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ASYMPTOTIC PROPERTIES OF RAO'S TEST FOR TESTING HYPOTHESES IN DISCRETE PARAMETER STOCHASTIC PROCESSES

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Summary

In this note some asymptotically optimum tests for testing hypotheses concerning parameters when the observations are dependent are obtained. Test statistics based on the score functions, similar to the one proposed by Rao in the case when the observations are i.i.d. are proposed. Asymptotically UMP tests for one sided hypotheses against one sided alternatives and asymptotically UMP unbiased test for a simple hypothesis against two sided alternatives are derived. In the multiparameter case tests for simple hypotheses that have asymptotically best constant power on some family of surfaces in the parameter space are derived.

1. Introduction

Rao [11] proposed a test statistic for testing hypotheses concerning several parameters when the observations are independent and identically distributed. In the present note, some test statistics similar to the one proposed by Rao are proposed and some asymptotic optimum properties of these tests are established, by using the concepts of contiguity of probability measures (Lecam [9]), when the observations are from a discrete parameter stochastic process without any assumptions about stationarity or independence. A similar test statistic was proposed in Sarma [14], when the observations are from a discrete parameter stationary Markov process and some asymptotic properties were established. Roussas [13] has studied the asymptotic properties of a test statistic, obtained using the concepts of differentiability in quadratic mean and contiguity of probability measures, when the observations are from a discrete parameter stationary Markov process. Basawa

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and Prakasa Rao [2] has given a statistic analogous to the one proposed in [14], when the observations are from a discrete parameter stochastic process without assumptions of stationarity or independence.

In the next section the preliminary notations, assumptions and some basic results are presented. Section 2 consists of establishing some asymptotic properties of the test for testing a single parameter while Section 3 deals with some asymptotic properties of the tests in the multiparameter case. It may be noted that proofs of some of the results appear to be similar to those in [13] and hence, some details are omitted. However the statistics proposed here seem to be easier to compute, are developed in a more general set up and the proofs are considerably different. Some remarks about other statistics are made. An example, where these results can be applied, is given very briefly.

2. Preliminaries

Let $\{X^{(n)}\}=\{(X_1,\cdots,X_n)\}$ be a sequence of random vectors defined on a probability space $(\Omega, \mathcal{B}, P(\cdot:\theta))$ and taking values in measure spaces $\{\mathcal{X}^{(n)}, \mathcal{A}^{(n)}, \mu^{(n)}\}$, $n\geqq 1$, μ being a σ-finite measure on $\{\mathcal{X}, \mathcal{A}\}$. The basic probability measure $P(\cdot:\theta)$ involves an unknown parameter $\theta\in\Theta$. In what follows $\mathcal{X}$ is taken to be $R^{(m)}$ for some fixed $m\geqq 1$ and Θ is an open set in $R^{(k)}$ for some fixed $k\geqq 1$. Suppose $X^{(n)}$ has the density function $P(x^{(n)}:\boldsymbol{\theta})=p(x_1,\cdots,x_n:\boldsymbol{\theta})$ with respect to the σ-finite measure $\mu^{(n)}$. The functional form of $p(x^{(n)}:\boldsymbol{\theta})$ is assumed to be known for $n\geqq 1$, except for the parameter value $\boldsymbol{\theta}$. In what follows, problems of testing simple hypotheses concerning $\boldsymbol{\theta}$ are considered and some test statistics are proposed. In all that follows it is assumed that the support of $X^{(n)}$ is independent of $\boldsymbol{\theta}$ for all $n\geqq 1$. Further $p(x^{(n)}:\boldsymbol{\theta})$ is assumed to be jointly measurable $\mathcal{A}^{(n)}\times C$ for each $n\geqq 1$, where C is the Borel σ-field on R^k restricted to Θ. Denote $p(x^{(0)}:\boldsymbol{\theta})\equiv 1$ for all $\boldsymbol{\theta}$. The following assumptions are made:

(A1) The conditional probability density of $X^{(n)}$ given $X^{(n-1)}$, denoted by $p_n(\boldsymbol{\theta})$, exists for all $n\geqq 1$ and $\boldsymbol{\theta}\in\Theta$ as a regular conditional probability density.

(A2) Denote $l_n(\boldsymbol{\theta})=\log p_n(\boldsymbol{\theta})$ and assume that $l_n(\boldsymbol{\theta})$ is twice differentiable with respect to $\boldsymbol{\theta}$ for all $x^{(n)}[\mu^{(n)}]$ and all $n\geqq 1$. Denote by $\dot{\boldsymbol{l}}_n(\boldsymbol{\theta})$ the $(k\times 1)$ vector of first partial derivatives and by $\ddot{l}_n(\boldsymbol{\theta})$ the $(k\times k)$ matrix of second partial derivatives. Assume that $\ddot{l}_n(\boldsymbol{\theta})$ is continuous in $\boldsymbol{\theta}$ uniformly in $x^{(n)}$.

Further, it is assumed that $p(x^{(n)}:\boldsymbol{\theta})$ is twice differentiable with respect to $\boldsymbol{\theta}$ under the integral sign for all $n\geqq 1$. Let $\mathcal{F}_n$ denote the σ-field generated by $X^{(n)}$, $n\geqq 1$ and $\mathcal{F}_0$ denote the trivial σ-field. Let

$$Y_n'(\boldsymbol{\theta})=\left(\frac{\partial}{\partial\theta_1}\log p(x^{(n)}:\boldsymbol{\theta}),\cdots,\frac{\partial}{\partial\theta_k}\log p(x^{(n)}:\boldsymbol{\theta})\right)=\sum \dot{\boldsymbol{l}}_r'(\boldsymbol{\theta}) . \tag{2.1}$$

Let $\boldsymbol{t}$ be any arbitrary fixed vector and set

$$S_n(\boldsymbol{\theta})=\boldsymbol{t}'\boldsymbol{Y}_n(\boldsymbol{\theta}) , \qquad Z_n(\boldsymbol{\theta})=S_n(\boldsymbol{\theta})-S_{n-1}(\boldsymbol{\theta})=\boldsymbol{t}'\dot{\boldsymbol{l}}_n(\boldsymbol{\theta}) . \tag{2.2}$$

Then $Z_n(\boldsymbol{\theta})$ is a zero mean martingale. Let $B_n(\boldsymbol{\theta})=\mathrm{E}\,(\boldsymbol{Y}_n(\boldsymbol{\theta})\cdot \boldsymbol{Y}_n'(\boldsymbol{\theta}):\boldsymbol{\theta})$ be positive definite for all $\boldsymbol{\theta}$ and all $n\geqq 1$. Denote $s_n^2(\boldsymbol{\theta})=\mathrm{E}\,(S_n^2(\boldsymbol{\theta}):\boldsymbol{\theta})=\boldsymbol{t}'B_n(\boldsymbol{\theta})\boldsymbol{t}$.

