Ann. Inst. Statist. Math.
39 (1987), Part A, 473-482

ON AKI'S NONPARAMETRIC TEST FOR SYMMETRY

SEIJI NABEYA

(Received May 26, 1986)

Summary

Aki (1987, *Ann. Inst. Statist. Math.*, **39**, 457–472) develops a theory of extending the test for symmetry about zero of a continuous distribution function F. In this paper we discuss the same testing problem in the case where the probability $F(0)$ of being negative is unknown, which is assumed to be known in Aki's paper.

1. Introduction

Aki [1] discusses an extension of the test for symmetry about zero of a continuous distribution function F. F is usually called symmetric about c, if it satisfies

$$F(c-x)+F(c+x)=1 \qquad \text{for all} \quad x . \tag{1.1}$$

Let $X_1, \cdots, X_n$ be independent random variables with a common continuous distribution function F, not necessarily symmetric, let α $(0<\alpha<1)$ be a given constant; and he considers testing for the hypothesis,

H_0: There exists a continuous distribution function G which is symmetric about zero and satisfies

$$F(x)=\begin{cases} 2\alpha G(x) , & \text{if} \quad x\leqq 0 , \\ \alpha+2(1-\alpha)\left(G(x)-\dfrac{1}{2}\right) , & \text{if} \quad x>0 . \end{cases} \tag{1.2}$$

If $\alpha=1/2$, then H_0 reduces to the hypothesis of symmetry about zero in the sense (1.1).

In this paper we consider testing for the symmetry in the sense of H_0 assuming that α is an unknown constant. The hypothesis we are going to test in this paper is thus,

H_0': There exist α $(0<\alpha<1)$ and a continuous distribution function G, symmetric about zero in the sense (1.1), which satisfy (1.2).

Key words and phrases: Brownian bridge, test for symmetry, weak convergence.

In the next section we shall give a test statistic and its limiting null distribution, and then prove that the test is consistent.

2. Extended test for symmetry

Let H be a continuous and strictly increasing distribution function symmetric about zero in the sense (1.1). Considering $Y_i=H(X_i)$ ($i=1,\cdots,n$) instead of X_i ($i=1,\cdots,n$) as in Aki [1], the testing problem of H_0' is reduced to testing for the symmetry about 1/2 of the continuous distribution function F defined on the unit interval (0, 1),

H_1': There exist α ($0<\alpha<1$) and a continuous distribution function G^* defined on (0, 1) which satisfy

$$F(t)=\begin{cases} \alpha G^*(2t), & \text{if } 0<t\leqq\frac{1}{2}, \\ \alpha+(1-\alpha)(1-G^*(2-2t)), & \text{if } \frac{1}{2}<t<1. \end{cases}$$

We give below a test statistic for H_1' and two theorems concerning it.

Let $X_1,\cdots,X_n$ be independent random variables on the unit interval (0, 1) with the same distribution function F. Define

$$\eta_i=I_{(0,1/2]}(X_i) \qquad (i=1,\cdots,n), \tag{2.1}$$

$$m=\sum_{i=1}^{n}\eta_i \quad \text{and} \quad \hat{\alpha}=\frac{m}{n}, \tag{2.2}$$

and put in the case $0<m<n$,

$$\xi_i=\sqrt{\frac{1-\hat{\alpha}}{\hat{\alpha}}}=\sqrt{\frac{n-m}{m}} \quad \text{and} \quad Y_i=2X_i, \qquad \text{if} \quad \eta_i=1,$$

whereas

$$\xi_i=-\sqrt{\frac{\hat{\alpha}}{1-\hat{\alpha}}}=-\sqrt{\frac{m}{n-m}} \quad \text{and} \quad Y_i=2(1-X_i), \qquad \text{if} \quad \eta_i=0.$$

Define for $t\in[0,1]$

$$u_n(t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i I_{(0,t]}(Y_i), \qquad \text{if} \quad 0<m<n,$$

and

$$u_n(t)=0, \qquad \text{if} \quad m=0 \quad \text{or} \quad m=n.$$

We consider as a test statistic for H_1' a functional of u_n,

$$T_n = \sup_{0 \leqq t \leqq 1} |u_n(t)| .$$

It is a natural substitute of the test statistic proposed by Aki [1] for the case where α is a known constant.

