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A COMPLETE CLASS FOR LINEAR ESTIMATION IN A GENERAL LINEAR MODEL

C. STĘPNIAK

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Summary

It is shown that in linear estimation, both unbiased and biased, all unique (up to equivalence with respect to risk) locally best estimators and their limits constitute a complete class.

1. Introduction

Olsen, Seely and Birkes [7], and LaMotte [6] have shown that in linear estimation, both unbiased and biased, all admissible estimators constitute a minimal complete class. Moreover, as they shown, any admissible linear estimator is *locally best* in something of a canonical parameter set. LaMotte [6] has presented also a procedure, based on so called trivial points of the parameter set, by which we can verify, in a finite number of steps, whether a linear estimator is admissible or not. However, this procedure is not convenient in practice.

Our plan is as follows. It is easy to show that any *unique* (up to equivalence with respect to risk) locally best estimator is admissible. In particular, any linear estimator being locally best in the *relative interior* of the canonical parameter set is admissible. Basing on a well-known necessary condition for admissibility, due to Farrell [1], we shall show that any admissible linear estimator may be presented as a *limit* of estimators being locally best in the relative interior. This fact is not surprising in the light of some known results in the Wald theory but it is unattainable, as yet, by the algebraic way, which is actually prevalent in linear estimation.

Our complete class is included, usually properly, in the set of *all* locally best estimators. However, as we shall show by examples, this class may also not be minimal complete, because some limits of the admissible estimators may be inadmissible.

Key words and phrases: Linear model, linear estimation, unbiased or not, complete class.

2. Notation and initial reduction

In this section the usual vector-matrix notation will be used. Among others, if A is a matrix, then A', $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote, respectively, the transpose, the range (column space) and the null space of A. By $\mathcal{S}_n$ is denoted the space of all symmetric matrices of size $n\times n$, and by $\mathcal{S}_n^+$ the cone of all non-negative definite matrices in $\mathcal{S}_n$. Symbol $\|\cdot\|$ stands for the usual norm in an Euclidean space $\boldsymbol{R}^n$, that is $\|x\|=x'x$, where x is a column of $n\times 1$. Moreover the symbol $ri(K)$, where K is a non-empty set in $\boldsymbol{R}^n$, denotes the relative interior of K, i.e.,

$$ri(K)=\{x\in K:\ S(x,\varepsilon)\cap \operatorname{aff}(K)\subseteqq K \text{ for some } \varepsilon>0\}\ ,$$

where $S(x,\varepsilon)$ is the open ball in $\boldsymbol{R}^n$ centered at x of radius ε, while $\operatorname{aff}(K)$ is the minimal affine set including K (see Rockafellar [9]).

In fact, $ri(K)$ is the interior of K relative to $\operatorname{aff}(K)$. We note that the relative interior of any non-empty convex set in $\boldsymbol{R}^n$ is non-empty. Moreover, by the relation $S(\lambda x_1+(1-\lambda)x_2, \lambda\varepsilon)=\lambda S(x_1,\varepsilon)+(1-\lambda)x_2$, we get

$$\lambda x_1+(1-\lambda)x_2\in ri(K) \quad \text{for all}\quad x_1\in ri(K),\ x_2\in K \text{ and } \lambda\in(0,1]\ . \tag{2.1}$$

Let Y be a random vector in $\boldsymbol{R}^n$ with expectation μ and variance-covariance matrix V, where (μ, V) is an unknown element of a given but arbitrary set Ω in $\boldsymbol{R}^n\times\mathcal{S}_n^+$. Consider estimation of a parameter $c'\mu$, $c\in\boldsymbol{R}^n$, by estimators of the form $d'Y$, where d belongs to a set D in $\boldsymbol{R}^n$. An estimator $d'Y$ will often be identified by its coefficient vector d. Our assumptions about the set D will be precised latter (see the condition (2.6)).

We shall compare different estimators according to their possible mean squared errors (MSE), where

$$\begin{aligned}\mathrm{MSE}\,(d\,|\,\mu, V)&=\mathrm{E}\,(d'Y-c'\mu)^2\\ &=d'(V+\mu\mu')d-2d'\mu\mu'c+c'\mu\mu'c\ .\end{aligned}$$

Olsen, Seely and Birkes [7] and LaMotte [6] noted that the mean squared error is a linear function of V and $\mu\mu'$. They treated the pair $(V, \mu\mu')$ as a canonical parameter. It will be more convenient for us to use a slightly modified parameter $\theta=(\theta_1,\theta_2)$, where $\theta_1=V+\mu\mu'$ and $\theta_2=\mu\mu'$ (see also LaMotte [4]).