(A3) Assume that

$$s_n^{-2}(\boldsymbol{\theta})\sum_1^n \mathrm{E}\,(Z_r^2(\boldsymbol{\theta})/\mathcal{F}_{r-1}:\boldsymbol{\theta})\rightarrow 1 \tag{2.3}$$

in $P(\,\cdot:\boldsymbol{\theta})$ probability and for any $\varepsilon>0$,

$$s_n^{-2}(\boldsymbol{\theta})\sum \mathrm{E}\,(Z_r^2(\boldsymbol{\theta})I\{|Z_r(\boldsymbol{\theta})|>\varepsilon s_n(\boldsymbol{\theta})\}:\boldsymbol{\theta})\rightarrow 0 , \tag{2.4}$$

where $I\{\cdot\}$ is the indicator function of the set in the brackets. The condition (2.4) can be replaced by a stronger but more easily verifiable Liapunov condition.

(A4) Suppose that there exists a monotonic increasing function $K(n)$ which tends to ∞ as n tends to ∞ such that

$$\lim_{n\rightarrow\infty} K^{-1}(n)B_n(\boldsymbol{\theta})=\Gamma(\boldsymbol{\theta}) , \tag{2.5}$$

where $\Gamma(\boldsymbol{\theta})$ is positive definite for all $\boldsymbol{\theta}$. Let $\Gamma_n(\boldsymbol{\theta})=\sum_1^n \mathrm{E}\,(\dot{\boldsymbol{l}}_r(\boldsymbol{\theta})\cdot\dot{\boldsymbol{l}}_r'(\boldsymbol{\theta})|\mathcal{F}_{r-1}:\boldsymbol{\theta})$. Then $B_n(\boldsymbol{\theta})=\mathrm{E}\,(\Gamma_n(\boldsymbol{\theta}):\boldsymbol{\theta})$. Assume that $K^{-1}(n)\Gamma_n(\boldsymbol{\theta})$ tends to $\Gamma(\boldsymbol{\theta})$ in $P(\,\cdot:\boldsymbol{\theta})$ probability. This would imply (2.3).

(A5) Let $\dot{Y}_n(\boldsymbol{\theta}_0,\boldsymbol{\theta})$ be the $(k\times k)$ matrix of the second derivatives of $\log p(x^{(n)}:\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ with rows possibly evaluated at different points on the line segment joining $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}$. Assume that $B_n^{-1}(\boldsymbol{\theta}_0)\cdot\dot{Y}_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0)$ tends to the identity matrix I_k in $P(\,\cdot:\theta_0)$ probability. Further, given any $\varepsilon>0$, let there exist a $\delta(\varepsilon)>0$ such that

$$P\{|\dot{Y}_n^{-1}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0)\dot{Y}_n(\boldsymbol{\theta}_0,\boldsymbol{\theta})-I_k|<\varepsilon:\boldsymbol{\theta}_0\}\rightarrow 1 , \tag{2.6}$$

when $|\boldsymbol{\theta}-\boldsymbol{\theta}_0|<\delta$.

The assumptions made here are similar to those made in [6]. Some discussion on these assumptions can be found there.

THEOREM 2.1. *Under the assumptions* (A1) *to* (A3)

$$\mathcal{L}\left(\frac{S_n(\boldsymbol{\theta})}{s_n(\boldsymbol{\theta})}:\boldsymbol{\theta}\right)\rightarrow N(0,1) . \tag{2.7}$$

Here and in what follows $\mathcal{L}(X)$ denotes the probability distribution of *the random variable X.*

This theorem is a direct consequence of the central limit theorem for martingales (see e.g., [2]) and the details are omitted. From this theorem one has the

COROLLARY 2.1.

$$(B_n^{-1/2}(\boldsymbol{\theta})Y_n(\boldsymbol{\theta}):\boldsymbol{\theta})\to N_k(0, I_k)\,. \tag{2.8}$$

Under the additional assumption (A4) one has

THEOREM 2.2.

$$\text{(i)}\quad \mathcal{L}(K_{(n)}^{-1/2}Y_n(\boldsymbol{\theta}):\boldsymbol{\theta})\to N_k(0, \Gamma(\theta))\,,$$

$$\text{(ii)}\quad \mathcal{L}(\Gamma_n^{-1/2}(\boldsymbol{\theta})Y_n(\boldsymbol{\theta}):\boldsymbol{\theta})\to N_k(0, I_k)\,.$$

PROOF. These results are immediate consequences of Theorem 2.1 and assumption (A4).

COROLLARY 2.2. *Let*

$$Q_n^{(1)}(\boldsymbol{\theta})=Y_n'(\boldsymbol{\theta})B_n^{-1}(\boldsymbol{\theta})Y_n(\boldsymbol{\theta})\,,$$

$$Q_n^{(2)}(\boldsymbol{\theta})=K^{-1}(n)Y_n'(\boldsymbol{\theta})\Gamma^{-1}(\boldsymbol{\theta})Y_n(\boldsymbol{\theta})\,,\quad \textit{and}$$

$$Q_n^{(3)}(\boldsymbol{\theta})=Y_n'(\boldsymbol{\theta})\Gamma_n^{-1}(\boldsymbol{\theta})Y_n(\boldsymbol{\theta})\,.$$

Then

$$\mathcal{L}(Q_n^{(i)}(\boldsymbol{\theta}):\boldsymbol{\theta})\to\chi^2(k)\qquad \textit{for}\qquad i=1, 2, 3\,.$$

Slightly more generally, if A is any matrix such that $A\Gamma(\boldsymbol{\theta})A=A$ and $Q_n^{(4)}(\boldsymbol{\theta})=K^{-1}(n)Y_n'(\boldsymbol{\theta})AY_n(\boldsymbol{\theta})$, then $\mathcal{L}(Q_n^{(4)}(\boldsymbol{\theta}):\boldsymbol{\theta})\to\chi^2$ (Rank A).

This corollary is a consequence of Theorem 2.2 and the results on the asymptotic distributions of quadratic forms (Rao [12]).

For problems of testing a simple hypothesis $H_0:\boldsymbol{\theta}=\boldsymbol{\theta}_0$, one can use any of the $Q_n^{(i)}(\boldsymbol{\theta}_0)$ as a test statistic as each of them is asymptotically distributed as a χ^2 variable under the hypothesis H_0. In order to investigate the asymptotic power properties of the tests proposed above, one needs to obtain their asymptotic distributions under the alternative hypotheses.

Consider a sequence $\{\boldsymbol{\theta}_n\}$ where $\boldsymbol{\theta}_n=\boldsymbol{\theta}_0+K^{-1/2}(n)\boldsymbol{\delta}_n$ where $\boldsymbol{\delta}_n$ tends to $\boldsymbol{\delta}$, a vector of constants, as n tends to ∞. Then it is known that if $P_n(\cdot:\boldsymbol{\theta})$ denotes the restriction of $P(\cdot:\boldsymbol{\theta})$ to $\mathcal{F}_n$, then $P_n(\cdot:\boldsymbol{\theta}_0)$ and $P_n(\cdot:\boldsymbol{\theta}_n)$ are contiguous (Roussas [13], Chapter II, Proposition 6.1).