THEOREM 2.1. *If F satisfies H_1', then $u_n(t)$ converges weakly to $W^\circ(G^*(t))$ where W° denotes Brownian bridge.*

PROOF. Assume that H_1' is satisfied. Then, $\eta_1, \cdots, \eta_n$ are independent with the same distribution

$$P(\eta_i = 1) = \alpha , \qquad P(\eta_i = 0) = 1 - \alpha ,$$

$Y_1, \cdots, Y_n$ are independent with the same distribution

$$P(Y_i \leqq t) = G^*(t) ,$$

furthermore $(\eta_1, \cdots, \eta_n)$ and $(Y_1, \cdots, Y_n)$ are independent. The distribution of m is binomial given by the expansion of $(\alpha + (1-\alpha))^n$.

We show first that $\{u_n\}$ is tight. For the purpose it suffices to prove that there exists a constant C for which

$$\text{(2.3)} \qquad E[|u_n(t) - u_n(t_1)|^2 |u_n(t_2) - u_n(t)|^2] \leqq C(G^*(t) - G^*(t_1))(G^*(t_2) - G^*(t))$$

holds for any n and for any t_1, t, t_2 such that $0 \leqq t_1 < t < t_2 \leqq 1$ (Billingsley [2]).

Put

$$\text{(2.4)} \qquad P(t_1 < Y_i \leqq t) = G^*(t) - G^*(t_1) = p$$

and

$$\text{(2.5)} \qquad P(t < Y_i \leqq t_2) = G^*(t_2) - G^*(t) = q ,$$

and evaluate the expectation on the left hand side of (2.3), conditionally upon a given m such that $0 < m < n$. Then we have

$$\text{(2.6)} \quad E[|u_n(t) - u_n(t_1)|^2 |u_n(t_2) - u_n(t)|^2 \mid m] = \frac{1}{n^2} \sum_{i,j,k,l} E[\xi_i \xi_j \xi_k \xi_l I_{(t_1,t]}(Y_i) I_{(t_1,t]}(Y_j) I_{(t,t_2]}(Y_k) I_{(t,t_2]}(Y_l) \mid m] .$$

The terms with $i=k$, $i=l$, $j=k$ or $j=l$ may be ignored in the above summation. Taking into account the fact that $Y_1, \cdots, Y_n$ are independent with (2.4) and (2.5), and that $(\xi_1, \cdots, \xi_n)$ and $(Y_1, \cdots, Y_n)$ are independent given m, we have

$$\text{(2.7)} \quad (2.6) = \frac{1}{n^2}\Big[\sum_{i,k}{}' E[\xi_i^2 \xi_k^2 \mid m] pq + \sum_{i,k,l}{}' E[\xi_i^2 \xi_k \xi_l \mid m] pq^2$$

$$+\sum_{i,j,k}{}' \mathrm{E}\,[\xi_i\xi_j\xi_k^2 \mid m]p^2q + \sum_{i,j,k,l}{}' \mathrm{E}\,[\xi_i\xi_j\xi_k\xi_l \mid m]p^2q^2\Big] ,$$

where $\sum'$ denotes the summation over all sets of different indices.

