left(\bigcup_{n=1}^{\infty}\bigcap_{i\geqq n} A(\theta_i)\right)\Delta A(\theta)\right)=\mu\left(\left(\bigcap_{n=1}^{\infty}\bigcup_{i\geqq n} A(\theta_i)\right)\Delta A(\theta)\right)=0\ ,$$

where $E\Delta F$ denotes the symmetric difference of two sets E and F.
(A.2.5) For any two points θ_1 and θ_2 in Θ with $\theta_1<\theta_2$, there exists a finite number of ξ_i $(i=1,\cdots,k)$ such that $\theta_1=\xi_0<\xi_1<\cdots<\xi_k=\theta_2$ and $\mu(A(\xi_i)\cap A(\xi_{i-1}))>0$ $(i=1,\cdots,k)$.

Then $\mathcal{P}$ is called to be a one-directional family of distributions if the conditions (A.2.1) to (A.2.5) hold.

3. The lower bound for the variance of unbiased estimators when X is a real random variable

Now suppose that X is a real random variable with a density function $f(x,\theta)$ whose support is an open interval $(a(\theta), b(\theta))$, then the condition of one-directionality means that $a(\theta)$ and $b(\theta)$ are both monotone and continuous functions. Without loss of generality we assume that $a(\theta)$ and $b(\theta)$ are monotone increasing functions. Therefore we can formulate the following problem. Let $\chi=\Theta=R^1$. Let X be a real random variable with a density function $f(x,\theta)$ (with respect to the Lebesgue measure μ) satisfying the following conditions (A.3.1) to (A.3.7):

(A.3.1)
$$f(x,\theta)>0 \quad \text{for} \quad a(\theta)<x<b(\theta)\ ,$$
$$f(x,\theta)=0 \quad \text{for} \quad x\leqq a(\theta),\ x\geqq b(\theta)\ ,$$

where $f(x,\theta)$ is continuous in x and θ for which $a(\theta)<x<b(\theta)$, and $(p+1)$-times continuously differentiable in θ for a.a. x $[\mu]$ for some non-negative integer p, both of functions $a(\theta)$ and $b(\theta)$ are p-times continuously differentiable and $a'(\theta)>0$, $b'(\theta)>0$ for all θ.

(A.3.2)
$$\lim_{x\to a(\theta)+0} f(x,\theta)=\lim_{x\to b(\theta)-0} f(x,\theta)=0\ ,$$

and for some positive integer p

$$\lim_{x\to a(\theta)+0}\frac{\partial^i}{\partial\theta^i}f(x,\theta)=\lim_{x\to b(\theta)-0}\frac{\partial^i}{\partial\theta^i}f(x,\theta)=0 \qquad (i=1,\cdots,p-1)\ ,$$

$$\lim_{x\to a(\theta)+0}\frac{\partial^p}{\partial\theta^p}f(x,\theta)=A_p(\theta)\ , \qquad \lim_{x\to b(\theta)-0}\frac{\partial^p}{\partial\theta^p}f(x,\theta)=B_p(\theta)\ ,$$

where $A_p(\theta)$ and $B_p(\theta)$ are non-zero, finite and continuous in θ.
(A.3.3) $(\partial^i/\partial\theta^i)f(x,\theta)$ $(i=1,\cdots,p)$ are linearly independent.
(A.3.4) For some $\theta_0\in\Theta$

$$0<\int_{a(\theta_0)}^{b(\theta_0)}\frac{\left\{\dfrac{\partial^i}{\partial\theta^i}f(x,\theta_0)\right\}^2}{f(x,\theta_0)}d\mu(x)$$

is finite for each $i=1,\cdots,k$, and

$$\int_{a(\theta_0)}^{b(\theta_0)} \frac{\left\{\sum_{i=k+1}^{p} c_i \frac{\partial^i}{\partial\theta^i} f(x,\theta_0)\right\}^2}{f(x,\theta_0)} d\mu$$

is infinite for each $i=k+1,\cdots,p$ unless $c_{k+1}=\cdots=c_p=0$.

(A.3.5) For $\theta_0\in\Theta$ there exists a positive number ε and a positive-valued measurable function $\rho(x)$ such that for every $x\in A(\theta)$ and every $\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)$, $\rho(x)>f(x,\theta)$, and for every $\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)$, $\int_{A(\theta)}|r(x)|\times f(x,\theta)d\mu<\infty$ implies $\int_{\bigcup_{\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)}A(\theta)}|r(x)|\rho(x)d\mu<\infty$.

(A.3.6) For each $i=1,\cdots,p+1$,

$$\overline{\lim_{h\to 0}} \sup_{x\in\bigcup_{j=1}^{i}A(\theta_0+jh)-A(\theta_0)} \frac{\left|\sum_{j=1}^{i}(-1)^j\binom{i}{j}f(x,\theta_0+jh)\right|}{|h|^i\rho(x)}<\infty\ .$$

(A.3.7) For each $i=1,\cdots,p+1$,

$$\overline{\lim_{h\to 0}} \sup_{x\in A(\theta_0)} \frac{\left|\frac{\partial^i}{\partial\theta^i}f(x,\theta_0+h)\right|}{\rho(x)}<\infty\ .$$

Note that the conditions (A.3.5), (A.3.6) and (A.3.7) are assumed to obtain the Bhattacharyya bound for the variance of unbiased estimators (see Akahira et al. [1]).

First we consider the special case when $p=0$, then we have to modify slightly the condition (A.3.2) as follows:

(A.3.2)′ $\lim_{x\to a(\theta)+0} f(x,\theta)=A_0(\theta)>0\ ,\quad \lim_{x\to b(\theta)-0} f(x,\theta)=B_0(\theta)>0\ .$

In the following theorem we shall have a lower bound.