THEOREM 2.3.

$$\mathcal{L}(K(n)^{-1/2}\boldsymbol{Y}_n(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)\to N_k(\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta},\,\Gamma(\theta_0)) \tag{2.9}$$

$$\mathcal{L}(B_n^{-1/2}(\boldsymbol{\theta}_0)\boldsymbol{Y}_n(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)\to N_k(\Gamma^{1/2}(\boldsymbol{\theta}_0)\boldsymbol{\delta},\,I_k)\quad \textit{and} \tag{2.10}$$

$$\mathcal{L}(\Gamma_n^{-1/2}(\boldsymbol{\theta}_0)\boldsymbol{Y}_n(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)\to N_k(\Gamma^{1/2}(\boldsymbol{\theta}_0)\boldsymbol{\delta},\,I_k)\,. \tag{2.11}$$

PROOF. (2.9) is a consequence of Theorem 7.2 and Corollary 7.3 of Chapter I in [13].

(2.10) follows from (2.9) and the fact that $K(n)^{-1}\cdot B_n(\boldsymbol{\theta}_0)$ tends to $\Gamma(\boldsymbol{\theta}_0)$ as n tends to ∞. Finally, (2.11) follows from (2.9) and Slutsky's Theorem.

As an immediate consequence of this theorem one has

COROLLARY 2.3.

$$\mathcal{L}(Q_n^{(i)}(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)\to\chi^2\left(k,\,\frac{1}{2}\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}\right)\qquad \textit{for}\quad i=1,2,3\qquad \textit{and}$$

$$\mathcal{L}(Q_n^{(4)}(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)\to\chi^2\left(\textit{Rank}\,(A),\,\frac{1}{2}\boldsymbol{\delta}'A\boldsymbol{\delta}\right)$$

where $\chi^2(m, c)$ *denotes the noncentral* χ^2 *distribution on* m *degrees of freedom and noncentrality parameter* c.

Consider the log likelihood ratio

$$\Lambda_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n)=\log\frac{p(x^{(n)}:\boldsymbol{\theta}_n)}{p(x^{(n)}:\boldsymbol{\theta}_0)}\,.$$

Expanding this in Taylor's series around $\boldsymbol{\theta}_0$, one has

$$\Lambda_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n)=K(n)^{-1/2}\boldsymbol{\delta}_n'\boldsymbol{Y}_n(\boldsymbol{\theta}_0)+\frac{K(n)^{-1}}{2}\boldsymbol{\delta}_n'\dot{\boldsymbol{Y}}_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n)\boldsymbol{\delta}_n'\,. \tag{2.12}$$

From the assumptions, one can conclude that the second term on the right hand side of (2.12) converges in $P(\cdot:\boldsymbol{\theta}_0)$ probability to $-\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}/2$. Further, $K(n)^{-1/2}(\boldsymbol{\delta}_n-\boldsymbol{\delta})'\boldsymbol{Y}_n(\boldsymbol{\theta}_0)$ converges to 0 in $P(\cdot:\boldsymbol{\theta}_0)$ probability. Thus one has $\Lambda_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n)-K(n)^{-1/2}\boldsymbol{\delta}'\boldsymbol{Y}_n(\boldsymbol{\theta}_0)$ converges to $-\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}/2$ in $P(\cdot:\boldsymbol{\theta}_0)$ probability. Hence one has

THEOREM 2.4.

$$\text{(i)}\quad \mathcal{L}(\Lambda_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n):\boldsymbol{\theta}_0)\to N\left(-\frac{1}{2}\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta},\,\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}\right),$$

$$\text{(ii)}\quad \mathcal{L}(\Lambda_n(\boldsymbol{\theta}_0,\boldsymbol{\theta}_n):\boldsymbol{\theta}_n)\to N\left(\frac{1}{2}\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta},\,\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}\right).$$

To prove (ii), *it is enough to notice that, since* $P(\cdot:\boldsymbol{\theta}_0)$ *and* $P(\cdot:\boldsymbol{\theta}_n)$

are contiguous, it follows that $\Lambda_n(\boldsymbol{\theta}_0, \boldsymbol{\theta}_n) - K(n)^{-1/2}\boldsymbol{\delta}'\boldsymbol{Y}_n(\boldsymbol{\theta}_0)$ *converges to* $-\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}/2$ *in* $P(\cdot : \boldsymbol{\theta}_n)$ *probability. Consequently, from Theorem* 2.2 *above, Lemma* 7.1 *and Corollary* 7.2 *in Chapter* I *of* [13], *one has the stated result.*

Finally, the following proposition proved in Hájek [8] is used in the sequel.

PROPOSITION 2.1. *Let* $\{\Delta_n\}$ *be a sequence of random vectors such that* $\mathcal{L}(\Delta_n) \to N(0, \Sigma)$. *Then, there exists a truncated version* Δ_n^* *of* Δ_n *such that* $|\Delta_n - \Delta_n^*|$ *converges to* 0 *in probability and* $\sup_{|\boldsymbol{h}|<c} \left| \mathrm{E}\left\{\exp\left(\boldsymbol{h}'\Delta_n^* - \frac{1}{2}\boldsymbol{h}'\Sigma\boldsymbol{h}\right)\right\} - 1\right|$ *tends to* 0 *as* n *tends to* ∞ *for any fixed* $c>0$.

In what follows let $\Delta_n(\boldsymbol{\theta}_0) = B_n^{-1/2}(\boldsymbol{\theta}_0)\boldsymbol{Y}_n(\boldsymbol{\theta}_0)$ and $\exp(C_n(\boldsymbol{\delta})) = \mathrm{E}\{\exp(\boldsymbol{\delta}' \cdot \Delta_n^*(\boldsymbol{\theta}_0)) : \boldsymbol{\theta}_0\}$, where $\Delta_n^*(\boldsymbol{\theta}_0)$ is a suitable truncation of $\Delta_n(\boldsymbol{\theta}_0)$ satisfying the conditions of the above proposition and $\boldsymbol{\delta} \in R^{(k)}$. Define the probability measures

$$R_{n,\boldsymbol{\delta}}(A) = \exp(-C_n(\boldsymbol{\delta})) \int_A \exp(\boldsymbol{\delta}'\Delta_n^*(\boldsymbol{\theta}_0)) dP_{n,\boldsymbol{\theta}_0} . \tag{2.13}$$

Then it can be shown that

$$\|P_n(\cdot : \boldsymbol{\theta}_n) - R_{n,\boldsymbol{\delta}}\| \to 0 , \tag{2.14}$$

where $\boldsymbol{\delta}$ is a bounded sequence and $\boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + K(n)^{-1/2}\boldsymbol{\delta}$. In fact, one can show that $\sup|\|P_n(\cdot : \boldsymbol{\theta}_n) - R_{n,\boldsymbol{\delta}}\| : \boldsymbol{\delta} \in B,\ \boldsymbol{\theta}_n = \theta_n + K(n)^{-1/2}\boldsymbol{\delta}| \to 0$, where B is a bounded subset of $R^{(k)}$ (see Roussas [13]).

