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A NOTE ON SOME TEST STATISTICS AGAINST HNBUE

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Summary

Some comments and extensions are given to the paper on statistics for testing exponentiality against the HNBUE property $\left(\text{i.e. } \int_t^\infty \bar{F}(x)dx \leqq \mu \exp(-t/\mu),\ t \geqq 0\right)$ by Basu and Ebrahimi (1985, *Ann. Inst. Statist. Math.*, **37**, 347–359).

1. Introduction and comments

A life distribution F with survival function $\bar{F}=1-F$ and finite mean $\mu=\int_0^\infty \bar{F}(x)dx$ is said to be *harmonic new better (worse) than used in expectation*, HNBUE (HNWUE) if $\int_t^\infty \bar{F}(x)dx \leqq (\geqq) \mu \exp(-t/\mu)$, $t \geqq 0$. Basu and Ebrahimi [2] discuss some test statistics based on a complete ordered sample $0=t(0)\leqq t(1)\leqq t(2)\leqq \cdots \leqq t(n)$ from a life distribution F for testing

$$H_0\colon F \text{ is exponential (i.e. } \bar{F}(t)=\exp(-\lambda t),\ t\geqq 0)$$

against

$$H_1\colon F \text{ is HNBUE (but not exponential)}$$

or

$$H_2\colon F \text{ is HNWUE (but not exponential)}\,.$$

For further notations and details see Basu and Ebrahimi [2] and for properties of HNBUE (HNWUE) life distributions see Klefsjö [9].

The authors present two test statistics denoted by A_n and B_n. The statistic A_n is scale dependent and rather complicated in different ways

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and will not be discussed here. The statistic

$$B_n=\sum_{j=1}^{n-1} \ln(1-S_j)+n ,$$

where S_j is the scaled total time on test (TTT-) statistic at time $t(j)$, is simpler and independent of scale and accordingly more interesting. In fact B_n is a linear function of the test statistic

$$\Delta_3=-2\sum_{j=1}^{n-1} \ln(1-S_j)$$

which was discussed by Klefsjö [8] in a partly unpublished report. Also Kochar and Deshpande [13] suggested a linear function of B_n, namely $\left(\sum_{j=1}^{n-1} \ln(1-S_j)\right)\Big/n$, as a test statistic for testing exponentiality against HNBUE (but they mainly discuss the consistency problem).

These test statistics are closely related to the test statistic $-2\cdot \sum_{j=1}^{n-1} \ln S_j$ (discussed e.g. by Epstein [6]), and widely used as an omnibus test for exponentiality.

One reason for studying B_n, or equivalently Δ_3, is that it is rather easy to get the percentiles under H_0. Since $(S_1, S_2, \cdots, S_{n-1})$ has the same distribution as an ordered sample from a uniform distribution on $[0, 1]$ (see e.g., Barlow and Proschan [1]), it follows that Δ_3 is distributed according to a $\chi^2(2(n-1))$-distribution. Accordingly we can, for larger values of n, use that $\sqrt{2\chi^2(k)}-\sqrt{2k-1}$ is asymptotically $N(0, 1)$-distributed (see e.g., Johnson and Kotz [7], p. 176), which is a better approximation than the usual approximation that $\{\chi^2(k)-k\}/\sqrt{2k}$ is asymptotically $N(0, 1)$-distributed.

Let $\chi^2_\alpha(k)$ be defined by $P(X>\chi^2_\alpha(k))=\alpha$, where X is $\chi^2(k)$-distributed. The critical values x_α of B_n, where $P(B_n>x_\alpha)=\alpha$, can then be obtained as

$$x_\alpha=\frac{1}{2}(2n-\chi^2_{1-\alpha}(2(n-1))) .$$

They seem not to be aware of this judging from Table 1 in Basu and Ebrahimi [2] on percentiles of B_n based on numerical integration.

Basu and Ebrahimi [2] present in Table 2 power estimates for $n=20$ against some different Weibull and gamma alternatives for A_n, B_n and two test statistics A_2 and B presented in detail in Klefsjö [11] and intended for testing exponentiality against increasing failure rate (IFR) and increasing failure rate in average (IFRA), respectively. The paper does not refer if they use percentiles from an asymptotic normal distribution or not and the power estimates presented seem to be inappropriate. In Klefsjö [8] the same power values (among others) were

Table 1. Power estimates for B_n, Δ_3 and B, all based on 2000 replications.

	B_n Basu and Ebrahimi [2] $n=20$	Δ_3 Klefsjö [8] $n=20$	Δ_3 Bergman and Klefsjö recently $n=20$	Δ_3 Klefsjö [8] $n=10$	B Basu and Ebrahimi [2] $n=20$	B Klefsjö [11] $n=20$
$\bar{F}(x)=\exp(-x^{1.5})$	0.68	0.46	0.47	0.24	0.59	0.62
$\bar{F}(x)=\exp(-x^{2.0})$	0.97	0.90	0.92	0.53	0.85	0.97
$\bar{F}(x)=\exp(-x^{0.8})$	0.59	0.29	0.24	0.19	0.43	0.32
$\bar{F}(x)=\int_x^\infty te^{-t}dt$	0.75	0.38	0.38	0.22	0.72	0.65

estimated for Δ_3 using χ^2-percentiles with quite different results. Also the case $n=10$ was studied by Klefsjö [8]. To be sure we recently made a new power simulation. All these power estimates can be found in Table 1. In that table also the power estimates for B presented in Basu and Ebrahimi [2] and Klefsjö [11] are included. The simulations in Klefsjö [11] are based on the exact percentiles which differ from the asymptotic percentiles from a normal distribution. The power estimates by Basu and Ebrahimi [2] and Klefsjö [8], [10] all differ more than 0.03 which is alarming since two independent power estimates based on 2000 replications should be within ± 0.03 with high probability.