THEOREM 3.1. *Let $g(\theta)$ be continuously differentiable over Θ. Let $\hat{g}(x)$ be an unbiased estimator of $g(\theta)$. If for $p=0$ and a fixed θ_0, the conditions* (A.3.1), (A.3.2)′, (A.3.5), (A.3.6) *and* (A.3.7) *hold, then*

$$\min_{\hat{g}:\text{ unbiased}} V_{\theta_0}(\hat{g})=0\ ,$$

where $V_{\theta_0}(\hat{g})$ denotes the variance of $\hat{g}$ at $\theta=\theta_0$.

PROOF. We first define

$$\hat{g}(x)=g(\theta_0)\qquad\text{for}\qquad a(\theta_0)<x<b(\theta_0)\ .$$

Since $a(\theta_0)<a(\theta)<b(\theta_0)<b(\theta)$ for $\theta>\theta_0$, it follows from the unbiasedness condition that

$$(3.1)\quad g(\theta)=\int_{a(\theta)}^{b(\theta)} \hat{g}(x)f(x, \theta)d\mu=\int_{a(\theta)}^{b(\theta_0)} \hat{g}(x)f(x, \theta)d\mu+\int_{b(\theta_0)}^{b(\theta)} \hat{g}(x)f(x, \theta)d\mu .$$

Putting

$$h(\theta)=\int_{a(\theta)}^{b(\theta_0)} \hat{g}(x)f(x, \theta)d\mu ,$$

we have by (3.1)

$$(3.2)\qquad \int_{b(\theta_0)}^{b(\theta)} \hat{g}(x)f(x, \theta)d\mu=g(\theta)-h(\theta) .$$

Differentiating both sides of (3.2) with respect to θ, we obtain

$$(3.3)\qquad b'(\theta)B_0(\theta)\hat{g}(b(\theta))+\int_{b(\theta_0)}^{b(\theta)} \hat{g}(x)\left\{\frac{\partial}{\partial\theta}f(x, \theta)\right\}d\mu=g'(\theta)-h'(\theta) .$$

Differentiation under the integral sign is admitted because of (A.3.7) with $p=0$. If $\hat{g}(x)$ satisfies (3.3), then it also satisfies (3.1) since $g(\theta_0)=h(\theta_0)$. Since by (A.3.1) and (A.3.2)$'$, $b'(\theta)B_0(\theta)>0$, it follows that the integral equation (3.3) is of Volterra's second type, hence the solution $\hat{g}(x)$ exists for $b(\theta)>x\geqq b(\theta_0)$. Similarly we can construct $\hat{g}(x)$ for $x< a(\theta_0)$. Repeating the same process we can define $\hat{g}(x)$ for all x. Hence we have

$$\min_{\hat{g}:\text{ unbiased}} V_{\theta_0}(\hat{g})=0 .$$

Thus we complete the proof.

The following useful lemma is a special case of the result by Takeuchi and Akahira [7].

LEMMA 3.1. *Let $g(\theta)$ be p-times differentiable over Θ. Suppose that the conditions* (A.3.3) and (A.3.4) *hold and G is the class of the all estimators $\hat{g}(x)$ of $g(\theta)$ for which*

$$\int_{a(\theta_0)}^{b(\theta_0)} \hat{g}(x)f(x, \theta_0)d\mu=g(\theta_0) ,$$

$$\int_{a(\theta_0)}^{b(\theta_0)} \hat{g}(x)\left\{\frac{\partial^i}{\partial\theta^i}f(x, \theta_0)\right\}d\mu=g^{(i)}(\theta_0)\qquad (i=1,\cdots, p) ,$$

where $g^{(i)}(\theta)$ is the i-th order derivative of $g(\theta)$ with respect to θ. Then

$$\inf_{\hat{g}\in G} V_{\theta_0}(\hat{g})=(g^{(1)}(\theta_0),\cdots, g^{(k)}(\theta_0))\Lambda^{-1}(g^{(1)}(\theta_0),\cdots, g^{(k)}(\theta_0))'$$

where Λ is a $k\times k$ matrix whose elements are

$$\lambda_{ij}=\int_{a(\theta_0)}^{b(\theta_0)} \frac{\dfrac{\partial^i}{\partial\theta^i}f(x, \theta_0)\dfrac{\partial^j}{\partial\theta^j}f(x, \theta_0)}{f(x, \theta_0)}d\mu\qquad (i, j=1,\cdots, k) .$$

The proof is omitted. In the following theorem we shall get a sharp lower bound.

THEOREM 3.2. *Let $g(\theta)$ be $(p+1)$-times differentiable over Θ. Let $\hat{g}(x)$ be an unbiased estimator of $g(\theta)$. If for $p\geqq 1$ and a fixed θ_0, the conditions* (A.3.1) *to* (A.3.7) *hold, then*

$$\inf_{\hat{g}:\text{ unbiased}} V_{\theta_0}(\hat{g})=v_k(\theta_0)\ ,$$

that is, the bound $v_k(\theta_0)$ is sharp, where

$$v_k(\theta_0)=(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))\Lambda^{-1}(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))'\ ,$$

with a $k\times k$ matrix Λ given in Lemma 3.1.

PROOF. From the unbiasedness condition of $\hat{g}(x)$, (A.3.2) and (A.3.7) we have

$$\int_{a(\theta_0)}^{b(\theta_0)}\hat{g}(x)f(x,\theta_0)d\mu=g(\theta_0)\ , \tag{3.4}$$

$$\int_{a(\theta_0)}^{b(\theta_0)}\hat{g}(x)\left\{\frac{\partial^i}{\partial\theta^i}f(x,\theta_0)\right\}d\mu=g^{(i)}(\theta_0)\qquad(i=1,\cdots,p)\ . \tag{3.5}$$

By Lemma 3.1 it follows that the sharp lower bound of $V_{\theta_0}(\hat{g})=\int_{a(\theta_0)}^{b(\theta_0)}\{\hat{g}(x)-g(\theta_0)\}^2f(x,\theta_0)d\mu(x)$ under (3.4) and (3.5) is given by $(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))\Lambda^{-1}(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))'$, i.e.,

$$\begin{aligned}\inf_{\hat{g}:\,(3.4),\,(3.5)} V_{\theta_0}(\hat{g})&=(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))\Lambda^{-1}(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))'\\&=v_k(\theta_0)\ \text{(say)}\ .\end{aligned} \tag{3.6}$$