Let $\boldsymbol{\Theta}_0$ be the set of all possible values of θ when (μ, V) runs over Ω. Then the risk of an estimator d may be written in the form

$$R(\theta, d)=d'\theta_1 d-2d'\theta_2 c+c'\theta_2 c\ , \qquad \theta \in \boldsymbol{\Theta}_0\ . \tag{2.2}$$

This risk defines a preordering $\succeq$ on the set D according to: $d_1 \succeq d_2$ if $R(\theta, d_1) \leqq R(\theta, d_2)$ for all $\theta \in \boldsymbol{\Theta}_0$. It is clear that this preordering remains unchanged if we replace $\boldsymbol{\Theta}_0$ by any other set $\boldsymbol{\Theta}$ such that

$$\text{Cone}\,(\boldsymbol{\Theta})=\text{Cone}\,(\boldsymbol{\Theta}_0)\ , \tag{2.3}$$

where Cone(K) denotes the minimal closed convex cone containing a set K. We recall that the canonical parameter set used by Olsen, Seely and Birkes [7] and LaMotte [6] corresponds to the choice $\boldsymbol{\Theta}=$ Cone$(\boldsymbol{\Theta}_0)$.

Throughout this paper $\boldsymbol{\Theta}$ will be an *arbitrary convex set* satisfying the condition (2.3).

Let G be any maximal element in the set $\{\theta_1 : (\theta_1, \theta_2) \in \boldsymbol{\Theta}\}$, in the sense that $\mathcal{R}(\theta_1) \subseteqq \mathcal{R}(G)$ for all $(\theta_1, \theta_2) \in \boldsymbol{\Theta}$, and let k be the rank of G. We define

$$F=[F_1 \vdots F_2]\ , \tag{2.4}$$

where the matrices F_1, of size $n\times k$, and F_2, of size $n\times(n-k)$, satisfy the conditions $\mathcal{R}(F_1)=\mathcal{R}(G)$, $\mathcal{R}(F_2)=\mathcal{N}(G)$, $F_1'F_1=I_k$ and $F_2'F_2=I_{n-k}$. It follows that F_1F_1' is the orthogonal projector on $\mathcal{R}(G)$. It is easy to check that $\mathcal{R}(\theta, d)$ depends on d only through $F_1F_1'd$. Thus, instead of D, we only need to consider the set

$$D_0=\{F_1F_1'd :\ d \in D\}\ . \tag{2.5}$$

We shall assume that D_0 is a closed convex set in $\mathcal{R}(G)$ and that there exists a non-empty set H in $\mathcal{N}(G)$ such that D may be presented as the direct sum

$$D=D_0 \oplus H\ . \tag{2.6}$$

It is easy to verify that for usual, that is unrestrictive, linear estimation both unbiased and biased the condition (2.6) is valid.

We need the following lemmas.

LEMMA 2.1. *If* $\theta=(\theta_1, \theta_2) \in ri(\boldsymbol{\Theta})$, *then* $\mathcal{R}(\theta_1)=\mathcal{R}(F_1)$.

LEMMA 2.2. *The risk* $R(\theta, \cdot)$ *is a convex function of* $d \in D_0$ *for all* $\theta \in \boldsymbol{\Theta}$, *and strictly convex for* $\theta \in ri(\boldsymbol{\Theta})$.

Remark 1. Lemma 2.1 is essentially due to LaMotte [4].

We shall give elementary proofs of these lemmas.

PROOF OF LEMMA 2.1. Let $\theta=(\theta_1, \theta_2) \in ri(\boldsymbol{\Theta})$. Suppose, by contradiction, that $\mathcal{R}(\theta_1)$ is a proper subset $\mathcal{R}(F_1)$. Then there exists a $\bar{\theta}=(\bar{\theta}_1, \bar{\theta}_2) \in \Theta$ and a vector $x \in \mathcal{N}(\theta_1)$ such that $x'\bar{\theta}_1 x>0$.

On the other hand, it follows from the conditions $\theta \in ri(\Theta)$ and $\bar{\theta} \in \Theta$ by (2.1), that $\bar{\theta}+\lambda(\theta-\bar{\theta}) \in ri(\Theta)$ for some $\lambda>1$. Thus we get $(1-\lambda)\cdot x'\bar{\theta}_1 x>0$. This contradicts the condition $\bar{\theta} \in \Theta$, because $\bar{\theta}_1$ is not non-negative definite.