3. Asymptotic properties of tests in one parameter case

In this section the parameter space is taken to be $\Theta \in R$, i.e., $k=1$; $B_n(\theta)$, $\Gamma_n(\theta)$ and $\Gamma(\theta)$ are denoted more conveniently by $b_n^2(\theta)$, $\sigma_n^2(\theta)$ and $\sigma^2(\theta)$, respectively. The following additional assumptions are made.

(A6) (1) For any sequence θ_n converging to θ, $\mathrm{E}(\dot{l}_m(\theta_n) : \theta) \to 0$ uniformly in m as n tends to ∞ and conversely.

(2) $\mathrm{E}(\dot{l}_m(\theta_1) : \theta)$ is a continuous function of θ and θ_1 for all $m \geqq 1$.

(3) $\mathrm{E}(\dot{l}_m^2(\theta_1) : \theta)$ is a bounded continuous function of θ and θ_1 for all $m \geqq 1$.

(4) $K(n) = O(n^{\frac{1}{2}+\varepsilon})$ for some $0<\varepsilon<1/2$.

Under the above assumptions, it can be shown in the first place that, given any $\delta>0$,

$$P\{K(n)^{-1/2}|Y_n(\theta^*)| > \delta : \theta\} \to 1 \quad \text{as} \quad n \to \infty \tag{3.1}$$

for any θ and θ^* such that $|\theta-\theta^*|\geqq\beta>0$, for an arbitrary constant β. It also follows that

$$P\{b_n^{-1}(\theta^*)|Y_n(\theta^*)|>\delta:\theta\}\rightarrow 1 \quad \text{as} \quad n\rightarrow\infty . \tag{3.2}$$

THEOREM 3.1. *For testing the hypothesis $H_0: \theta=\theta_0$ against $H_1: \theta>\theta_0$, consider the tests given by*

$$\varphi_n(X^{(n)})=\begin{cases} 1 & \textit{if} \quad \Delta_n(\theta_0)>C_n \\ \lambda_n & \textit{if} \quad \Delta_n(\theta_0)=C_n \\ 0 & \textit{otherwise} \end{cases} \tag{3.3}$$

or

$$\varphi_n^{(1)}(X^{(n)})=\begin{cases} 1 & \textit{if} \quad K(n)^{-1/2}Y_n(\theta_0)>C_n^{(1)} \\ \lambda_n^{(1)} & \textit{if} \quad K(n)^{-1/2}Y_n(\theta_0)=C_n^{(1)} \\ 0 & \textit{otherwise} \end{cases} \tag{3.4}$$

where C_n, λ_n or $C_n^{(1)}$, $\lambda_n^{(1)}$ are determined for each n such that $\mathrm{E}(\varphi_n:\theta_0)=\mathrm{E}(\varphi_n^{(1)}:\theta_0)=\alpha$, the preassigned size of the test. Then $\{\varphi_n\}$ or $\{\varphi_n^{(1)}\}$ are asymptotically uniformly most powerful for testing H_0 against H_1.

PROOF. At the outset, it may be noted that $\{\varphi_n\}$ and $\{\varphi_n^{(1)}\}$ are asymptotically equivalent. So the proof will be given for $\{\varphi_n\}$ and the same goes through with obvious modifications for $\{\varphi_n^{(1)}\}$ also.

Suppose $\{\varphi_n\}$ is not asymptotically uniformly most powerful. Then there exists a sequence of tests, say $\{\psi_n\}$ of size α, such that

$$\limsup\left[\sup_{\theta>\theta_0}\mathrm{E}(\psi_n:\theta)-\mathrm{E}(\varphi_n:\theta)\right]=\varepsilon>0 . \tag{3.5}$$

Thus there exists a subsequence $\{m\}$ of $\{n\}$ and a sequence $\{\theta_m\}$ of parameter values $\theta_m>\theta_0$ such that

$$\lim_{m\rightarrow\infty}\{\mathrm{E}(\psi_m:\theta_m)-\mathrm{E}(\varphi_m:\theta_m)\}=\varepsilon . \tag{3.6}$$

Let $\theta_m=\theta_0+K(m)^{-1/2}\delta_m$.

Case 1. Suppose $\{\delta_m\}$ is bounded. Without loss of generality let $\delta_m\rightarrow\delta\geqq 0$. Otherwise, a subsequence for which the limit exists can be considered and the arguments be made for this subsequence. By Theorem 2.3, $\mathcal{L}(\Delta_n(\theta_0):\theta_n)\rightarrow N(\sigma(\theta_0)\delta, 1)$ and consequently

$$\mathrm{E}(\varphi_n:\theta_n)\rightarrow 1-\Phi(Z_\alpha-\delta\sigma(\theta_0)) \quad \text{where} \quad \Phi(Z_\alpha)=\int_{-\infty}^{Z_\alpha}dN(0,1)=\alpha .$$

Consider the sequence of tests given by

$$
(3.7) \qquad W_n(X^{(n)}) = \begin{cases} 1 & \text{if} \quad \Lambda(\theta_0, \theta_n) > C_n^* \\ \lambda_n^* & \text{if} \quad \Lambda(\theta_0, \theta_n) = C_n^* \\ 0 & \text{otherwise} \end{cases}
$$

where C_n^* and λ_n^* are determined such that $\mathrm{E}(W_n: \theta_0) = \alpha \, \forall \, n \geqq 1$. It is known from Theorem 2.4 that $\mathrm{E}(W_n: \theta_n) \to 1 - \Phi \, (Z_\alpha - \delta\sigma(\theta_0))$.
(a) Suppose $\delta > 0$. Since $\mathrm{E}(W_n: \theta_n) - \mathrm{E}(\varphi_n: \theta_n) \to 0$, $\mathrm{E}(\psi_n: \theta_n) - \mathrm{E}(W_n: \theta_n) \to \varepsilon > 0$ or $\mathrm{E}(W_n: \theta_n) < \mathrm{E}(\psi_n: \theta_n) - \frac{1}{2}\varepsilon$ for all $n > n_1(\varepsilon)$ for a suitable $n_1(\varepsilon)$.
However W_n is a most powerful test for testing $\theta = \theta_0$ against $\theta = \theta_n$ for any n and this leads to a contradiction.
(b) Suppose $\delta = 0$. It follows that $\Lambda(\theta_0, \theta_n) \to 0$ in $P(\cdot : \theta_n)$ probability and in this case $\mathrm{E}(\varphi_n: \theta_n) \to \alpha$ and $\mathrm{E}(W_n: \theta_n) \to \alpha$, as $n \to \infty$ which leads to a contradiction.