Note that the right-hand side of (3.6) is the Bhattacharyya bound for the variance of unbiased estimators at $\theta=\theta_0$ (see Akahira et al. [1]). From (3.6) it follows that for any $\varepsilon>0$ there exists $\hat{g}_\varepsilon(x)$ in the interval $(a(\theta_0), b(\theta_0))$ satisfying (3.4), (3.5) and

$$V_{\theta_0}(\hat{g}_\varepsilon)\leqq v_k(\theta_0)+\varepsilon\ .$$

We can extend $\hat{g}_\varepsilon(x)$ for x outside $(a(\theta_0), b(\theta_0))$ from the unbiasedness condition

$$\int_{a(\theta)}^{b(\theta)}\hat{g}_\varepsilon(x)f(x,\theta)d\mu(x)=g(\theta)\qquad\text{for all}\qquad\theta\in\Theta\ . \tag{3.7}$$

For $\theta>\theta_0$, i.e., $b(\theta)\geqq b(\theta_0)$ we put

$$h(\theta)=\int_{a(\theta)}^{b(\theta_0)}\hat{g}_\varepsilon(x)f(x,\theta)d\mu(x)\ .$$

By (3.7) we obtain

$$\int_{b(\theta_0)}^{b(\theta)} \hat{g}_\epsilon(x) f(x, \theta) d\mu(x) = g(\theta) - h(\theta) . \tag{3.8}$$

Differentiating $(p+1)$-times both sides of (3.8) with respect to θ, recursively, we have by (A.3.1), (A.3.2), (A.3.5), (A.3.6) and (A.3.7)

$$\int_{b(\theta_0)}^{b(\theta)} \hat{g}_\epsilon(x) \left\{ \frac{\partial}{\partial\theta} f(x, \theta) \right\} d\mu(x) = g^{(1)}(\theta) - h^{(1)}(\theta) ,$$

$$\int_{b(\theta_0)}^{b(\theta)} \hat{g}_\epsilon(x) \left\{ \frac{\partial^p}{\partial\theta^p} f(x, \theta) \right\} d\mu(x) = g^{(p)}(\theta) - h^{(p)}(\theta) ,$$

$$B_p(\theta) b^{(1)}(\theta) \hat{g}_\epsilon(b(\theta)) + \int_{b(\theta_0)}^{b(\theta)} \hat{g}_\epsilon(x) \left\{ \frac{\partial^{p+1}}{\partial\theta^{p+1}} f(x, \theta) \right\} d\mu = g^{(p+1)}(\theta) - h^{(p+1)}(\theta) . \tag{3.9}$$

Differentiation under the integral sign is admitted because of (A.3.7). If $\hat{g}_\epsilon(x)$ satisfies (3.9), then it also satisfies (3.8) since $g^{(i)}(\theta_0) = h^{(i)}(\theta_0)$ $(i=1, \cdots, p)$ and $g(\theta_0) = h(\theta_0)$. Note that $h^{(p+1)}(\theta)$ is determined by the values of $\hat{g}_\epsilon(x)$ for $a(\theta) < x < b(\theta_0)$, where it is already given. Since the integral equation (3.9) is of Volterra's second type, it follows that the solution $\hat{g}_\epsilon(x)$ exists for $b(\theta) > x \geqq b(\theta_0)$. Similarly we can construct $\hat{g}_\epsilon(x)$ for $x < a(\theta_0)$. Repeating the same process we can define $\hat{g}_\epsilon(x)$ for all x. Hence we have

$$\inf_{\hat{g}:\ \text{unbiased}} V_{\theta_0}(\hat{g}) = v_k(\theta_0)$$

i.e., $v_k(\theta_0)$ is a sharp bound. Thus we have completed the proof.

We shall give one example corresponding to each of the situations where the conditions (A.3.1), (A.3.2)/(A.3.2)′ to (A.3.7) are assumed.

Example 3.1. Let X be a real random variable with a density function $f(x, \theta)$ (with respect to the Lebesgue measure μ) satisfying each case.

(i) Location parameter case. The density function is of the form $f(x-\theta)$ and satisfies the following:

$$f(x) > 0 \quad \text{for} \quad a < x < b ,$$

$$f(x) = 0 \quad \text{for} \quad x \leqq a,\ x \geqq b ,$$

and $\lim_{x \to a+0} f(x) > 0$, $\lim_{x \to b-0} f(x) > 0$ and $f(x)$ is continuously differentiable in the open interval (a, b).

(ii) The case on estimation of $g(\theta) = \theta$ with a density function

$$f(x-\theta)=\begin{cases} c(1-(x-\theta)^2)^{q-1} & \text{for} \quad |x-\theta|<1\,, \\ 0 & \text{for} \quad |x-\theta|\geqq 1\,, \end{cases}$$

is discussed in Akahira et al. [1], where $q>1$ and c is some constant.

(iii) Scale parameter case. The density function of the form $f(x/\theta)/\theta$ satisfies the following:

$$f(x)>0 \quad \text{for} \quad 0<a<x<b\,,$$
$$f(x)=0 \quad \text{otherwise}\,,$$

and satisfies the same condition in (i).

(iv) Selection parameter case (e.g., Morimoto and Sibuya [6]). Consider a family of density functions whose supporting intervals depend on a selection parameter θ and are of the form $(\theta, b(\theta))$, where $-\infty<\theta<b(\theta)<\infty$ and $b(\theta)$ is a nondecreasing function of θ and almost everywhere differentiable. Such a family of density functions is specified by

$$f(x,\theta)=\begin{cases} \dfrac{p(x)}{F(\theta)} & \text{for} \quad \theta\leqq x\leqq b(\theta)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $p(x)>0$ a.e. and $F(\theta)=\int_{\theta}^{b(\theta)} p(x)d\mu(x)$. Note that the cases (i), (iii) and (iv) correspond to the case $p=0$, and (ii) corresponds to the case $p=q-1$, where q is an integer.