PROOF OF LEMMA 2.2. We need to show

$$R\left(\theta, \frac{d_1+d_2}{2}\right) \leqq \frac{1}{2} R(\theta, d_1)+\frac{1}{2} R(\theta, d_2) \tag{2.7}$$

with the strict inequality if $\theta \in ri(\Theta)$ and $d_1 \neq d_2$. Note that

$$R\left(\theta, \frac{d_1+d_2}{2}\right)=\frac{1}{4} R(\theta, d_1)+\frac{1}{4} R(\theta, d_2)+\frac{1}{2}[d_1'\theta_1 d_2-c'\theta_2(d_1+d_2)+c'\theta_2 c] .$$

On the other hand, by inequalities $ab \leqq (a^2+b^2)/2$ and of Cauchy-Schwarz, we get

$$d_1'\theta_1 d_2 \leqq \sqrt{d_1'\theta_1 d_1}\,\sqrt{d_2'\theta_1 d_2} \leqq \frac{1}{2}(d_1'\theta_1 d_1+d_2'\theta_1 d_2)$$

with the strict inequality unless

$$d_1'\theta_1 d_1=d_2'\theta_1 d_2 \quad \text{and} \quad d_2=\lambda d_1 \qquad \text{for a positive scalar } \lambda . \tag{2.8}$$

The condition (2.8) contradicts at least one of the assumptions $\theta \in ri(\Theta)$ or $d_1 \neq d_2$. Now the desired result is evident.

In this way the problem of linear estimation is reduced to a statistical game (Θ, D, R) satisfying the assumptions (A1)-(A5) listed in the next section.

3. A necessary condition for admissibility in a statistical game with a strictly convex risk

Consider a statistical game (Θ, D, R), where Θ is the parameter set, D is the set of the decision rules and $R(\theta, d)$ is the risk of a rule $d \in D$ with respect to a parameter $\theta \in \Theta$. In a similar way as in Olsen, Seely and Birkes ([7], Proposition 3.3), we can prove

LEMMA 3.1. *Assume that D is a closed subset of $\boldsymbol{R}^n$, $R(\theta, \cdot)$ is a continuous function of $d \in D$ for all $\theta \in \Theta$ and there exists a $\theta_0 \in \Theta$ such that $\lim_{k\to\infty} R(\theta_0, d_k)=\infty$ for any sequence $\{d_k\}$ in D such that $\lim_{k\to\infty} \|d_k\|=\infty$. Then all admissible rules in the game (Θ, D, R) constitute a minimal complete class.*

In this section we shall assume that

(A1) Θ is a convex set in $\boldsymbol{R}^m$,

(A2) D is a closed convex set in $\boldsymbol{R}^n$,

(A3) $R(\cdot, d)$ is non-negative and continuous for all $d \in D$,

(A4) $R(\theta, \cdot)$ is convex and continuous for all $\theta \in \Theta$ and strictly convex for some $\theta = \theta_0$,

(A5) For any scalar $C > 0$ and for any $\theta \in ri(\Theta)$ there exists a constant $M = M(C, \theta)$ such that $R(\theta, d) > C$ providing $\|d\| > M$, $d \in D$.

A very general necessary condition for admissibility has been given by Farrell ([1], Theorem 3.7). This condition is expressed in terms of risk functions being relative to some prior distributions. However it will be more convenient for us to handle the original statistical rules than their risks. Theorem 3.1 is just an adaptation of the result by Farrell to purposes of linear estimation.

Denote by $r(\tau, d)$ the Bayes risk of a rule $d \in D$ with respect to a prior distribution τ on Θ, i.e.,

$$\tau(\tau, d) = \int_{\Theta} R(\theta, d) d\tau(\theta) .$$

A rule d_0 is called Bayes with respect to τ (or τ-Bayes, for short) if it minimizes $r(\tau, d)$ among all $d \in D$.

THEOREM 3.1. *Under the assumptions* (A1)-(A5), *for any admissible rule* $d_0 \in D$ *there exists a sequence* $\{\tau_k\}$ *of prior distributions on* Θ *and a corresponding sequence* $\{d_k\}$ *of decision rules such that*:

(i) *The expectation* E_{τ_k} *of* τ_k *exists and belongs to* $ri(\Theta)$, $k = 1, 2, \cdots$.