Now, for a given m such that $0<m<n$, $(\eta_1, \cdots, \eta_n)$ is distributed with equal probabilities over the set of $\binom{n}{m}$ points in the n-dimensional space, where m of the coordinates are 1 while $n-m$ other coordinates are 0. Hence, if $i \neq k$, we have

$$\mathrm{P}\,(\eta_i=\eta_k=1 \mid m)=\frac{m(m-1)}{n(n-1)} ,$$

$$\mathrm{P}\,(\eta_i=1,\ \eta_k=0 \mid m)=\mathrm{P}\,(\eta_i=0,\ \eta_k=1 \mid m)=\frac{m(n-m)}{n(n-1)} ,$$

$$\mathrm{P}\,(\eta_i=\eta_k=0 \mid m)=\frac{(n-m)(n-m-1)}{n(n-1)} .$$

By using the value of $\xi_i^2\xi_k^2$ in each case we obtain

$$\mathrm{E}\,[\xi_i^2\xi_k^2 \mid m]=\left(\frac{n-m}{m}\right)^2\frac{m(m-1)}{n(n-1)}+2\frac{m(n-m)}{n(n-1)} + \left(\frac{m}{n-m}\right)^2\frac{(n-m)(n-m-1)}{n(n-1)} = \frac{A-B}{n(n-1)m(n-m)} , \tag{2.8}$$

putting

$$A=n^2m(n-m) \qquad \text{and} \qquad B=m^3+(n-m)^3 .$$

In the same way we have for different sets of indices that

$$\mathrm{E}\,[\xi_i^2\xi_k\xi_l \mid m]=\frac{-A+2B}{n(n-1)(n-2)m(n-m)} , \tag{2.9}$$

$$\mathrm{E}\,[\xi_i\xi_j\xi_k^2 \mid m]=\frac{-A+2B}{n(n-1)(n-2)m(n-m)} , \tag{2.10}$$

and

$$\mathrm{E}\,[\xi_i\xi_j\xi_k\xi_l \mid m]=\frac{3A-6B}{n(n-1)(n-2)(n-3)m(n-m)} . \tag{2.11}$$

In (2.7) terms of the types (2.8), (2.9), (2.10) and (2.11) appear $n(n-1)$, $n(n-1)(n-2)$, $n(n-1)(n-2)$ and $n(n-1)(n-2)(n-3)$ times, respectively, hence we have

(2.12) $$(2.7)=\frac{A-B}{n^2m(n-m)}pq+\frac{-A+2B}{n^2m(n-m)}(pq^2+p^2q)+\frac{3A-6B}{n^2m(n-m)}p^2q^2$$
$$=pq\Big[\frac{A}{n^2m(n-m)}(1-p-q+3pq)$$
$$+\frac{B}{n^2m(n-m)}(-1+2p+2q-6pq)\Big] .$$

Since

$$\frac{A}{n^2m(n-m)}=1 \quad \text{and} \quad \frac{B}{n^2m(n-m)}=\frac{m^2}{n^2(n-m)}+\frac{(n-m)^2}{n^2m}\leqq 1$$

for $1\leqq m\leqq n-1$, we have

$$(2.12)\leqq pq(|1-p-q+3pq|+|-1+2p+2q-6pq|) .$$

As the expression in the parentheses on the right hand side is bounded for

(2.13) $$p\geqq 0 , \qquad q\geqq 0 \qquad \text{and} \qquad p+q\leqq 1 ,$$

so denoting by C its upper bound over the region (2.13) we have

(2.14) $$\mathrm{E}\,[|u_n(t)-u_n(t_1)|^2|u_n(t_2)-u_n(t)|^2 \mid m]$$
$$\leqq Cpq=C(G^*(t)-G^*(t_1))(G^*(t_2)-G^*(t)) .$$

In the case $m=0$ or $m=n$ we defined $u_n(t)$ to be $=0$, hence (2.14) holds trivially. Thus (2.3) is proved, establishing the tightness of $\{u_n\}$.

Next we shall find the limiting distribution of the finite dimensional random variable

(2.15) $$(u_n(t_1),\cdots, u_n(t_k))$$

for $0<t_1<\cdots<t_k<1$. Note that $u_n(0)=u_n(1)=0$. First we deal with the case $k=2$ and evaluate the limiting characteristic function of $(u_n(t_1), u_n(t_2)-u_n(t_1))$.