Case 2. Suppose δ_m is unbounded. Without loss of generality let $\delta_m \uparrow \infty$, as $m \to \infty$.
(a) Suppose $K(m)^{-1/2}\delta_m \geqq \beta > 0$ for some β and sufficiently large m. Then, $P\{\Delta_m(\theta_0) > C: \theta_m\} \to 1$, as $m \to \infty$ for any $C > 0$. However this contradicts (3.6).
(b) Suppose $\delta_m K(m)^{-1/2} \to 0$, as $m \to \infty$. This also leads to a contradiction as in Case 1-(b).

Remark. By exactly similar arguments, the corresponding result for testing $H_0: \theta = \theta_0$ against $H_1: \theta < \theta_0$ can be obtained.

In the case where the alternative is $H_1: \theta \neq \theta_0$, asymptotically uniformly most powerful unbiased tests are obtained in Theorem 3.2 below. Before giving the details, it may be pointed out that the following results can be obtained as in [13] and hence their details are omitted.

PROPOSITION 3.1. *Let $\{Z_n\}$ be a sequence of random variables such that $|Z_n| \leqq 1$ for all $n \geqq 1$ and let $\bar{Z}_n(\theta_0) = \mathrm{E}(Z_n | \Delta_n(\theta_0): \theta_0)$ a.s. Then* $\sup |\mathrm{E}(Z_n: \theta_n) - \mathrm{E}(\bar{Z}_n: \theta_n)| \to 0$ *where $\theta_n = \theta_0 + K(n)^{-1/2}h$ and for each n the supremum is taken over all random variables Z_n such that $|Z_n| \leqq 1$ and over all h in any bounded subset B of R.*

PROPOSITION 3.2. *Let $\{Z_n\}$ be any sequence of test functions. Let $\Delta_n^*(\theta_0)$ be a suitable truncation of $\Delta_n(\theta_0)$ as described in Proposition* 2.1 *and let $\theta_n = \theta_0 + K(n)^{-1/2}h$. Then for any bounded subset B of R*

$$
\sup_{h \in B} |\mathrm{E}(Z_n(\Delta_n(\theta_0)): \theta_n) - \mathrm{E}(Z_n(\Delta_n^*(\theta_0)): \theta_n)| \to 0 \, , \quad as \quad n \to \infty \, .
$$

Define a sequence of tests

$$\varphi_n(\Delta_n(\theta_0))=\begin{cases} 1 & \textit{if} \quad \Delta_n(\theta_0)<C_{1n} \textit{ or } >C_{2n} \\ \lambda_{in} & \textit{if} \quad \Delta_n(\theta_0)=C_{in}\,, \quad i=1,2\,, \\ 0 & \textit{otherwise} \end{cases} \tag{3.8}$$

where C_{in} *and* λ_{in}, $i=1,2$ *are so chosen that* $\mathrm{E}\,(\varphi_n:\theta_0)=\alpha$ *and* $\mathrm{E}\,(\varphi_n(\theta_0)\cdot\Delta_n(\theta_0))=\alpha\,\mathrm{E}\,(\Delta_n(\theta_0):\theta_0)$.

THEOREM 3.2. *The sequence of tests* $\{\varphi_n\}$ *defined by* (3.8) *is asymptotically uniformly most powerful unbiased of size* α *for testing* $H_0:\theta=\theta_0$ *against* $H_1:\theta\neq\theta_0$.

PROOF. At the outset it may be observed that $C_{1n}\to -Z_{\alpha/2}$ and $C_{2n}\to Z_{\alpha/2}$ where $\int_{Z_{\alpha/2}}^{\infty} dN(0,1)=\frac{\alpha}{2}$. It can also be seen that $\{\varphi_n\}$ is asymptotically unbiased. This can be done by a contradiction argument. Suppose $\liminf_{n\to\infty}\left\{\inf_{\theta}\mathrm{E}\,(\varphi_n:\theta)\right\}=\eta<\alpha$. There exists a sequence $\{\theta_m\}$ such that $\mathrm{E}\,(\varphi_m:\theta_m)\to\eta$, as $m\to\infty$. Let $\theta_m=\theta_0+K(m)^{-1/2}\cdot\delta_m$. Suppose δ_m is unbounded and without loss of generality δ_m increases to ∞ (or decreases to $-\infty$). Then it can be seen that $\mathrm{E}\,(\varphi_m:\theta_m)\to 1$, as $m\to\infty$, which is a contradiction. Suppose $\{\delta_m\}$ is bounded and without loss of generality $\delta_m\to\delta$, as $m\to\infty$. Then $P_m(\,\cdot:\theta_0)$ and $P_m(\,\cdot:\theta_m)$ are contiguous so that $\mathcal{L}(\Delta_m(\theta_0):\theta_m)\to N(\delta\sigma(\theta_0),1)$ by Theorem 2.3. If $\delta=0$, then $\mathrm{E}\,(\varphi_m:\theta_m)\to\alpha$ which is a contradiction. If $\delta\neq 0$, $\mathrm{E}\,(\varphi_m:\theta_m)$ tends to a quantity larger than α which is again a contradiction.

Let $\{W_n\}$ be a sequence of asymptotically unbiased tests of size α. It will be shown that

$$\limsup_{n\to\infty}\left\{\sup_{\theta}\left[\mathrm{E}\,(W_n:\theta)-\mathrm{E}\,(\varphi_n:\theta)\right]\right\}\leqq 0\,. \tag{3.9}$$

This is also done by a contradiction argument. Suppose there exists a sequence $\{\theta_m\}$ such that

$$\lim_{m\to\infty}\left[\mathrm{E}\,(W_m:\theta_m)-\mathrm{E}\,(\varphi_m:\theta_m)\right]=\varepsilon>0\,. \tag{3.10}$$

Let $\theta_m=\theta_0+\delta_m K(m)^{-1/2}$. (a) Let δ_m be unbounded. Without loss of generality let $\delta_m\to\pm\infty$. Then $\mathrm{E}\,(\varphi_m:\theta_m)\to 1$, as $m\to\infty$ which contradicts (3.10). (b) Suppose δ_m is bounded and $\delta_m\to\delta$, as $m\to\infty$. Now $P_m(\,\cdot:\theta_0)$ and $P_m(\,\cdot:\theta_m)$ are contiguous. By Proposition 3.1, the sequence of tests $\{W_n\}$ can be replaced by $\bar{\bar{W}}(\Delta_n(\theta_0))$ where $\bar{\bar{W}}(\Delta_n(\theta_0))=\mathrm{E}\,(W_n\,|\,\Delta_n(\theta_0):\theta_0)$ by noting that $\{\Delta_n(\theta_0)\}$ is asymptotically sufficient for $\{P_n(\,\cdot:\theta),\ \theta\in\Theta\}$ at θ_0. Hence for convenience in notation, it is assumed that W_n itself is based on $\Delta_n(\theta_0)$. Moreover one can determine $\Delta_n^*(\theta_0)$ such that $\Delta_m^*(\theta_0)-\Delta_m(\theta_0)\to 0$ in $P_m(\,\cdot:\theta_m)$ probability, $|\mathrm{E}\,(W_m:\theta_m)-\mathrm{E}\,(W_m(\Delta_m^*):\theta_m)|\to 0$, and $|\mathrm{E}\,(\varphi_m(\Delta_m):\theta_m)-\mathrm{E}\,(\varphi_m(\Delta_m^*):\theta_m)|\to 0$, by contiguity.

