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A NOTE ON TESTING TWO-DIMENSIONAL NORMAL MEAN

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Summary

For the problem of testing a composite hypothesis with one-sided alternatives of the mean vector of a two-dimensional normal distribution, a characterization of similar tests is presented and an unbiased test dominating the likelihood ratio test is proposed. A sufficient condition for admissibility is given, which implies the result given by Cohen et al. (1983, *Studies in Econometrics, Time Series and Multivariate Statistics*, Academic Press): the admissibility of the likelihood ratio test.

1. Introduction

In the present paper we treat the problem of testing the null hypothesis H_0: $\min(\mu, \nu)=0$, against the alternative H_1: $\min(\mu, \nu)>0$, where (μ, ν) is a mean vector of two-dimensional normal random vector (X, Y) with the identity covariance matrix. The likelihood ratio test (LRT) derived by Inada [6] and Sasabuchi [10] rejects H_0 if $\min(X, Y)>z_\alpha$ where z_α is the upper $\alpha\times 100\%$ point of the standard normal distribution.

Cohen et al. [3] have proposed a test for the hypothesis that the difference between two marginal probabilities is zero in the $2\times 2\times 2$ contingency table with conditional independence. Their problem is reduced to ours when the sample size is large.

In Section 2, we will present unbiased tests. For this purpose we first give a necessary and sufficient condition for a test to be similar. Then we show that any non-randomized tests whose rejection region lies in one side of a line cannot be similar, and that any non-trivial monotone test cannot be similar. We also give a sufficient condition for a test to be unbiased for testing H_0 vs. H_1. A test proposed by Lehmann [7] is an example of unbiased tests in our testing problem. Note that, treating a hypothesis which is slightly different from ours,

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he showed that there exists no non-trivial unbiased test.

On the other hand it may be seen that Lehmann's test is also an example of tests whose power functions uniformly dominate the power function of the LRT with the same level. Analogous phenomena are reported by several authors, Berger [1], Berger et al. [2], and Warrack et al. [11]. We will not go further on this problem in this paper, but it will be interesting to compare this with the results of Cohen et al. [4] that the LRT is admissible and uniformly most powerful among monotone tests, which is defined in Section 2.

In Section 3, we will present a class of admissible tests containing the LRT. A class of inadmissible tests is also given. In Section 4, we will relax the assumption of normality and introduce exponential type distributions. The results in Sections 2 and 3 are extended for these distributions.

2. Unbiased tests

Denote by ψ a test function for testing H_0: $\min(\mu, \nu)=0$ against H_1: $\min(\mu, \nu)>0$. The expectation of ψ is denoted by $\mathrm{E}_{(\mu,\nu)}(\psi)$, and the conditional expectation of ψ given $X=x$ is denoted by $\mathrm{E}_{(\mu,\nu)}(\psi\mid X=x)$. The standard normal density function is denoted by ϕ.

At first we determine the class of similar tests since unbiased tests are necessarily similar in our problem.

THEOREM 2.1. *A test ψ with size α is similar if and only if $\mathrm{E}_{(0,0)}(\psi\mid X=x)=\alpha$ a.e. x and $\mathrm{E}_{(0,0)}(\psi\mid Y=y)=\alpha$ a.e. y.*

PROOF. If ψ is similar, then

$$\iint \psi(x, y)\phi(x-\mu)\phi(y)dxdy=\alpha \qquad \text{for all} \quad \mu>0\,,$$

and the completeness of X (cf. Lehmann [8], p. 132) implies

$$\int \psi(x, y)\phi(y)dy=\alpha \qquad \text{for almost all} \quad x\,.$$

Thus the conditional expectation of ψ given $X=x$ is α almost everywhere. The same is true given $Y=y$. The converse is obvious.

COROLLARY 2.1. *If the rejection region denoted by $[(x, y): \psi(x, y)>0]$ lies in one side of a straight line, then the test ψ is not similar.*

PROOF. Assume, without loss of generality, the straight line is not parallel to the X-axis. For $(\mu, \nu)=(0, 0)$ the conditional expectation given $X=x$ tends to 0 when x goes to $+\infty$ or $-\infty$, and hence it cannot be constant for $-\infty<x<\infty$. Hence cannot be similar from Theorem 2.1.

A test ψ is said to be monotone if $x \leqq x'$ and $y \leqq y'$ imply $\psi(x, y) \leqq \psi(x', y')$.