(ii) *The rule* d_k *is Bayes with respect to* τ_k, $k = 1, 2, \cdots$.

(iii) *The sequence* $\{d_k\}$ *is convergent in* $\boldsymbol{R}^n$ *and* $\lim_{k \to \infty} d_k = d_0$.

PROOF. Suppose a rule d_0 is admissible. Then, by (A3), it is also admissible in the game (Ω, D, R_0), where $\Omega = ri(\Theta)$ and R_0 is the restriction of R to the set $\Omega \times D$.

Denote by $\mathcal{R}$ the family of all risk functions corresponding to possible randomized rules in the game (Ω, D, R_0). To prove the theorem we shall use Farrell ([1], Theorem 3.7) with the set Ω and the family $\mathcal{R}$ defined as above and with the normative function $a(\theta) = 1$ for all $\theta \in \Omega$. At first we shall verify the assumptions of the Farrell's theorem.

It is clear that the set Ω is σ-compact. We shall show that the family $\mathcal{R}$ is weakly subcompact and sequentially weakly subcompact.

Given a directed index set A consider a sequence $\{f_a, a \in A\}$, where $f_a \in \mathcal{R}$. If $\limsup_{a \in A} f_a(\theta) = \infty$ for all $\theta \in \Omega$, then the condition $f(\theta) \leqq \limsup_{a \in A} f_a(\theta)$ is trivially satisfied by arbitrary $f \in \mathcal{R}$. Now otherwise, let

$$\limsup_{a \in A} f_a(\theta_0) < \infty \qquad \text{for some} \quad \theta_0 \in \Omega . \tag{3.1}$$

By essentially completeness of the nonrandomized rules among all rules in the game (Ω, D, R_0), (see e.g., Ferguson [2], p. 78), to each $a \in A$ corresponds a rule $d_a \in D$ such that

$$R_0(\theta, d_a) \leqq f_a(\theta) \qquad \text{for all} \quad \theta \in \Omega . \tag{3.2}$$

It follows from (3.1) and (3.2), via assumption (A5), that $\limsup_{a \in A} \|d_a\| < \infty$. Thus there exists a subsequence $\{d_b,\ b \in B\}$ of $\{d_a,\ a \in A\}$ such that $\lim_{b \to B} d_b = d$ for some $d \in D$. Therefore, by (3.2) and by continuity of $R_0(\theta, \cdot)$, we get

$$R_0(\theta, d) = \lim_{b \in B} R_0(\theta, d_b) \leqq \limsup_{a \in A} R_0(\theta, d_a) \leqq \limsup_{a \in A} f_a(\theta) .$$

Thus $\mathcal{R}$ is weakly subcompact.

Now let us consider a countable sequence $\{f_k\}$, where $f_k \in \mathcal{R}$, $k = 1, 2, \cdots$. The case when $\liminf_{k \to \infty} f_k(\theta) = \infty$ for all $\theta \in \Omega$ is trivial, so suppose that $\liminf_{k \to \infty} f_k(\theta_0) < \infty$ for some $\theta_0 \in \Omega$. Then, as above, there exists a subsequence $\{f_{k_r}\}$ of $\{f_k\}$ and a corresponding sequence $\{d_{k_r}\}$ in D such that $R_0(\theta, d_{k_r}) \leqq f_{k_r}(\theta)$ for all $\theta \in \Omega$ and $\lim_{r \to \infty} d_{k_r} = d$ for some $d \in D$. The rule d satisfies the condition $R_0(\theta, d) \leqq \liminf_{r \to \infty} f_{k_r}(\theta)$ for all $\theta \in \Omega$ and hence the family $\mathcal{R}$ is sequentially weakly subcompact. Thus the assumptions of the theorem by Farrell are met.

This theorem yields that there exists a sequence $\{\tau_k\}$ of priors on Ω supported on compact subsets F_k, $k = 1, 2, \cdots$, and a sequence $\{f_k\}$ in $\mathcal{R}$ such that f_k is Bayes with respect to τ_k and

$$\lim_{k \to \infty} f_k(\theta) = R_0(\theta, d_0) \qquad \text{for all} \quad \theta \in \Omega .$$

Thus, by essentially completeness of nonrandomized rules, there exists a sequence $\{d_k\}$ in D such that

$$\lim_{k \to \infty} R_0(\theta, d_k) \leqq R_0(\theta, d_0) . \tag{3.3}$$

By (A5) this sequence is bounded, and therefore, we can take a convergent subsequence. Without loss of generality assume that $\lim_{k \to \infty} d_k = \bar{d}$ for some $\bar{d} \in D$. Then by (3.3) and by continuity of $R(\cdot, d)$ we get $R(\theta, \bar{d}) = R(\theta, d)$ for all $\theta \in \Theta$.