4. The lower bound for the variance of unbiased estimators for a sample of size n of real-valued observations

Now suppose that we have a sample of size n, $(X_1,\cdots,X_n)$ of which X_i's are independently and identically distributed according to the distribution characterized in the previous section. Then we can define statistics

$$Y=\max_{1\leqq i\leqq n} X_i+\min_{1\leqq i\leqq n} X_i\,, \qquad Z=\max_{1\leqq i\leqq n} X_i-\min_{1\leqq i\leqq n} X_i$$

and we may concentrate our attention on the estimators depending only on Y and Z. Since they are not sufficient statistics, we may lose some information by doing so. More generally, for a sample of size n, $(X_1,\cdots,X_n)$ from a population in a one-directional family of densities with a support $A(\theta)$, we can define two statistics

$$\bar{\theta}=\sup\{\theta\,|\,X_i\in A(\theta)\ \ (i=1,\cdots,n)\}\,,$$
$$\underline{\theta}=\inf\{\theta\,|\,X_i\in A(\theta)\ \ (i=1,\cdots,n)\}$$

and also define

$$Y=\frac{1}{2}(\bar{\theta}+\underline{\theta}), \qquad Z=\frac{1}{2}(\bar{\theta}-\underline{\theta}).$$

There are various ways of defining the pair of statistics Y and Z, but disregarding their construction we assume that there exists a pair (Y, Z) which satisfies the following.

Let Y and Z be real-valued statistics based on a sample $(X_1, \cdots, X_n)$ of size n for $n \geqq 2$. We assume that (Y, Z) has a joint probability density function $f_\theta(y, z)$ (with respect to the Lebesgue measure $\mu_{y,z}$) satisfying

$$f_\theta(y, z)=f_\theta(y|z)h_\theta(z), \qquad \text{a.e.,}$$

where $f_\theta(y|z)$ is a conditional density function of y given z with respect to the Lebesgue measure μ_y and $h_\theta(z)$ is a density function of z with respect to the Lebesgue measure μ_z. Note that if Z is ancillary, $h_\theta(z)$ is independent of θ. We assume the following condition:

(A.4.1) For almost all z $[\mu_z]$

$$f_\theta(y|z)>0 \quad \text{for} \quad a_z(\theta)<y<b_z(\theta),$$

$$f_\theta(y|z)=0 \quad \text{for} \quad y \leqq a_z(\theta),\ y \geqq b_z(\theta),$$

where $a_z(\theta)$ and $b_z(\theta)$ are strictly monotone increasing functions of θ for almost all z $[\mu_z]$ which depend on z, and

$$h_\theta(z)>0 \quad \text{for} \quad c<z<d,$$

$$h_\theta(z)=0 \quad \text{for} \quad z \leqq c,\ z \geqq d,$$

where c and d are constants independent of θ. We also assume that for almost all $z\,[\mu_z]$, $f_\theta(y|z)$ instead of $f(x, \theta)$ satisfies the conditions (A.3.2) to (A.3.7), and we call the corresponding conditions (A.4.2) to (A.4.7).

Let $\hat{g}(y, z)$ be any unbiased estimator of $g(\theta)$. We define

$$\phi_z(\theta)=\int_{a_z(\theta)}^{b_z(\theta)} \hat{g}(y, z)f_\theta(y|z)d\mu_y \quad \text{for} \quad \text{a.a.} \quad z\,[\mu_z]. \tag{4.1}$$

Further we assume the following condition:

(A.4.8)
$$\phi_z(\theta_0)=g(\theta_0) \quad \text{for} \quad \text{a.a.} \quad z\,[\mu_z],$$

$$\phi_z^{(i)}(\theta_0)=0 \quad \text{for} \quad \text{a.a.} \quad z\,[\mu_z]\ (i=1, \cdots, k),$$

where $\phi_z^{(i)}(\theta)$ is the k-th order derivative of $\phi_z(\theta)$ with respect to θ.

In the following theorem we shall show that the sharp bound is equal to zero.

THEOREM 4.1. *Let $g(\theta)$ be $(p+1)$-times differentiable over Θ. Let $\hat{g}(X_1,\cdots,X_n)$ be an unbiased estimator of $g(\theta)$. If $n\geqq 2$ and for a fixed θ_0, the conditions* (A.4.1) *to* (A.4.8) *hold, then,*

$$\inf_{\hat{g}:\,\text{unbiased}} V_{\theta_0}(\hat{g})=0 .$$

PROOF. From the unbiasedness condition we obtain

$$\int_c^d \phi_z(\theta)h_\theta(z)d\mu_z=g(\theta) \qquad \text{for all} \qquad \theta\in\Theta . \tag{4.2}$$

First we assume that $\phi_z(\theta)$ is given, then under (A.4.1) to (A.4.7) we have by Lemma 3.1

$$\begin{aligned}\inf_{\hat{g}:(4.1)} &\int_{a_z(\theta_0)}^{b_z(\theta_0)} \{\hat{g}(y,z)-\phi_z(\theta_0)\}^2 f_{\theta_0}(y|z)d\mu_y \\ &=(\phi_z^{(1)}(\theta_0),\cdots,\phi_z^{(k)}(\theta_0))\Lambda^{-1}(\phi_z^{(1)}(\theta_0),\cdots,\phi_z^{(k)}(\theta_0))' \\ &=v_k(\theta_0|z) \qquad (\text{say}) ,\end{aligned} \tag{4.3}$$

where $\phi_z^{(i)}(\theta)$ is the i-th order derivative of $\phi_z(\theta)$ with respect to θ and Λ is a $k\times k$ matrix whose elements are

$$\lambda_{ij}=\int_{a_z(\theta_0)}^{b_z(\theta_0)} \frac{1}{f_{\theta_0}(y|z)}\left\{\frac{\partial^i}{\partial\theta^i}f_{\theta_0}(y|z)\right\}\left\{\frac{\partial^j}{\partial\theta^j}f_{\theta_0}(y|z)\right\}d\mu_y .$$