Consider the one parameter exponential family of measures R_{m,δ_m} given by (2.13). For each m, consider the problem of testing $H_0: \delta_m = 0$ against the alternative $H_1: \delta_m \neq 0$ at level $\mathrm{E}(W_m(\Delta_m^*): \theta_0) = \alpha_m$ for this exponential family. The UMPUT of size α_m is given by

$$\bar{\varphi}_m^* = \begin{cases} 1 & \text{if} \quad \Delta_m^* < C_{1m}^* \text{ or } > C_{2m}^* \\ \lambda_{im}^* & \text{if} \quad \Delta_m^* = C_{im}^*, \quad i=1,2, \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

where the constants are determined such that

$$\mathrm{E}(\bar{\varphi}_m^*: \delta_m = 0) = \alpha_m \quad \text{and} \quad \mathrm{E}(\Delta_m^* \bar{\varphi}_m^*: \delta_m = 0) = \alpha_m \mathrm{E}(\Delta_m^*: \delta_m = 0) .$$

It may be noted that $C_{1m}^* \to -Z_{\alpha/2}$ and $C_{2m}^* \to Z_{\alpha/2}$. Further

$$\begin{aligned} \mathrm{E}(W_m: \theta_m) - \mathrm{E}(\varphi_m: \theta_m) = & [\mathrm{E}(W_m: \theta_m) - \mathrm{E}(W_m(\Delta_m^*): \theta_m)] \\ & + [\mathrm{E}(W_m(\Delta_m^*): \theta_m) - \mathrm{E}(W_m(\Delta_m^*): R_{m,\delta_m})] \\ & + [\mathrm{E}(W_m(\Delta_m^*): R_{m,\delta_m}) - \mathrm{E}(\bar{\varphi}_m^*: R_{m,\delta_m})] \\ & + [\mathrm{E}(\bar{\varphi}_m^*: R_{m,\delta_m}) - \mathrm{E}(\bar{\varphi}_m^*: \theta_m)] \\ & + [\mathrm{E}(\bar{\varphi}_m^*: \theta_m) - \mathrm{E}(\varphi_m(\Delta_m^*): \theta_m)] \\ & + [\mathrm{E}(\varphi_m(\Delta_m^*): \theta_m) - \mathrm{E}(\varphi_m: \theta_m)] . \end{aligned} \tag{3.12}$$

All the terms except $\mathrm{E}(W_m(\Delta_m^*): R_{m,\delta_m}) - \mathrm{E}(\bar{\varphi}_m^*: R_{m,\delta_m})$, on the right hand side of (3.12) tends to 0 and hence one has that $\mathrm{E}(W_m(\Delta_m^*): R_{m,\delta_m}) - \mathrm{E}(\bar{\varphi}_m^*: R_{m,\delta_m}) \to \varepsilon > 0$, as $m \to \infty$ which is a contradiction. This proves the stated result.

Remark 1. The exponential approximation used above can be used to obtain asymptotically UMP Tests for

(a) $H_0: \theta \leqq \theta_0$ against $H_1: \theta > \theta_0$,

(b) $H_0: \theta \geqq \theta_0$ against $H_1: \theta < \theta_0$.

Remark 2. In the above discussion sequences of tests of size α for each n are considered. However tests of asymptotic size α can also be considered as in [13] as the determination of the constants is much simpler and it can be shown that these tests will also be AUMP or AUMU in the class of sequences of tests of asymptotic size α.

Remark 3. In view of Remark 2, since it is not practical to determine the constants defining the tests above for each n and since only asymptotic properties of tests are considered, the constant C_n in (3.3) can be replaced by Z_α and C_{1n} and C_{2n} in (3.11) can be replaced by $-Z_{\alpha/2}$ and $Z_{\alpha/2}$ respectively, with the sign of equality incorporated in the defining inequalities there by omitting λ's. The resulting tests will be asymptotically equivalent to the tests considered in (3.3) and (3.11).

Remark 4. The test statistics used in the above results are based on $\Delta_n(\theta_0)$. Same results can also be proved taking test statistics based on $K(n)^{-1/2}Y_n(\theta_0)$ and $\sigma(\theta_0)^{-1}Y_n(\theta_0)$.

4. Multiparameter case

In this section it is assumed that $\boldsymbol{\theta}'=(\theta_1,\cdots,\theta_k)$, $k>1$. In order to test the simple hypothesis $H_0: \boldsymbol{\theta}=\boldsymbol{\theta}_0$ any of the $Q_n^{(i)}(\boldsymbol{\theta}_0)$'s defined in Corollary 2.2 can be used as a test statistic. Some asymptotic properties of the test using $Q_n^{(1)}(\boldsymbol{\theta}_0)$ are obtained below. As remarked above, one can restrict the attention to tests with asymptotic level of significance $0<\alpha<1$. The sequence of tests can be shown to be consistent against non local alternatives under the assumptions of the type made in the one parameter case. Thus in what follows sequences of local alternatives are considered.

Consider the sequence of measures $\{R_{n,\boldsymbol{\delta}}\}$ as defined in (2.13). Since $P_{n,\boldsymbol{\theta}_0}=R_{n,0}$, one has $\boldsymbol{\theta}=\boldsymbol{\theta}_0$ iff $\boldsymbol{\delta}=\mathbf{0}$. For any sequence of bounded functions $\{\psi_n\}$,

$$\sup_{\boldsymbol{\delta}\in B}[\mathrm{E}(\psi_n(\Delta_n^*(\boldsymbol{\theta}_0)):\boldsymbol{\theta}_n)-\mathrm{E}(\psi_n(\Delta_n^*(\boldsymbol{\theta}_0)):R_{n,\boldsymbol{\delta}})] \quad \text{and}$$
$$\sup_{\boldsymbol{\delta}\in B}[\mathrm{E}(\psi_n(\Delta_n^*(\boldsymbol{\theta}_0)):\boldsymbol{\theta}_n)-\mathrm{E}(\psi_n(\Delta_n(\boldsymbol{\theta}_0)):\boldsymbol{\theta}_n)] \tag{4.1}$$