Suppose $\bar{d} \neq d_0$. Then by (A4) the rule $d_1 = (d_0 + \bar{d})/2$ is better than d_0. This contradicts the assumption that d_0 is admissible. Now it remains to prove that the priors τ_k, $k = 1, 2, \cdots$, satisfy the condition (i), but it follows from the fact that each τ_k is supported on a subset F_k of $ri(\Theta)$. This completes the proof.

4. A complete class in linear estimation

Return to the problem of linear estimation of $c'\mu$ in a general linear model $\mathrm{E}\,Y=\mu$, $\mathrm{Cov}\,Y=V$, where $(\mu, V)\in\Omega$. In Section 2 this problem was reduced to a statistical game $(\boldsymbol{\Theta}, D, R)$ such that the assumptions (A1)–(A5) were satisfied. These assumptions may be completed by:

(A6) $R(\lambda\theta+(1-\lambda)\bar{\theta}, d)=\lambda R(\theta, d)+(1-\lambda)R(\bar{\theta}, d)$ for all $\theta, \bar{\theta}\in\boldsymbol{\Theta}$ and $\lambda\in[0, 1]$.

We shall say that an estimator d_0 is θ-best for some $\theta\in\boldsymbol{\Theta}$ if $R(\theta, d_0)\leqq R(\theta, d)$ for all $d\in D$. Similarly, we shall say that d_0 is *locally best* (LB) in $\boldsymbol{\Theta}_0$, where $\boldsymbol{\Theta}_0$ is a subset of $\boldsymbol{\Theta}$, if d_0 is θ-best for some $\theta\in\boldsymbol{\Theta}_0$. An estimator d_0 is said to be *unique locally best* (ULB) if it is θ-best for some $\theta\in\boldsymbol{\Theta}$ such that all θ-best estimators have the same risk.

It follows from the assumptions (A2), (A4) and (A5) that for any $\theta\in ri(\boldsymbol{\Theta})$ there exists a θ-best estimator and this estimator is ULB. For linear unbiased estimation this estimator may be presented explicitly (see e.g., Stępniak [10]). For linear biased estimation such explicite form is given in

PROPOSITION 4.1. *For linear estimation of $c'\mu$ in a linear model $\mathrm{E}\,Y=\mu$, $\mathrm{Cov}\,Y=V$, $(\mu, V)\in\Omega$, with the decision set $D=\boldsymbol{R}^n$:*

(a) *An estimator $d'Y$ is θ-best for some $\theta=(\theta_1, \theta_2)$ if and only if*

$$\theta_1 d=\theta_2 c\ . \tag{4.1}$$

(b) *For a given $\theta=(\theta_1, \theta_2)\in\Omega$, all θ-best estimators have the same risk if and only if*

$$\mathcal{R}(\theta_1)=\mathcal{R}(F_1)\ , \tag{4.2}$$

where F_1 is defined by (2.4).

(c) *An estimator $d'Y$ is θ-best for some $\theta=(\theta_1, \theta_2)$ satisfying the condition* (4.2) *if and only if*

$$d=F\begin{bmatrix} b_1 \\ \cdots \\ b_2 \end{bmatrix}\ ,$$

where $b_1=(F_1'\theta_1 F_1)^{-1}F_1'\theta_2 c$ and b_2 is an arbitrary vector of size $(n-k)\times 1$.

PROOF. Condition (a) was shown by LaMotte ([5] and [6]). We note that to each $d\in\boldsymbol{R}^n$ corresponds a unique vector $b=(b_1' \vdots b_2')'$, such that $d=Fb$. Let us put in (4.1) $d=Fb$. Then we get

$$\theta_1 F_1 b_1=\theta_2 c\ . \tag{4.3}$$

The equation (4.3) possesses a unique solution with respect to b_1 if and only if $\mathcal{R}(\theta_1)=\mathcal{R}(F_1)$, and if so, then this solution is just $b_1=(F_1'\theta_1F_1)^{-1}\cdot F_1'\theta_2c$. This implies both (c) and sufficiency of (4.2).