For a given m such that $0<m<n$, m of η_i's are 1 and $n-m$ of η_i's are 0, and $(\eta_1,\cdots,\eta_n)$ and $(Y_1,\cdots,Y_n)$ are independent; therefore the conditional distribution of $(u_n(t_1), u_n(t_2)-u_n(t_1))$ given m is the same as the conditional distribution given

(2.16) $$\eta_1=\cdots=\eta_m=1 \qquad \text{and} \qquad \eta_{m+1}=\cdots=\eta_n=0 .$$

Assuming (2.16) we have

$$u_n(t_1)=\frac{1}{\sqrt{n}}\Big[\sqrt{\frac{n-m}{m}}\sum_{j=1}^{m}I_{(0,t_1]}(Y_j)-\sqrt{\frac{m}{n-m}}\sum_{j=m+1}^{n}I_{(0,t_1]}(Y_j)\Big]$$

and

$$u_n(t_2)-u_n(t_1)=\frac{1}{\sqrt{n}}\left[\sqrt{\frac{n-m}{m}}\sum_{j=1}^{m} I_{(t_1,t_2]}(Y_j)-\sqrt{\frac{m}{n-m}}\sum_{j=m+1}^{n} I_{(t_1,t_2]}(Y_j)\right].$$

Put

$$\mathrm{P}(0<Y_j\leqq t_1)=p, \qquad \mathrm{P}(t_1<Y_j\leqq t_2)=q \qquad \text{and} \qquad \mathrm{P}(t_2<Y_j<1)=r,$$

then we have clearly

$$p\geqq 0, \qquad q\geqq 0, \qquad r\geqq 0 \qquad \text{and} \qquad p+q+r=1,$$

and the random variables defined by

$$(U_1, U_2, U_3)=\left(\sum_{j=1}^{m} I_{(0,t_1]}(Y_j),\ \sum_{j=1}^{m} I_{(t_1,t_2]}(Y_j),\ \sum_{j=1}^{m} I_{(t_2,1)}(Y_j)\right)$$

and

$$(V_1, V_2, V_3)=\left(\sum_{j=m+1}^{n} I_{(0,t_1]}(Y_j),\ \sum_{j=m+1}^{n} I_{(t_1,t_2]}(Y_j),\ \sum_{j=m+1}^{n} I_{(t_2,1)}(Y_j)\right)$$

are independent with the multinomial distribution given by the expansion of $(p+q+r)^m$ and $(p+q+r)^{n-m}$, respectively.

Furthermore we have

$$u_n(t_1)=\frac{1}{\sqrt{n}}\left(\sqrt{\frac{n-m}{m}}U_1-\sqrt{\frac{m}{n-m}}V_1\right) \tag{2.17}$$

and

$$u_n(t_2)-u_n(t_1)=\frac{1}{\sqrt{n}}\left(\sqrt{\frac{n-m}{m}}U_2-\sqrt{\frac{m}{n-m}}V_2\right),$$

hence we obtain the conditional characteristic function of $(u_n(t_1), u_n(t_2)-u_n(t_1))$ given m as in the following,

$$\begin{aligned}
\phi_n(\theta_1, \theta_2\,|\,m) &= \mathrm{E}\left[\exp\{i\theta_1 u_n(t_1)+i\theta_2(u_n(t_2)-u_n(t_1))\}\ |\ m\right] \\
&= \mathrm{E}\left[\exp\left\{\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}(\theta_1 U_1+\theta_2 U_2)\right.\right. \\
&\qquad \left.\left.-\frac{i}{\sqrt{n}}\sqrt{\frac{m}{n-m}}(\theta_1 V_1+\theta_2 V_2)\right\}\,\middle|\, m\right] \\
&= \left[p\exp\left(\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}\theta_1\right)+q\exp\left(\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}\theta_2\right)+r\right]^m \\
&\qquad \times\left[p\exp\left(\frac{-i}{\sqrt{n}}\sqrt{\frac{m}{n-m}}\theta_1\right)\right. \\
&\qquad\qquad \left.+q\exp\left(\frac{-i}{\sqrt{n}}\sqrt{\frac{m}{n-m}}\theta_2\right)+r\right]^{n-m} \\
&= \phi_n^{(1)}(\theta_1, \theta_2\,|\,m)\,\phi_n^{(2)}(\theta_1, \theta_2\,|\,m), \qquad \text{say.}
\end{aligned}$$