From (4.3) it follows that for any $\varepsilon>0$ there exists $\hat{g}_\varepsilon(y,z)$ such that

$$\int_{a_z(\theta)}^{b_z(\theta)} \hat{g}_\varepsilon(y,z)f_\theta(y|z)d\mu_y=\phi_z(\theta) \qquad \text{for all} \qquad \theta\in\Theta , \tag{4.4}$$

$$\int_{a_z(\theta_0)}^{b_z(\theta_0)} \{\hat{g}(y,z)-\phi_z(\theta_0)\}^2 f_{\theta_0}(y|z)d\mu_y<v_k(\theta_0|z)+\varepsilon . \tag{4.5}$$

Since by (4.2)

$$\int_c^d\int_{a_z(\theta)}^{b_z(\theta)} \hat{g}_\varepsilon(y,z)f_\theta(y|z)h_\theta(z)d\mu_y d\mu_z=g(\theta)$$

for all $\theta\in\Theta$, it follows from (4.5) that

$$\begin{aligned}\int_c^d\int_{a_z(\theta_0)}^{b_z(\theta_0)} &\{\hat{g}_\varepsilon(y,z)-\phi_z(\theta_0)\}^2 f_{\theta_0}(y|z)h_{\theta_0}(z)d\mu_y d\mu_z \\ &<\int_c^d v_k(\theta_0|z)h_{\theta_0}(z)d\mu_z+\varepsilon .\end{aligned} \tag{4.6}$$

By the condition (A.4.8)

$$v_k(\theta_0|z)=0 \qquad \text{for} \qquad \text{a.a.} \quad z\,[\mu_z] .$$

From (4.6) we obtain

$$\int_c^d \int_{a_z(\theta_0)}^{b_z(\theta_0)} \{\hat{g}_\varepsilon(y, z)-\theta_0\}^2 f_{\theta_0}(y|z)h_{\theta_0}(z)d\mu_y d\mu_z<\varepsilon \ . \tag{4.7}$$

Putting $\hat{g}_\varepsilon(x_1,\cdots, x_n)=\hat{g}_\varepsilon(y, z)$, we have by (4.4) and (4.7)

$$E_\theta(\hat{g}_\varepsilon)=g(\theta) \qquad \text{for all} \qquad \theta\in\Theta ,$$

$$V_{\theta_0}(\hat{g}_\varepsilon)<\varepsilon \ .$$

Letting $\varepsilon\to 0$ we obtain

$$\inf_{\hat{g}:\ \text{unbiased}} V_{\theta_0}(\hat{g})=0 \ .$$

Thus we complete the proof.

Now we shall find a function $\phi_z(\theta)$ satisfying the condition (A.4.8) when $\Theta=R^1$, $g(\theta)=\theta$ and μ_z is a Lebesgue measure. Without loss of generality, we put $\theta_0=0$. We define

$$\phi_z(\theta)=\frac{M \operatorname{sgn}\theta|\theta|^{k+2}(z-c)^k}{h_\theta(z)\{(d-z)^{k+2}+|\theta|^{k+2}(z-c)^{k+2}\}} \ , \tag{4.8}$$

where M is a constant and k is a positive integer. Then we have

$$\begin{aligned}\int_c^d \phi_z(\theta)h_\theta(z)dz&=M\int_c^d \frac{\operatorname{sgn}\theta|\theta|^{k+2}(z-c)^k}{(d-z)^{k+2}+|\theta|^{k+2}(z-c)^{k+2}}dz\\ &=\frac{M}{d-c}\operatorname{sgn}\theta|\theta|^{k+2}\int_0^\infty \frac{u^k}{1+|\theta|^{k+2}u^{k+2}}du\\ &\qquad\left(\text{after transformation } u=\frac{z-c}{d-z}\right)\\ &=\frac{M}{d-c}\operatorname{sgn}\theta|\theta|\int_0^\infty \frac{v^k}{1+v^{k+2}}dv=\frac{MK}{d-c}\theta \ ,\end{aligned}$$

where $K=\int_0^\infty \frac{v^k}{1+v^{k+2}}dv$ is a constant. If we put $M=(d-c)/K$, then

$$\int_c^d \phi_z(\theta)h_\theta(z)dz=\theta \ .$$

And it is easily seen that

$$\phi_z(0)=0 \qquad \text{for} \qquad \text{a.a.}\ \ z ,$$

$$\phi_z^{(i)}(0)=0 \qquad \text{for} \qquad \text{a.a.}\ \ z\ (i=1,\cdots, k) .$$

Thus it is shown that $\phi_z(\theta)$ given by (4.8) satisfies the condition (A.4.8).

We consider the estimation on the location parameter θ. Let X_1 and X_2 be independently and identically distributed with a density function $f(x, \theta)$ of the form $f(x-\theta)$ which satisfies the following:

$$f(x)>0 \quad \text{for} \quad a<x<b\,,$$

$$f(x)=0 \quad \text{for} \quad x\leqq a,\ x\geqq b\,,$$

and $\lim_{x\to a+0} f(x)>0$, $\lim_{x\to b-0} f(x)>0$ and $f(x)$ is continuously differentiable in the open interval (a, b). We define

$$Y=\frac{1}{2}(X_1+X_2)\,, \qquad Z=\frac{1}{2}(X_1-X_2)\,.$$

Then if the conditions (A.4.1) to (A.4.8) are assumed, we have

$$\inf_{\hat{\theta}:\,\text{unbiased}} V_{\theta_0}(\hat{\theta})=0\,.$$

Note that Z is an ancillary statistic, but that (Y, Z) is not sufficient unless $f(x)$ is constant for $a<x<b$.