In order to show necessity of (4.2) suppose, by contradiction, that all θ-best estimators have the same risk but $\mathcal{R}(\theta_1)$ is a proper subset of $\mathcal{R}(F_1)$. Then there exists a vector x such that $\theta_1F_1x=0$ but $F_1x\neq 0$. Consider the estimators $d_0'Y$ and $d'Y$, where $d_0=F_1b_1$ and $d=F_1(b_1+x)$, while b_1 is a solution of (4.3). We note that $d, d_0\in\mathcal{R}(F_1)$, $d\neq d_0$ and they both are admissible. On the other hand, by (A4), the estimator $(d+d_0)/2$ is better than d and d_0 alike. This contradiction completes the proof.

Now we shall state the main result of this paper.

THEOREM 4.1. *For linear estimation of* $c'\mu$ *in a linear model* $\mathrm{E}\,Y=\mu$, $\mathrm{Cov}\,Y=V$, $(\mu, V)\in\Omega$, *under a decision set* D *and a convex set* Θ *such that the conditions* (2.3) *and* (2.6) *are valid*

(a) *Any admissible estimator d may be presented as a limit of* LB *estimators in* $ri(\Theta)$.

(b) *The class* C *of all such limits coincides with the minimal closed set containing all* ULB *estimators.*

PROOF. Consider a game (Θ, D, R), where Θ satisfies the condition (2.3), D satisfies the condition (2.6) and R is defined by (2.2). Then the assumptions (A1)–(A6) are valid. Thus, by Theorem 3.1, for the proof of the part (a) we only need to show that any linear estimator being Bayes with respect to a prior having a finite expectation is locally best. Really, by the assumption (A6), via Jensen's inequality, we get $r(\tau, d)=R(\mathrm{E}_\tau, d)$ for such a prior τ and for arbitrary $d\in D$. This implies the desired condition and completes the proof of (a).

For (b) we notice that the set C is included in the closure of all ULB estimators as any linear estimator being LB in $ri(\Theta)$ is ULB. On the other hand, any ULB estimator is a member of C because one is admissible. This implies the desired coincidence and completes the proof.

5. Examples

Example 1. Linear estimation in the model $\mathrm{E}\,Y=\beta 1_n$, $\mathrm{Cov}\,Y=\sigma I_n$, $\beta\in R$, $\sigma>0$.

Let us put $\Theta=\{\theta=(\theta_1, \theta_2)\colon \theta_1=xI_n+y1_n1_n', \ \theta_2=y1_n1_n', \ x>0, \ y\geqq 0\}$, where 1_n denotes the column of n ones. We note that θ_1 is always positive definite. Hence, by Proposition 4.1 (c), the class $\mathcal{U}$ of all ULB estimators for β, is

$$\mathcal{U}=\left\{d=\alpha 1_n:\ 0\leqq\alpha<\frac{1}{n}\right\}.$$

On the other hand, our parameter set Θ may be equivalently replaced by a two-element set $\Theta_0=\{(I_n, 0), (1_n1_n', 1_n1_n')\}$. Thus, by Theorem 2 in Stępniak [12] any limit of the admissible estimators is admissible, and, in a consequence, our complete class

$$C=\left\{d=\alpha 1_n:\ 0\leqq\alpha\leqq\frac{1}{n}\right\}$$

coincides with the minimal complete class.

Example 2. (Klonecki and Zontek [3], p. 49). Quadratic estimation of the variance component σ_1 in the normal linear model

$$\mathrm{E}\,Y=0\,,\qquad \mathrm{Cov}\,Y=\sigma_1 I_2+\sigma_2\begin{bmatrix}0&0\\0&1\end{bmatrix},\qquad \text{where}\quad \sigma_1>0,\ \sigma_2\geqq 0\,.$$

This problem reduces to linear estimation of σ_1 in the model

$$Z=\begin{bmatrix}Y_1^2\\Y_2^2\\Y_1Y_2\end{bmatrix},\qquad \mathrm{E}\,Z=\begin{bmatrix}\sigma_1\\\sigma_1+\sigma_2\\0\end{bmatrix},\qquad \mathrm{Cov}\,Z=2\begin{bmatrix}\sigma_1^2&0&0\\0&(\sigma_1+\sigma_2)^2&0\\0&0&0\end{bmatrix},$$

and next to the same problem in the model

$$U=\begin{bmatrix}Y_1^2\\Y_2^2\end{bmatrix},\qquad \mathrm{E}\,U=\begin{bmatrix}\sigma_1\\\sigma_1+\sigma_2\end{bmatrix},\qquad \mathrm{Cov}\,U=2\begin{bmatrix}\sigma_1^2&0\\0&(\sigma_1+\sigma_2)^2\end{bmatrix}.$$