COROLLARY 2.2. *If a test ψ is both monotone and similar, then its test function $\psi(x, y)$ is constant.*

PROOF. For fixed y and y' s.t. $y < y'$, we have from the similarity

$$\int (\psi(x, y') - \psi(x, y))\phi(x)dx = 0 .$$

As the integrand is nonnegative, $\psi(x, y) = \psi(x, y')$ for almost all x. From the monotonicity there exist f and g such that $f(x) = \lim_{y \to -\infty} \psi(x, y)$ and $g(x) = \lim_{y \to +\infty} \psi(x, y)$, and $f(x) \leqq \psi(x, y) \leqq g(x)$ for all x and y. Thus there is a set N of measure zero such that $f(x) = g(x) = \psi(x, y)$ for all $x \notin N$ and for all y. By symmetry it now follows that ψ must be constant.

A real valued function $f(x, y)$ on $R \times R$ is said to be Schur-concave if $x + y = x' + y'$ and $\max(x, y) > \max(x', y')$ imply $f(x, y) \leqq f(x', y')$.

THEOREM 2.2. *A test is unbiased size α if it is both similar and Schur-concave.*

PROOF. The power function f of ψ is

$$f(\mu, \nu) = \int\int \psi(x, y)\phi(x - \mu)\phi(y - \nu)dxdy .$$

Then f becomes Schur-concave (cf. Marshall et al. [9], p. 296) so that $f(\mu, \nu) \geqq f(\mu + \nu, 0)$ for $\mu, \nu > 0$. Each $f(\mu + \nu, 0)$ is also equal to the size of ψ since ψ is similar. These complete the proof.

Since the constant test is always unbiased in any testing problem, hence there is at least one unbiased test. Lehmann [7] (cf. also problem 7 of Chapter 4 in Lehmann [8]) discussed another testing problem where the only unbiased test is constant. For $-\infty \leqq \varepsilon < 0$, let us consider the testing problem $H_0(\varepsilon)$: $\varepsilon < \min(\mu, \nu) \leqq 0$ against H_1, the same alternative as before. Note that $H_0(-\infty)$ corresponds to the alternative hypothesis of Lehmann's problem and H_1 to the null. By a similar argument in Lehmann [7], it is easy to show that there is no non-trivial unbiased test for the testing problem $H_0(\varepsilon)$ vs. H_1 with $-\infty \leqq \varepsilon < 0$. Though our original null hypothesis H_0 is the limit of $H_0(\varepsilon)$ as ε tends to zero, we can give an example of non-trivial unbiased tests by means of the Theorem 2.2. This is essentially due to Lehmann [7] who, of course, thought it similar but not unbiased in his problem.

$$\phi(x, y)=\begin{cases} 1 & \text{if} \quad z_{(i/m)}<x,\, y<z_{((i+1)/m)} \ (i=0, \cdots, m=1) \\ 0 & \text{otherwise} \end{cases}$$

where $z_{(i/m)}$ is the lower $(i/m)\times 100\%$ point of the standard normal distribution. This test ϕ has the size $\alpha=1/m$ and is similar. It is also Schur-concave, and therefore it is unbiased. In this example the size α is limited to $\alpha=1/m$, $m=2, 3, \cdots$ but it is easily extended to arbitrary value of α by the randomization.

Consider the above test for $m=20$, the rejection region is divided into 20 subsets, and the last one is of the form $\min(x, y)>z_{(19/20)}$. This is nothing but the rejection region of the LRT of size 0.05. As the rejection region of the above test is larger than that of the LRT, it is uniformly more powerful than the LRT.

3. Admissibility

The following theorem gives a class of admissible tests which contains the LRT. The admissibility of the LRT has been proved by Cohen et al. [4]. Note that our proof is simpler than their proof.

THEOREM 3.1. *Any non-randomized test ϕ whose rejection region is of the form $A\times B$ or $(A\times B)^c$ is admissible, where A and B are measurable sets in R.*

PROOF. Let ϕ have the rejection region of the form $A\times B$, and let ϕ' be better than ϕ. From the continuity of the power function, we have

$$E_{(\mu,0)}\,\phi=E_{(\mu,0)}\,\phi' \qquad \text{for} \quad \mu>0\,,$$

and

$$E_{(0,\nu)}\,\phi=E_{(0,\nu)}\,\phi' \qquad \text{for} \quad \nu>0\,.$$

Putting $f=\phi'-\phi$, similarly as in the proof of Theorem 2.1, we have

$$\int f(x, y)\phi(x)dx=0 \qquad \text{a.e.} \quad y\,,$$

and

$$\int f(x, y)\phi(y)dy=0 \qquad \text{a.e.} \quad x\,.$$

On the other hand, f is nonnegative on the sets $A^c\times R$ and $R\times B^c$ so that $f=0$ a.e. on these sets. In the same manner we also have $f=0$

a.e. on $A \times B$. Therefore ψ' is equivalent to ψ. It completes the proof in the case that rejection region is of the form $A \times B$. The proof for the other case is clear.