5. A second type approach to obtain the lower bound for the variance of unbiased estimators

Suppose that (X, Y) is a pair of real random variables according to a joint density function $f(x, y, \theta)$ (with respect to the Lebesgue measure μ) which has the product set $(0, a(\theta))\times(0, b(\theta))$ of two open intervals as its support $A(\theta)$, where $a(\theta)$ is a monotone increasing function and $b(\theta)$ is a monotone decreasing function. We assume the condition

(A.5.1) $$\inf_{(x,y)\in A(\theta)} f(x, y, \theta)>0\,.$$

Let the marginal density functions of X and Y be $f_1(x, \theta)$ and $f_2(y, \theta)$ with respect to the Lebesgue measures μ_x and μ_y, respectively. We further make the following assumption:

(A.5.2) The density functions $f_1(x, \theta)$ and $f_2(y, \theta)$ are continuously differentiable in θ and satisfy the conditions (A.3.5), (A.3.6) and (A.3.7) for $k=1$ when $f_1(x, \theta)$ and $f_2(y, \theta)$ are substituted instead of $f(x, \theta)$ in them.

In the following theorem we shall show that the sharp bound is equal to zero.

THEOREM 5.1. *Let $\Theta=R^1$. Suppose that X and Y are random variables with a joint density function $f(x, y, \theta)$ (with respect to a σ-finite measure μ) satisfying* (A.5.1) *and* (A.5.2) *for a fixed θ_0. Let $g(\theta)$ be continuously differentiable over Θ. Let $\hat{g}(x, y)$ be an unbiased estimator of $g(\theta)$. Then*

$$\inf_{\hat{g}:\,\text{unbiased}} V_{\theta_0}(\hat{g})=0\,.$$

PROOF. We first define

$$\hat{g}(x, y)=g(\theta_0) \qquad \text{for} \qquad (x, y) \in A(\theta_0) .$$

In order to extend $\hat{g}(x, y)$ for (x, y) outside $A(\theta_0)$ using the unbiasedness condition, we consider unbiased estimators $\hat{g}_1(x)$ and $\hat{g}_2(y)$ of $g(\theta)$ with respect to $f_1(x, \theta)$ and $f_2(y, \theta)$, respectively, such that $\hat{g}_1(x)=g(\theta_0)$ for $0<x<a(\theta_0)$ and $\hat{g}_2(y)=g(\theta_0)$ for $0<y<b(\theta_0)$. For $\theta>\theta_0$, i.e., $a(\theta)\geqq a(\theta_0)$ we put

$$h_1(\theta)=g(\theta_0)\int_0^{a(\theta_0)} f_1(x, \theta)d\mu_x$$

and also for $\theta\leqq\theta_0$, i.e., $b(\theta)\geqq b(\theta_0)$

$$h_2(\theta)=g(\theta_0)\int_0^{b(\theta_0)} f_2(y, \theta)d\mu_y .$$

Since $\hat{g}_1(X)$ and $\hat{g}_2(Y)$ are unbiased estimator of $g(\theta)$, it follows that

$$(5.1) \qquad \begin{aligned} &\int_{a(\theta_0)}^{a(\theta)} \hat{g}_1(x)f_1(x, \theta)d\mu_x=g(\theta)-h_1(\theta) \qquad \text{for all} \qquad \theta\geqq\theta_0 , \\ &\int_{b(\theta_0)}^{b(\theta)} \hat{g}_2(y)f_2(y, \theta)d\mu_y=g(\theta)-h_2(\theta) \qquad \text{for all} \qquad \theta\leqq\theta_0 . \end{aligned}$$

Since the supports of the density functions $f_1(x, \theta)$ and $f_2(y, \theta)$ are open intervals $(0, a(\theta))$ and $(0, b(\theta))$, respectively, it follows from (A.5.1) that

$$(5.2) \qquad \begin{aligned} &0<\lim_{x\to a(\theta)-0} f_1(x, \theta)=\alpha_1(\theta) \qquad \text{(say)} , \\ &0<\lim_{y\to b(\theta)-0} f_2(y, \theta) . \end{aligned}$$

Differentiating both sides of (5.1) we have by (A.5.2)

$$(5.3) \qquad \hat{g}_1(a(\theta))\alpha_1(\theta)+\int_{a(\theta_0)}^{a(\theta)} \hat{g}_1(x)\left\{\frac{\partial}{\partial\theta}f_1(x, \theta)\right\}d\mu_x=g'(\theta)-h_1'(\theta) ,$$

for all $\theta\geqq\theta_0$. Since the equation (5.3) is of Volterra's second type, it follows that the solution $\hat{g}_1(x)$ exists for all $x\geqq a(\theta_0)$. If $\hat{g}_1(x)$ satisfies (5.3), then it also satisfies (5.1) since $g(\theta_0)=h_1(\theta_0)$. Similarly we can construct the unbiased estimator $\hat{g}_2(y)$ for all $y\geqq b(\theta_0)$. We define an estimator

$$(5.4) \qquad \hat{g}(x, y)=\begin{cases} g(\theta_0) & \text{for} \quad 0<x<a(\theta_0),\ 0<y<b(\theta_0) , \\ \hat{g}_1(x) & \text{for} \quad a(\theta_0)\leqq x,\ 0<y<b(\theta_0) , \\ \hat{g}_2(y) & \text{for} \quad 0<x<a(\theta_0),\ b(\theta_0)\leqq y . \end{cases}$$

Then $\hat{g}(X, Y)$ is an unbiased estimator of $g(\theta)$ with variance 0 at $\theta=\theta_0$.