both tend to zero as n tends to ∞ for any bounded subset $B\in R^k$ and $\boldsymbol{\theta}_n=\boldsymbol{\theta}_0+\boldsymbol{\delta}K(n)^{-1/2}$. Thus the problem of testing $H_0: \boldsymbol{\theta}=\boldsymbol{\theta}_0$ can be replaced by the equivalent problem of testing $H_0: \boldsymbol{\delta}=\mathbf{0}$ in the family $R_{n,\boldsymbol{\delta}}$ and the tests can be based on $\Delta_n^*(\boldsymbol{\theta}_0)$. From the earlier discussion, it can be seen that each of $\mathcal{L}(\Delta_n(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)$, $\mathcal{L}(\Delta_n^*(\boldsymbol{\theta}_0):\boldsymbol{\theta}_n)$ and $\mathcal{L}(\Delta_n^*(\boldsymbol{\theta}_0):R_{n,\boldsymbol{\delta}})$, denoted by $\mathcal{L}_{n,\boldsymbol{\theta}_n}$, $\mathcal{L}^*_{n,\boldsymbol{\theta}_n}$ and $\mathcal{L}^{**}_{n,\boldsymbol{\delta}}$ respectively, converges to $N_k(\Gamma^{1/2}(\boldsymbol{\theta}_0)\boldsymbol{\delta}, I_k)$, denoted by $\mathcal{L}_{\boldsymbol{\delta}}$. Let $p_n(\boldsymbol{Z};\boldsymbol{\delta})=\exp(\boldsymbol{\delta}'\boldsymbol{Z}-C_n(\boldsymbol{\delta}))$. Consider the sequence of regions

$$W_n^*=\{\Delta_n^*(\boldsymbol{\theta}_0):\Delta_n^*(\boldsymbol{\theta}_0)'\Delta_n^*(\boldsymbol{\theta}_0)>d_n\}\,, \tag{4.2}$$

where the constants d_n are so chosen that $P(W_n^*:R_{n,0})\to\alpha$, as $n\to\infty$. Consider the family of concentric circles

$$S_{n,C}:(\boldsymbol{\theta}-\boldsymbol{\theta}_0)'(\boldsymbol{\theta}-\boldsymbol{\theta}_0)=K(n)^{-1}\cdot C \qquad \text{for} \qquad C>0\,. \tag{4.3}$$

For sufficiently large n, $S_{n,C}\subset\Theta$. It can be seen that for any n, there exists a constant $a_n(C)$ such that

$$(4.4) \qquad M(Z)=\int_{S_{n,C}} p_n(Z:\delta)dA \geqq a_n(C) \quad \text{for} \quad Z\in W_n^* \quad \text{and}$$
$$< a_n(C) \quad \text{for} \quad Z \notin W_n^*,$$

where the integral is the surface integral over $S_{n,C}$. This is done by showing that $M(Z)$ is a function depending on Z only through $r_Z=(\Sigma Z_i^2)^{1/2}$ and then showing that M is a monotonic increasing function of r_Z.

From this discussion it can be concluded that $\varphi_n^*=I(W_n^*)$, the indicator function of W_n^*, is a most powerful test of size $\alpha_n=\mathrm{E}(\varphi_n^*:0)$ for testing the hypothesis H_0 that the p.d.f. is $p_n(Z:0)$ against the alternative H_1 that it is const. $\int_{S_{n,C}} p_n(Z,\delta)dA$.

The following proposition is proved in [13].

PROPOSITION 4.1. *Let T be an open set in R^k and V a k-dimensional random vector with probability measure $Q(\cdot:\xi)$ which is absolutely continuous with respect to $Q(\cdot:\xi_0)$ for some fixed $\xi_0\in T$ and $dQ(\cdot:\xi)/dQ(\cdot:\xi_0)=C(\xi)\exp(\xi'\Sigma v)$, where Σ is a symmetric positive definite matrix. For testing the hypothesis $\xi=\xi_0$, the class $\mathcal{D}$ of tests of the form $\varphi(v)=1$ if $v\in D^c$ and 0 if $v\in D^0$ (interior of D) and is defined in an arbitrarily measurable way on the boundary of D, where D is a closed convex set of R^k, is essentially complete.*

In view of the earlier discussion and Proposition 4.1, it suffices to consider tests belonging to the class $\mathcal{D}$ for the problem under consideration.

THEOREM 4.1. *For testing the hypothesis $H_0:\theta=\theta_0$, consider the sequence of tests given by*

$$(4.5) \qquad \varphi_n(\Delta_n(\theta_0))=I(W_n),$$

where $W_n=\{X^{(n)}:Q_n^{(1)}(\theta_0)\geqq d_n\}$, d_n being chosen such that $\mathrm{E}(\varphi_n:\theta_0)\to\alpha$, as $n\to\infty$. Consider the family of surfaces

$$(4.6) \qquad S_{n,C}^*:(\theta-\theta_0)'\Gamma(\theta_0)(\theta-\theta_0)=K(n)^{-1}C,$$

where $C>0$ is a constant. The surfaces $S_{n,C}^*\subset\Theta$ for sufficiently large n. The sequences of tests $\{\varphi_n\}$ defined by (4.5) has asymptotically best constant power (Wald [16]) on the family of surfaces $S_{n,C}^*$, $C\in B$ where B is any bounded set.

PROOF. Consider $W=\{Z:Z'\Gamma(\theta_0)Z\geqq d\}$ where Z is $N_k(\delta,\Gamma^{-1}(\theta_0))$ and $P(W:\delta=0)=\alpha$. Then it is known that $P(W:\delta)$ is constant for any δ on the surface $S^{**}(C):\delta'\Gamma(\theta_0)\delta=C$ and $C>0$ is arbitrary. This is immediate since $\boldsymbol{Z}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{Z}$ has the noncentral χ^2 with k degrees of freedom and noncentrality parameter $h(\boldsymbol{\delta})=\boldsymbol{\delta}'\Gamma(\boldsymbol{\theta}_0)\boldsymbol{\delta}$. Let $\boldsymbol{\delta}\in S^{**}(C)$ and $\boldsymbol{\theta}_n=\boldsymbol{\theta}_0+K(n)^{-1/2}\boldsymbol{\delta}$ so that $\boldsymbol{\theta}_n\in S^*_{n,C}$. Thus $\theta_n\in S^*_{n,C}\Longleftrightarrow\boldsymbol{\delta}\in S^{**}(C)$. Let W^*_n be as defined by (4.1) and $\varphi^*_n=I(W^*_n)$. Then

$$\mathrm{E}\,(\varphi^*_n: R_{n,\boldsymbol{\delta}})=P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\ .$$

