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CHARACTERIZATION OF A MARSHALL-OLKIN TYPE CLASS OF DISTRIBUTIONS*

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Summary

A class of bivariate distributions that generalize Marshall-Olkin's one is characterized through a functional equation which involves two associative operations. The obtained distributions concentrate positive mass on the line $x=y$ when the two associative operations coincide; otherwise a positive mass is concentrated on a continuous monotone function.

1. Introduction

Marshall and Olkin [8] characterize a bivariate distribution, assuming that it has exponential marginals and the following functional equation holds:

$$\bar{F}(s_1+t, s_2+t)=\bar{F}(s_1, s_2)\bar{F}(t, t) \tag{1.1}$$

where $\bar{F}(s, t)=\mathrm{P}(X>s, Y>t)$. Equation (1.1) represents a particular type of the Lack of Memory Property. This distribution is a mixture of an absolutely continuous and a singular component, that concentrates its mass on the line $x=y$.

In this paper we generalize Marshall-Olkin's results considering a lack-of-memory-property-type functional equation which involves operations different than the addition:

$$\bar{F}(s_1 * t, s_2 * t)=\bar{F}(s_1, s_2)\bar{F}(t, t) \tag{1.2}$$

and analogous equations for the marginals. In particular we shall consider an associative, binary operation $*$. We obtain a class of bivariate distributions whose marginals are not necessarily exponential; their form depends on the associative operation. These distributions con-

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centrate positive mass on the line $x=y$ (like Marshall-Olkin's one). If we consider two different associative operations and we assume a particular relation between two variables in the functional equation, we obtain a class of bivariate distributions each of whose marginals depends on one of the associative operations. These distributions concentrate positive mass on a monotone continuous function $y=\psi(x)$.

Thus we provide a way of generating bivariate distributions of nonindependent random variables. Dependence of these random variables is obtained by depositing positive mass on a set of null Lebesgue measure (the graph of a continuous monotone function). The marginals can be arbitrary obtained in the class of absolutely continuous distributions, by choosing suitable associative operations in the functional equation (1.2).

Furthermore, we shall show that all the distributions that we obtain have a common structure. All of them have indeed the same type of copula that does not depend on the chosen associative operations.

In Section 2 some univariate characterizations are considered. In Section 3 bivariate distributions of independent random variables are characterized. Section 4 is devoted to the characterization of Marshall-Olkin type class of distributions. In Section 5 some properties of these distributions are examined. In Section 6 the case of two different associative operations is considered. Copulae of these distributions are presented in Section 7. In Section 8 a possible generalization to the n-dimensional case is suggested.

2. Univariate characterization

Let X be a nonnegative random variable with distribution function $F(x)=\mathrm{P}(X\leqq x)$. Then X is said to have the Lack of Memory Property if:

$$\mathrm{P}(X>s+t \mid X>s)=\mathrm{P}(X>t) \tag{2.1}$$

for all $s, t>0$. Now, if $\mathrm{P}(X>s)>0$ for all $s>0$, then (2.1) can be written as

$$\bar{F}(s+t)=\bar{F}(s)\bar{F}(t) \tag{2.2}$$

where $\bar{F}(x)=1-F(x)$ is the survival function and $s, t>0$ are arbitrary.

Equation (2.2) is one of the four types of equations that are called Cauchy equations (see Aczél [1], Section 2.1.2). Standard techniques (see Aczél [1], Theorem 1, p. 38) lead to the (continuous) solution:

$$\bar{F}(s)=\exp(-\lambda s)\ , \qquad \lambda>0,\ s>0$$

that is, F is exponential (see Galambos and Kotz [5] for a complete bibliography and detailed proofs).

We shall extend the Lack of Memory Property assuming that the following holds:

$$\mathrm{P}(X>s*t\,|\,X>s)=\mathrm{P}(X>t) \tag{2.3}$$

where the binary operation $*$ is associative, i.e. such that

$$(x*y)*z=x*(y*z)\ . \tag{2.4}$$

The general reducible (i.e. $x*y=x*z$ or $y*w=z*w$ only if $z=y$), continuous solution of the functional equation (2.4) is

$$x*y=g^{-1}(g(x)+g(y)) \tag{2.5}$$

with g continuous and strictly monotone, provided x, y, $x*y$ always lie in a fixed (possible infinite) interval (see Aczél [1], p. 253 ff.).