Let an arbitrary $\varepsilon>0$ be given, then by the central limit theorem there exist $\gamma>0$ and a natural number n_0 such that for any n satisfying $n\geqq n_0$ we have

$$\mathrm{P}\,(n\alpha-\sqrt{n}\,\gamma<m<n\alpha+\sqrt{n}\,\gamma)>1-\varepsilon\ . \tag{2.18}$$

We shall show below that $\phi_n(\theta_1, \theta_2\,|\,m)$ tends, as $n\to\infty$, to a limiting function $\phi(\theta_1, \theta_2)$ uniformly in m for which

$$n\alpha-\sqrt{n}\,\gamma<m<n\alpha+\sqrt{n}\,\gamma \tag{2.19}$$

holds.

Note that $(n-m)/m$ and $m/(n-m)$ are bounded for n satisfying $n\geqq n_0$ and for m satisfying (2.19), and apply Taylor expansion to obtain

$$\begin{aligned}
&\log \phi_n^{(1)}(\theta_1, \theta_2\,|\,m)\\
&\quad=m\log\Big[1+p\Big\{\exp\Big(\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}\,\theta_1\Big)-1\Big\}\\
&\qquad\qquad+q\Big\{\exp\Big(\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}\,\theta_2\Big)-1\Big\}\Big]\\
&\quad=m\log\Big[1+\frac{i}{\sqrt{n}}\sqrt{\frac{n-m}{m}}(p\theta_1+q\theta_2)-\frac{n-m}{2nm}(p\theta_1^2+q\theta_2^2)+O(n^{-3/2})\Big]\\
&\quad=i\sqrt{\frac{m(n-m)}{n}}(p\theta_1+q\theta_2)-\frac{n-m}{2n}(p\theta_1^2+q\theta_2^2-(p\theta_1+q\theta_2)^2)+O(n^{-1/2})\ ,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\log \phi_n^{(2)}(\theta_1, \theta_2\,|\,m)\\
&\quad=-i\sqrt{\frac{m(n-m)}{n}}(p\theta_1+q\theta_2)-\frac{m}{2n}(p\theta_1^2+q\theta_2^2-(p\theta_1+q\theta_2)^2)+O(n^{-1/2})\ ,
\end{aligned}$$

uniformly in m satisfying (2.19).

Adding both equations side by side we get

$$\log \phi_n(\theta_1, \theta_2\,|\,m)=-\frac{1}{2}(p\theta_1^2+q\theta_2^2-(p\theta_1+q\theta_2)^2)+O(n^{-1/2})\ ,$$

from which it follows that

$$\phi_n(\theta_1, \theta_2\,|\,m)=\phi(\theta_1, \theta_2)+O(n^{-1/2})\ , \tag{2.20}$$

uniformly in m satisfying (2.19), where we put

$$\phi(\theta_1, \theta_2)=\exp\Big[-\frac{1}{2}(p\theta_1^2+q\theta_2^2-(p\theta_1+q\theta_2)^2)\Big]\ .$$

Therefore, if we take some integer n_1 such that $n_1\geqq n_0$, then the

$O(n^{-1/2})$ term in (2.20) is less than ε in absolute value for any n such that $n \geqq n_1$ and any m satisfying (2.19), that is,

$$|\phi_n(\theta_1, \theta_2 | m) - \phi(\theta_1, \theta_2)| < \varepsilon . \tag{2.21}$$