Indeed, we have from (5.1) and (5.4)

$$E_\theta[\hat{g}(X, Y)]=\int_0^{b(\theta)}\int_0^{a(\theta_0)} g(\theta_0)f(x, y, \theta)d\mu+\int_0^{b(\theta)}\int_{a(\theta_0)}^{a(\theta)} \hat{g}_1(x)f(x, y, \theta)d\mu$$
$$=g(\theta_0)\int_0^{a(\theta_0)} f_1(x, \theta)d\mu_x+\int_{a(\theta_0)}^{a(\theta)} \hat{g}_1(x)f_1(x, \theta)d\mu_x ,$$

for all $\theta\geqq\theta_0$. Similarly we have that $E_\theta[\hat{g}(X, Y)]=g(\theta)$ for all $\theta\leqq\theta_0$. Hence we see that $\hat{g}(X, Y)$ is an unbiased estimator of $g(\theta)$. We also have, for all $\theta\geqq\theta_0$

$$V_\theta(\hat{g}(X, Y))=g^2(\theta_0)\int_0^{a(\theta_0)} f_1(x, \theta)d\mu_x-\int_{a(\theta_0)}^{a(\theta)} \hat{g}_1^2(x)f_1(x, \theta)d\mu_x-g^2(\theta) .$$

When $\theta=\theta_0$, we obtain

$$V_{\theta_0}(\hat{g}(X, Y))=0 .$$

Thus we complete the proof.

We can give the following example.

Example 5.1. Let $X_1,\cdots, X_n$ and $Y_1,\cdots, Y_n$ be independently and identically distributed random variables with the uniform distributions $U(0, \theta)$ and $U(0, 1/\theta)$, respectively. Put

$$T_1=\max_{1\leqq i\leqq n} X_i , \qquad T_2=\max_{1\leqq i\leqq n} Y_i .$$

Then the unbiased estimator $\hat{\theta}(T_1, T_2)$ of θ with variance 0 at $\theta=1$ is given by

$$\hat{\theta}(t_1, t_2)=\begin{cases} \hat{\theta}_1(t_1) & \text{for} \quad 1\leqq t_1,\ 0<t_2<1 , \\ \hat{\theta}_2(t_2) & \text{for} \quad 1\leqq t_2,\ 0<t_1<1 , \\ 1 & \text{for} \quad t_1<1,\ t_2<1 , \end{cases} \tag{5.5}$$

where

$$\hat{\theta}_1(t_1)=\left(1+\frac{1}{n}\right)t_1 \qquad \text{for} \quad 1\leqq t_1 ,$$

$$\hat{\theta}_2(t_2)=\left(1-\frac{1}{n}\right)t_2 \qquad \text{for} \quad 1\leqq t_2 .$$

Indeed, we can easily see that the estimator $\hat{\theta}(T_1, T_2)$ is unbiased. We also have, for $0<\theta\leqq1$

$$V_\theta(\hat{\theta}(T_1, T_2))=\left\{1-\frac{(n-1)^2}{n(n-2)}\right\}(\theta^n-\theta^2) . \tag{5.6}$$

We also have, for $\theta\geqq1$

$$(5.7) \qquad V_\theta(\hat{\theta}(T_1, T_2)) = \left\{1 - \frac{(n+1)^2}{n(n-2)}\right\}(\theta^{-n} - \theta^2) .$$

From (5.6) and (5.7), we obtain

$$V_1(\hat{\theta}(T_1, T_2)) = 0 .$$

Thus we see that the unbiased estimator $\hat{\theta}(T_1, T_2)$ given by (5.5) with variance 0 at $\theta=1$.

As a further case of this situation, let X and Y be independent real random variables according to density functions $(1/\theta)f(x/\theta)$ and $\theta f(\theta y)$ (with respect to the Lebesgue measure μ) with a positive valued parameter θ, respectively, which satisfy the following:

$$(\text{A.5.3}) \qquad \begin{aligned} &f(x)>0 \quad \text{for} \quad 0<x<1 , \\ &f(x)=0 \quad \text{otherwise} , \end{aligned}$$

and $f(x)$ is $(p+1)$-times continuously differentiable in the open interval $(0, 1)$ and for each $i=0, 1, \cdots, p$

$$0 < \lim_{x\to 0+0} \left|\frac{f^{(i)}(x)}{x^{p-i}}\right| < \infty , \qquad 0 < \lim_{x\to 1-0} \left|\frac{f^{(i)}(x)}{(1-x)^{p-i}}\right| < \infty .$$

By Theorem 4.1 we have the following:

THEOREM 5.2. *Let $g(\theta)$ be an estimable function which is $(p+1)$-times differentiable over R^1. Let $\hat{g}(X, Y)$ be an unbiased estimator of $g(\theta)$. If the conditions* (A.5.3) *and* (A.4.8) *on $p_\theta(x|t)$ hold, then*

$$\inf_{\hat{g}:\,\text{unbiased}} V_{\theta_0}(\hat{g}) = 0 ,$$

where $p_\theta(x|t)$ denotes the conditional density function of X given $XY=t$.

PROOF. Letting $T=XY$, we have the conditional density function $p_\theta(x|t)$ of X given $T=t$:

$$(5.8) \qquad p_\theta(x|t) = \begin{cases} \dfrac{\dfrac{1}{x}f\left(\dfrac{x}{\theta}\right)f\left(\dfrac{\theta t}{x}\right)}{\displaystyle\int_{\theta t}^{\theta} \frac{1}{x}f\left(\frac{x}{\theta}\right)f\left(\frac{\theta t}{x}\right)dx} & \text{for} \quad \theta t<x<\theta , \\ 0 & \text{otherwise} , \end{cases}$$

for almost all $t\,[\mu]$. Since for $\theta=1$ and almost all $t\,[\mu]$

$$p_1(x|t) = \begin{cases} \dfrac{c_t}{x}f(x)f\left(\dfrac{t}{x}\right) & \text{for} \quad t<x<1 , \\ 0 & \text{otherwise} , \end{cases}$$

we obtain from (5.8)

$$p_\theta(x\,|\,t)=\frac{1}{\theta}p_1\left(\frac{x}{\theta}\,\middle|\,t\right) \qquad \text{for all} \qquad \theta, \text{ a.a. } t\ [\mu]$$