Let Θ be the minimal closed convex set including

$$\Theta_0=\left\{\theta_\rho=(\theta_1,\theta_2):\ \theta_1=\begin{bmatrix}3&\rho+1\\\rho+1&3(\rho+1)^2\end{bmatrix},\ \theta_2=\begin{bmatrix}1&\rho+1\\\rho+1&(\rho+1)^2\end{bmatrix},\ \rho\geqq 0\right\}.$$

Then

$$\Theta=\left\{\theta_{t,u}=(\theta_1,\theta_2):\ \theta_1=\begin{bmatrix}3&t\\t&3(t^2+u)\end{bmatrix},\ \theta_2=\begin{bmatrix}1&t\\t&t^2+u\end{bmatrix},\ t\geqq 1,\ u\geqq 0\right\}.$$

By Proposition 4.1 (c) a linear form $d'Z$ is a ULB estimator of σ_1 if and only if

$$d=\begin{bmatrix}a(t,u)\\v\end{bmatrix},\qquad \text{where}\quad a(t,u)=\theta_1^{-1}\theta_2\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{8t^2+9u}\begin{bmatrix}2t^2+3u\\2t\end{bmatrix},$$

for some $t\geqq 1$, $u\geqq 0$ and $v\in R$. Hence in this case our complete class C is defined as the minimal closed set including

$$\mathcal{U}=\left\{d=d(t, u, v):\ d=\frac{1}{8t^2+9u}\begin{bmatrix}2t^2+3u\\ 2t\\ v\end{bmatrix},\ t\geqq 1,\ u\geqq 0,\ v\in R\right\}.$$

It appears that this complete class is not minimal. In particular

$$h=\lim_{t\to\infty} d(t, 0, 0)=\left(\frac{1}{4}, 0, 0\right)'$$

belongs to C but $h'Y=Y_1^2/4$ is inadmissible for σ_1 because it is dominated by $\hat{\sigma}_1=Y_1^2/3$.

Example 3. Linear estimation in the Gauss-Markov model. Consider linear estimation of the parameter μ_1 in the model

$$\mathrm{E}\,Y=\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix},\qquad \mathrm{Cov}\,Y=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$

where $\mu_1, \mu_2\in R$. It is known (see Rao [8] and Stępniak [11]) that the minimal complete class for this problem is defined by

$$C_0=\{d=(d_1, d_2)':\ 0\leqq d_1+d_2<1 \text{ or } d=(1, 0)'\}.$$

On the other hand the complete class C is defined by

$$C=\{d=(d_1, d_2)':\ 0\leqq d_1+d_2\leqq 1\}.$$

Thus C is not minimal complete.

Example 4. Linear unbiased estimation in one-way random model. Consider linear unbiased estimation of the parameter β in the model

$$\mathrm{E}\,Y=\beta 1_n,\qquad \mathrm{Cov}\,Y=\sigma_1 I_n+\sigma_2\,\mathrm{diag}\,(1_{n_1}1'_{n_1},\cdots, 1_{n_k}1'_{n_k}),$$

where $n=\sum_{i=1}^{k} n_i$, $\beta\in R$, $\sigma_1>0$ and $\sigma_2\geqq 0$. It was shown by Stępniak [12] that the class of all ULB estimators for this problem is defined by

$$\mathcal{U}=\left\{d_\rho:\ d_\rho=\left(\sum_{i=1}^{k}\frac{n_i}{1+n_i\rho}\right)^{-1}\left[\frac{1}{1+n_1\rho}1'_{n_1},\cdots,\frac{1}{1+n_k\rho}1'_{n_k}\right]',\ \rho\geqq 0\right\}.$$

Moreover, by Theorem 4 (c) in Stępniak [12] the limit

$$\bar{d}=\lim_{\rho\to\infty} d_\rho$$

is also admissible. Thus our complete class $C=\mathcal{U}\cup\{\bar{d}\}$ coincides with the minimal complete class.

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AGRICULTURAL UNIVERSITY OF LUBLIN

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