Let

$$\phi_n(\theta_1, \theta_2) = \sum_{m=0}^{n} \binom{n}{m} \alpha^m (1-\alpha)^{n-m} \phi_n(\theta_1, \theta_2 | m) = \Sigma' + \Sigma'' ,$$

where Σ' denotes the summation over all values of m satisfying (2.19), while Σ'' denotes the summation over all other values of m. Taking into account of (2.21) for Σ' and of (2.18) we have

$$|\Sigma' - \phi(\theta_1, \theta_2)| < \mathrm{P}\,(|m - n\alpha| \geqq \sqrt{n}\,\gamma) + \varepsilon < 2\varepsilon$$

and

$$|\Sigma''| < \mathrm{P}\,(|m - n\alpha| \geqq \sqrt{n}\,\gamma) < \varepsilon ,$$

therefore we get

$$|\phi_n(\theta_1, \theta_2) - \phi(\theta_1, \theta_2)| < 3\varepsilon$$

for all n such that $n \geqq n_1$.

Thus we have proved

$$\lim_{n\to\infty} \phi_n(\theta_1, \theta_2) = \phi(\theta_1, \theta_2)$$

for all θ_1 and θ_2. $\phi(\theta_1, \theta_2)$ is clearly the characteristic function of $(W^\circ(p), W^\circ(q) - W^\circ(p))$, where W° denotes Brownian bridge. In particular it has been shown that the variances and the covariance of the limiting distribution of $(u_n(t_1), u_n(t_2))$ coincide with those of $(W^\circ(p), W^\circ(q)) = (W^\circ(G^*(t_1)), W^\circ(G^*(t_2)))$.

By using a similar argument we can find the limit of the characteristic function of

$$(u_n(t_1), u_n(t_2) - u_n(t_1), \cdots, u_n(t_k) - u_n(t_{k-1})) \tag{2.22}$$

for any k and any $t_1, \cdots, t_k$ such that $0 < t_1 < \cdots < t_k < 1$, and we can see that it is of the form

$$\exp\left[-\frac{1}{2}(\text{quadratic form in } \theta\text{'s})\right] ,$$

which shows that the limiting distribution of (2.22) is k-variate normal with zero means. Since the variances and covariances of the limiting distribution of (2.15) have been already shown to coincide with those of

$$(W^\circ(G^*(t_1)), \cdots, W^\circ(G^*(t_k))) , \tag{2.23}$$

the limiting distribution of (2.15) is proved to be that of (2.23).

The proof of Theorem 2.1 is thus completed by Theorem 15.6 of Billingsley [2].

By Theorem 2.1 the limiting distribution of T_n is given by

$$\lim_{n\to\infty} \mathrm{P}\,(T_n \leqq z) = \mathrm{P}\,(\sup_{0\leqq t\leqq 1} |W^{\circ}(t)| \leqq z) = 1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-2k^2z^2} .$$

See Billingsley [2], p. 85.

We prove next the consistency of the test given by the critical region $T_n \geqq c$ for any constant c. We give a proof of a slightly different character from that of Theorem 3.2 in Aki [1].

THEOREM 2.2. *If a continuous distribution function F defined on $(0, 1)$ does not satisfy H_1', then $\mathrm{P}\,(T_n \geqq c)$ converges to 1 for any constant c, provided that $0 < F(1/2) < 1$.*

PROOF. Put $\alpha = F(1/2)$ $(0<\alpha<1)$ and define for any t such that $0 \leqq t \leqq 1$,

$$G_1^*(t) = \frac{1}{\alpha} F\left(\frac{t}{2}\right) \quad \text{and} \quad G_2^*(t) = \frac{1}{1-\alpha}\left(1 - F\left(1 - \frac{t}{2}\right)\right) ,$$

then we have

$$G_1^*(t) = \mathrm{P}\,(Y_i \leqq t \mid \eta_i = 1) \quad \text{and} \quad G_2^*(t) = \mathrm{P}\,(Y_i \leqq t \mid \eta_i = 0) .$$

By the assumption of Theorem 2.2 there exists t_0 such that

$$0 < t_0 < 1 \quad \text{and} \quad G_1^*(t_0) \neq G_2^*(t_0) .$$

Put $G_1^*(t_0) = p_1$ and $G_2^*(t_0) = p_2$.