where

$$c_t=\left(\int_t^1 \frac{1}{x}f(x)f\left(\frac{t}{x}\right)dx\right)^{-1}.$$

Putting

$$g_t(x)=\begin{cases} \dfrac{c_t}{x}f\left(\dfrac{t}{x}\right) & \text{for} \quad t<x<1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

for almost all t $[\mu]$, we have

$$p_1(x\,|\,t)=f(x)g_t(x)\,, \qquad \text{a.a. } t\ [\mu]\,, \tag{5.9}$$

hence the same conditions on $p_1(x\,|\,t)$ as (A.4.1) and (A.4.2) hold. Indeed, we have by (A.5.3) and (5.9)

$$\begin{aligned} &\lim_{x\to t+0} p_1(x\,|\,t)=f(t+0)g_t(t+0)=\frac{c_t}{t}f(t+0)f(1+0)=0\,, \\ &\lim_{x\to 1-0} p_1(x\,|\,t)=f(1-0)g_t(1-0)=0 \end{aligned} \tag{5.10}$$

for almost all t $[\mu]$. By the Leibniz's formula, we obtain

$$\begin{aligned} \frac{\partial^i p_t(x\,|\,t)}{\partial x^i}&=\sum_{j=0}^{i}\binom{i}{j}f^{(j)}(x)g_t^{(i-j)}(x) \\ &=\sum_{j=0}^{i}\binom{i}{j}f^{(i-j)}(x)g_t^{(j)}(x)\,, \qquad \text{a.a. } t\ [\mu]\,. \end{aligned} \tag{5.11}$$

By (A.5.3) we have for $i=1,\cdots,p-1$ and a.a. t $[\mu]$

$$\lim_{x\to 1-0}\frac{\partial^i p_1(x\,|\,t)}{\partial x^i}=0\,, \qquad \lim_{x\to 1-0}\frac{\partial^p p_1(x\,|\,t)}{\partial x^p}=c_t\neq 0 \tag{5.12}$$

where c_t is finite. Since

$$g^{(i)}(x)=c_t\sum_{j=0}^{i}(-1)^j j\,!\frac{1}{x^{j+1}}\frac{\partial^{j-i}}{\partial x^{j-i}}f\left(\frac{t}{x}\right)\,,$$

it follows by (A.5.3) and (5.11) that for almost all t $[\mu]$,

$$\begin{aligned} &\lim_{x\to t+0}\frac{\partial^i p_1(x\,|\,t)}{\partial x^i}=0 \qquad (i=1,\cdots,p-1)\,, \\ &\lim_{x\to t+0}\frac{\partial^p p_1(x\,|\,t)}{\partial x^p}=D_t\neq 0\,, \end{aligned} \tag{5.13}$$

where D_t is finite. It is seen by (5.9), (5.10), (5.12) and (5.13) that the same condition on $p_1(x|t)$ as (A.4.2) holds.

For $i=1,\cdots,[p/2]$

$$0<\int_t^1 \frac{\left\{\frac{\partial^i}{\partial x^i} p_1(x|t)\right\}^2}{p_1(x|t)} dx<\infty\,, \qquad \text{a.a.}\quad t\ [\mu]$$

and

$$\int_t^1 \frac{\left\{\sum_{i=[p/2]+1}^{p} c_i \frac{\partial^i}{\partial x^i} p_1(x|t)\right\}^2}{p_1(x|t)} dx\,, \qquad \text{a.a.}\quad t\ [\mu]$$

is infinite unless $c_{[p/2]+1}=\cdots=c_p$, where $[s]$ denotes the largest integer less than or equal to s, since when $x\to 0+0$ or $x\to t-0$ the numerator of the integrand approaches to a polynomial in x or $t-x$ of the degree $p-i^*$ if $c_{i^*}\neq 0$ and $c_{i^*+1}=\cdots=c_p=0$ and the denominator tends to that of the degree p. Hence the same condition on $p_1(x|t)$ as (A.4.4) holds for $k=[p/2]$. And also from (A.5.1) it is seen that when $x\to 0+0$ or $x\to t-0$, $\{(\partial^i/\partial x^i)p_1(x|t)\}/p_1(x|t)$ $(i=0,1,\cdots,k)$ approaches to polynomials of different degrees, hence they are linearly independent. Putting

$$\rho_t(x)=\sup_{x':|x'-x|<c} p_1(x'|t)$$

for appropriate $c>0$, the same conditions on $p_1(x|t)$ as (A.4.5) to (A.4.7) hold. By Theorem 4.1 we obtain the conclusion of Theorem 5.2.

When $g(\theta)=\theta$, Example 5.1 can be also an example of Theorem 5.2.

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REFERENCES

[1] Akahira, M., Puri, M. L. and Takeuchi, K. (1984). Bhattacharyya bound of variances of unbiased estimators in non-regular cases, *Ann. Inst. Statist. Math.*, **38**, 35-44.

[2] Chapman, D. G. and Robbins, H. (1951). Minimum variance estimation without regularity assumptions, *Ann. Math. Statist.*, **22**, 581-586.

[3] Fraser, D. A. S. and Guttman, I. (1952). Bhattacharyya bounds without regularity assumptions, *Ann. Math. Statist.*, **23**, 629-632.

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[4] Kiefer, J. (1952). On minimum variance in non-regular estimation, *Ann. Math. Statist.*, **23**, 627-629.

[5] Móri, T. F. (1983). Note on the Cramér-Rao inequality in the non-regular cases: The family of uniform distributions, *J. Statist. Plann. Inference*, **7**, 353-358.

[6] Morimoto, H. and Sibuya, M. (1967). Sufficient statistics and unbiased estimation of restricted selection parameter, *Sankhyā*, A**27**, 15-40.

[7] Takeuchi, K. and Akahira, M. (1983). A note on minimum variance, *Metrika*, **33**, 85-91.

[8] Víncze, I. (1979). On the Cramér-Fréchet-Rao inequality in the non-regular case, In *Contributions to Statistics*, the Jaroslav Hájek Memorial Volume, Academia, Prague, 253-262.