Thus in view of (4.1), it suffices to show that the sequence $\{\varphi^*_n\}$ has asymptotically best constant power on the surface $S^{**}(C)$ for $C\in B$. Since $\mathcal{L}^{**}_{n,\boldsymbol{\delta}}\to\mathcal{L}_{\boldsymbol{\delta}}$, the asymptotic distribution of $\Delta^{*\prime}_n(\boldsymbol{\theta}_0)\Delta^*_n(\boldsymbol{\theta}_0)$ will be noncentral $\chi^2(k, h(\boldsymbol{\delta}))$. Thus, $P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})$ is asymptotically constant for $\boldsymbol{\delta}\in S^{**}(C)$. To be more precise, suffices to show that

$$\text{(4.7)}\qquad \lim_{n\to\infty}\left\{\sup_{\boldsymbol{\delta}\in S^{**}(C)} P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-\inf_{\boldsymbol{\delta}\in S^{**}(C)} P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\right\}=0\ .$$

This is seen by noting that

$$\left|\sup_{\boldsymbol{\delta}\in S^{**}(C)} P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-\inf_{\boldsymbol{\delta}\in S^{**}(C)} P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\right|$$
$$\leqq 2\sup_{\boldsymbol{\delta}\in S^{**}(C)}|P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-P(W^*_n: \mathcal{L}_{\boldsymbol{\delta}})|\ ,$$

and the right hand side expression goes to zero, as $n\to\infty$. Now it will be shown that for any sequence of tests $\{\psi_n\}$ for which (1) $\mathrm{E}\,(\psi_n: \boldsymbol{\theta}_0)\to\alpha$, as $n\to\infty$, (2) $\lim\left[\sup_{C\in B}\left\{\sup_{\boldsymbol{\theta}\in S^*_{n,C}}\mathrm{E}\,(\psi_n:\boldsymbol{\theta})-\inf_{\boldsymbol{\theta}\in S^*_{n,C}}\mathrm{E}\,(\psi_n:\boldsymbol{\theta})\right\}\right]=0$, one has

$$\text{(4.8)}\qquad \liminf\left[\inf_{C\in B}\left\{\inf_{\boldsymbol{\theta}\in S^*_{n,C}}[\mathrm{E}\,(\varphi_n:\boldsymbol{\theta})-\mathrm{E}\,(\psi_n:\boldsymbol{\theta})]\right\}\right]\geqq 0\ .$$

In view of Proposition 4.1, it is enough to consider $\psi_n=I(V_n)$ where V_n^C is a convex set in R^k. In view of (4.1) it is sufficient to prove that

$$\text{(4.9)}\qquad \liminf\left[\inf_{\boldsymbol{\delta}\in S^{**}(C)}\{P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-P(V_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\}\right]\geqq 0\ ,$$

for all $C\in B$. Consider

$$\liminf\left[\sup_{\boldsymbol{\delta}\in S^{**}(C)}\{P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-P(V_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\}\right]$$

with $C\in B$ and suppose it is $\varepsilon<0$. Then for sufficiently large n and all $\boldsymbol{\delta}\in S^{**}(C)$

$$P(W^*_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})-P(V_n: \mathcal{L}^{**}_{n,\boldsymbol{\delta}})\leqq\frac{\varepsilon}{2}<0\ .$$

Thus

$$\int_{S^{**}(C)} P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})dA - \int_{S^{**}(C)} P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})dA < 0 \tag{4.10}$$

or

$$\int_{W_n^*}\int_{S^{**}(C)} \exp(\boldsymbol{\delta}'\Delta_n^*(\boldsymbol{\theta}_0) - C(\boldsymbol{\delta}))dA dP_{n,0}$$
$$-\int_{V_n}\int_{S^{**}(C)} \exp(\boldsymbol{\delta}'\Delta_n^*(\boldsymbol{\theta}_0) - C(\boldsymbol{\delta}))dA dP_{n,0} < 0\,,$$

which contradicts (4.4) for a suitable choice of C. Finally

$$\begin{aligned}
&\inf_{\boldsymbol{\delta}}\,[P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\quad \geqq \inf_{\boldsymbol{\delta}} P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \sup_{\boldsymbol{\delta}} P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) \\
&\quad = [\sup P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \inf P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\qquad -[\sup P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \inf P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\qquad -[\sup P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \inf P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\quad \geqq \sup\,[P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\qquad -[\sup P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \inf P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})] \\
&\qquad -[\sup P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - \inf P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})]
\end{aligned}$$

and consequently

$$\liminf\left[\inf_{\boldsymbol{\delta}\in S^{**}(C)} P(W_n^*: \mathcal{L}_{n,\boldsymbol{\delta}}^{**}) - P(V_n: \mathcal{L}_{n,\boldsymbol{\delta}}^{**})\right] \geqq 0\,, \tag{4.11}$$

which proves the stated result.

Remark 1. As was done in [13] or in [16] it can be shown that the sequence of tests given by (4.5) has asymptotically best average power on the surfaces given by (4.6) under suitable weignt function. Theorem 4.1 can also be derived from this result. It can also be shown that the sequence of tests is asymptotically most stringent in the sense of Wald [16].

Remark 2. It can be shown that the sequence of tests based on $Q_n^{(2)}(\boldsymbol{\theta}_0)$ also have the same asymptotic properties as $Q_n^{(1)}(\boldsymbol{\theta}_0)$. It may also be seen that $Q_n^{(4)}(\boldsymbol{\theta}_0)$ has similar asymptotic properties. The statistics $Q_n^{(2)}(\boldsymbol{\theta}_0)$ in the case of i.i.d. observations were considered in [11]. Finally, the likelihood ratio statistic or Wald's type statistic ([16]) can also be considered as the maximum likelihood estimators exist as roots of corresponding equations under suitable regularity conditions and are asymptotically normal (see Sarma [14]).

Remark 3. The sequences of tests considered above and mentioned in Remark 2 above, can be shown to be consistent against non-local alternatives under assumptions of the type made in the one parameter case.

Remark 4. Since all the tests mentioned above have the same asymptotic properties, it is desirable to examine the asymptotic properties of the tests further by considering their rates of convergence or probabilities of large deviations and Bahadur efficiencies, at least in some special cases on the lines of [7]. These aspects are being investigated and will be reported separately.

Remark 5. Tests for some composite hypotheses are considered and their asymptotic properties are derived in the above set up and they will be reported separately.

Remark 6. In [3] the authors have considered the case where the log likelihood function converges to a mixture of normal distributions under local alternatives and have derived some properties of minimax tests.

Application. The above procedure can be applied to tests of hypotheses for exponential families of stochastic processes. In this case the likelihood function will be of the form

$$p(x^{(n)}:\boldsymbol{\theta})=a_n(x^{(n)})\exp\left\{\sum_{i=1}^{k}\theta_i A_{i,n}(x^{(n)})-\sum_{i=1}^{l}f_i(\boldsymbol{\theta})B_{i,n}(x^{(n)})\right\}$$

where $\boldsymbol{\theta}\in\Theta\subset R^k$ and $f_i(\boldsymbol{\theta})$ are twice differentiable functions from Θ to R. An account of such processes indicating special cases can be found in [15] and in references given there. Markov process in discrete time, some birth and death processes, possibly with immigration can have likelihood function of this type.

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