As (2.17) in the proof of Theorem 2.1 we have

$$u_n(t_0) = \frac{1}{\sqrt{n}}\left(\sqrt{\frac{n-m}{m}}\, U_0 - \sqrt{\frac{m}{n-m}}\, V_0\right) ,$$

where m is defined by (2.2) and is assumed to be $0<m<n$. For a given m, U_0 and V_0 are independent random variables with the binomial distribution given by the expansion of $(p_1 + (1-p_1))^m$ and $(p_2 + (1-p_2))^{n-m}$, respectively. Hence we have, if $0<m<n$,

$$\mathrm{E}\,[u_n(t_0) \mid m] = \sqrt{\frac{m(n-m)}{n}}\,(p_1 - p_2)$$

and

$$\mathrm{V}\,[u_n(t_0) \mid m] = \frac{n-m}{n} p_1(1-p_1) + \frac{m}{n} p_2(1-p_2) .$$

If $m=0$ or $m=n$, we have clearly

$$\mathrm{E}\,[u_n(t_0)\,|\,m]=\mathrm{V}\,[u_n(t_0)\,|\,m]=0\ .$$

Since $\sqrt{m(n-m)/n^2}$ is the sample standard deviation calculated from $\eta_1,\cdots,\eta_n$, its mean and variance are derived from the formulae given in Cramér [3], p. 353. Using the results we have

$$\mathrm{E}\,[u_n(t_0)]=\mathop{\mathrm{E}}_{m}\mathrm{E}\,[u_n(t_0)\,|\,m]=\sqrt{n\alpha(1-\alpha)}(p_1-p_2)(1+O(n^{-1}))$$

and

$$\begin{aligned}\mathrm{V}\,[u_n(t_0)]&=\mathop{\mathrm{V}}_{m}\mathrm{E}\,[u_n(t_0)\,|\,m]+\mathop{\mathrm{E}}_{m}\mathrm{V}\,[u_n(t_0)\,|\,m]\\&=\left(\frac{1}{4}-\alpha(1-\alpha)\right)(p_1-p_2)^2+(1-\alpha)p_1(1-p_1)\\&\qquad+\alpha p_2(1-p_2)+O(n^{-1})\\&=O(1)\ .\end{aligned}$$

By $p_1-p_2\neq 0$, $|\mathrm{E}\,[u_n(t_0)]|$ tends to infinity whereas $\mathrm{V}\,[u_n(t_0)]$ is bounded as $n\to\infty$; therefore by using Tchebychev's inequality we have $\mathrm{P}\,(|u_n(t_0)|\geqq c)\to 1$ for any c so that

$$\lim_{n\to\infty}\mathrm{P}\,(T_n\geqq c)=\lim_{n\to\infty}\mathrm{P}(\sup_{0\leqq t\leqq 1}|u_n(t)|\geqq c)=1\ ,$$

completing the proof.

Acknowledgement

The author had a chance of attending Aki's seminar in advance of submitting his paper [1], which has been the stimulus to the author's present research. The author expresses sincere thanks to him.

Hitotsubashi University

References

[1] Aki, S. (1987). On nonparametric tests for symmetry, *Ann. Inst. Statist. Math.*, **39**, 457-472.
[2] Billingsley, P. (1968). *Convergence of Probability Measures*, Wiley, New York.
[3] Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press, Princeton.