Since the rejection region of the LRT is of the form $A \times B$, we have

COROLLARY 3.1. *The LRT is admissible.*

It would be interesting to note that tests with unreasonable rejection regions may be admissible. For example the test with size α given by

$$\psi(x, y) = \begin{cases} 1 & \text{if } x < z \text{ and } y < z \\ 0 & \text{otherwise} \end{cases}$$

seems most unreasonable for testing H_0 against H_1, where z is the lower $\alpha^{1/2} \times 100\%$ point of the standard normal distribution. However this test belongs to the class of admissible tests in Theorem 3.1.

Farrell [5] presented an example of testing problem, due to L. D. Brown, where all tests are admissible. Next proposition is motivated by this example.

PROPOSITION 3.1. *Let ψ be any test and $A = [(x, y): x > 0, y > 0, \psi(x, y) < 1, \psi(-x, y) > 0, \psi(x, -y) > 0, \psi(-x, -y) < 1]$. If the Lebesgue measure of A is positive, ψ is not admissible.*

PROOF. Consider the following function

$$f(x, y) = \operatorname{sgn}(x) \operatorname{sgn}(y) \min [\psi(-x, y), \psi(x, -y), 1 - \psi(x, y), 1 - \psi(-x, -y)],$$

where sgn is the sign function. Simple calculations lead us to that the value of $\phi(x-\mu)\phi(y-\nu) + \phi(-x-\mu)\phi(-y-\nu) - \phi(x-\mu)\phi(y-\nu) - \phi(x-\mu) \cdot \phi(-y-\nu)$ is positive when $x, y, \mu, \nu > 0$ and zero when $\mu = 0$ or $\nu = 0$. From this, if the Lebesgue measure of A is positive,

$$E_{(\mu, \nu)} f = \int\int f(x, y) \phi(x-\mu) \phi(y-\nu) dx dy$$

is positive when $\mu > 0$, $\nu > 0$ and zero when $\mu = 0$ or $\nu = 0$. Clearly $\psi + f$ is between 0 and 1 so that it is a test dominating ψ. It completes the proof.

It is clear that any non-randomized test whose rejection region is contained in the second or the forth quadrant and is symmetric around the origin is inadmissible.

4. Extension to exponential type distribution

A random variable X is said to be exponential type distributed if it possesses a density function, for some finite measure τ, of the following type

$$f_\mu(x)=c(\mu)\exp(xd(\mu)),$$

where $c(\cdot)$, $d(\cdot)$ are real functions and satisfy

$$c(\mu)=\left(\int \exp(xd(\mu))d\tau\right)^{-1}.$$

Assume d is monotone increasing and continuous, which implies the continuity of the inverse function d^{-1}.

Let X and Y be independently distributed in the exponential type distribution with parameters μ and ν, respectively. Then our problem is generally formalized as testing H_0: $\min(\mu, \nu)=b$ against H_1: $\min(\mu, \nu)>b$ for some b. Since $d^{-1}((b, \infty))$ is open, i.e. containing an interval, it is clear from the completeness for exponential type distribution (cf. Lehmann [8], p. 132) that Theorems 2.1 and 3.1 hold true. Since the exponential distribution satisfies the monotone likelihood ratio property (cf. Lehmann [8], p. 68), the non-randomized LRT test is of the form $[a, \infty)\times[a, \infty)$ for some a, that is, the LRT test is admissible.

Theorem 2.2 does not generally hold true, because we use, in the proof of this theorem, the property that the simultaneous density function of (X, Y) is parametrized to preserve Schur-convexity (cf. Marshall et al. [9], p. 296). But, after reparametrization of natural parameter if necessary, some distributions are known to satisfy the property, e.g. Poisson, gamma. Therefore, under these distributions, Theorem 2.2 is valid, and hence the same test as in Section 2 is unbiased.

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