Some deviations can be seen in the results by Basu and Ebrahimi [2] in the light of the asymptotic Pitman efficacy values for B_n (or Δ_3) and B; see Table 2.

Table 2. Asymptotic Pitman efficacy values of B_n and B.

	B_n (or Δ_3)	B
Weibull	1.00	1.25
Gamma	0.25	0.50

Basu and Ebrahimi [2] mention the scaled TTT-transform as a basis for their idea to get the test statistics A_n and B_n. It is indeed interesting to analyse B_n, or equivalently Δ_3, together with scaled TTT-transforms of different life distributions.

It is apparent that B_n and Δ_3 are sensitive to values of S_j which are near to one. Looking at the scaled TTT-transforms in Figure 1 it is rather obvious that B_n and Δ_3 are more sensitive to Weibull distributions with shape parameter $\beta>1$ than to Weibull distributions with $\beta<1$. Also in this light the power estimate 0.59 for B_n when $\beta=0.8$ does not seem to be proper; see Table 1.

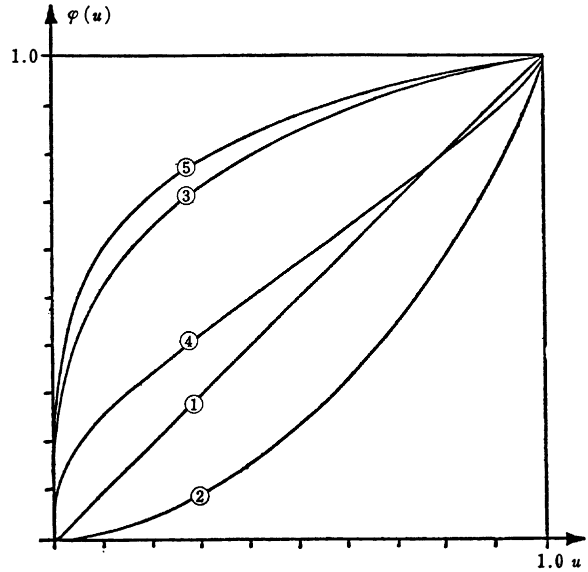

Fig. 1. Scaled TTT-transforms from five different life distributions: 1. Exponential; 2. Weibull $\beta=0.6$; 3. Weibull $\beta=3.0$; 4. Lognormal, i.e. ${}^{e}\log Y$ normal with $\mu=0$, $\sigma=0.8$; 5. Normal with $\mu=1$, $\sigma=0.3$ restricted to be positive (from Bergman [4]).

2. Some other test statistics

We here also want to mention that the test statistics presented by Basu and Ebrahimi [2] are not the only existing statistics for testing exponentiality against HNBUE. Klefsjö [12] studied two families of test statistics intended for the same problem. In Bergman and Klefsjö [5] these families were extended to the situation when the sample is right censored. Two of the statistics discussed in these papers are

$$Q_1=\sum_{j=1}^{n}\left(-\frac{1}{3}+3\left(1-\frac{j}{n}\right)^2\right)t(j)/S_n$$

and

$$Q_2=\sum_{j=1}^{n}\left(\frac{11}{6}-3\left(\frac{j}{n}\right)^2\right)t(j)/S_n\,.$$

It is of interest to compare the power estimates in Table 1 and the asymptotic Pitman efficacy values in Table 2 with the corresponding values for Q_1 and Q_2. These values can be found in Tables 3 and 4, respectively. The values are reprinted from Klefsjö [12].

Table 3. Power estimates for Q_1 and Q_2 based on 2000 simulations and exact percentiles when $n=20$.

	Q_1	Q_2
$\bar{F}(x)=\exp(-x^{1.5})$	0.66	0.63
$\bar{F}(x)=\exp(-x^{2.0})$	0.98	0.97
$\bar{F}(x)=\exp(-x^{0.8})$	0.36	0.33
$\bar{F}(x)=\int_x^\infty te^{-t}dt$	0.69	0.58

Table 4. Asymptotic Pitman efficacy values for Q_1 and Q_2.

	Q_1	Q_2
Weibull	1.51	1.34
Gamma	0.52	0.39

3. Testing against a distribution which is HNBUE but not NBUE

The Weibull and gamma alternatives mentioned in this paper all have monotone failure rate (IFR or DFR), a much stronger condition than HNBUE or HNWUE (see e.g., Klefsjö [9]). Therefore we have made a power study for some of the mentioned test statistics also against a piecewise exponential distribution which is HNBUE but not NBUE (and accordingly not IFR). The scaled TTT-transform of this distribution can be found in Klefsjö [10]. Here we only mention that F is NBUE exactly when its scaled TTT-transform nowhere is below the diagonal (see e.g., Bergman [3]). A life distribution which is HNBUE but not NBUE accordingly has a scaled TTT-transform which is below the diagonal somewhere in $0 \leqq t \leqq 1$. However, calculations prove that the scaled TTT-transform is above the diagonal in the beginning and the end of this interval. The power estimates are presented in Table 5. Table 5 indicates that B_n (and Δ_3) is rather effective in detecting life distributions which are HNBUE but not NBUE.

Table 5. Power estimates against a life distribution which is HNBUE but not NBUE based on 2000 simulations and exact percentiles.

	B_n (or Δ_3)	A_2	B	Q_1	Q_2
$n=20$	0.58	0.08	0.48	0.65	0.38
$n=10$	0.28	0.07	0.24	0.21	0.14

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