The function g occurring in (2.5) is determined up to a multiplicative constant, so that

$$g_1^{-1}(g_1(x)+g_1(y))=g_2^{-1}(g_2(x)+g_2(y))$$

for all x, y in a fixed interval when $g_2(x)=\alpha g_1(x)$ with $\alpha\neq 0$. This allows us to consider the function g strictly increasing. From now on (unless otherwise stated) the binary operation $*$ will be assumed reducible and associative, i.e. such that the representation (2.5) holds with g strictly increasing. Furthermore we assume that an identity element $e\in\bar{R}$ exists, such that

$$x*e=x\ .$$

We recall that every continuous, reducible, associative operation defined on a real interval is commutative (see Aczél [1], p. 267). If $\mathrm{P}(X>s)>0$ for all $s>e$, then (2.3) can be written:

$$\bar{F}(s*t)=\bar{F}(s)\bar{F}(t)\ . \tag{2.6}$$

PROPOSITION 1. *The (continuous) solution of* (2.6) *is*

$$\bar{F}(s)=\exp(-\lambda g(s))$$

with $\lambda>0$, $e=g^{-1}(0)<s<g^{-1}(\infty)$.

PROOF. Combining (2.5) and (2.6) we have

$$\bar{F}(g^{-1}(g(s)+g(t)))=\bar{F}(s)\bar{F}(t)\ . \tag{2.7}$$

If $g(s)=u$, $g(t)=v$, $F\circ g^{-1}=H$, then (2.7) becomes

$$H(u+v)=H(u)H(v)\ ,$$

that is a Cauchy equation. Therefore $H(u)=\exp(-\lambda u)$ with $\lambda>0$, and

$$\bar{F}(s)=\exp(-\lambda g(s)) .$$

From now on, unless otherwise stated, the support of the random variables will be assumed $(e, g^{-1}(\infty))$.

Remark 1. If we particularize the operation $*$ we can obtain different classes of distributions.

Example 1. When $x*y=x+y$, $g(x)=x$; therefore $\bar{F}(x)=\exp(-\lambda x)$ $(\lambda>0,\ x>0)$, this is the usual lack of memory characterization of the exponential distribution.

Example 2. When $x*y=xy$, $g(x)=\log x$; therefore $\bar{F}(x)=x^{-\lambda}$ $(\lambda>0,\ x>1)$. F is the distribution function of the Pareto distribution with parameters λ and 1.

Example 3. When $x*y=(x^\alpha+y^\alpha)^{1/\alpha}$, $g(x)=x^\alpha$; therefore $\bar{F}(x)=\exp(-\lambda x^\alpha)$ $(\alpha>0,\ \lambda>0,\ x>0)$. This characterizes the Weibull distribution (see Wang [14]).

Related results may be found in Castagnoli [3], Muliere [10] and Castagnoli and Muliere [4].

3. Bivariate characterizations

If we write $\bar{F}(x, y)=\mathrm{P}(X>x, Y>y)$, then a direct extension of (2.2) might be

$$\bar{F}(s_1+t_1, s_2+t_2)=\bar{F}(s_1, s_2)\bar{F}(t_1, t_2) \tag{3.1}$$

where $s_1, s_2, t_1, t_2>0$.

Setting $s_1=t_1=0$ and then $s_2=t_2=0$, we obtain that both X and Y are exponential. Then, taking $t_1=s_2=0$ leads to the independence of X and Y.

Consequently, the only solution of equation (3.1) is

$$\bar{F}(s, t)=\exp(-\lambda_1 s-\lambda_2 t)$$

for some $\lambda_1>0$, $\lambda_2>0$ (see Marshall and Olkin [8], Galambos and Kotz [5]).

We consider an analogous equation for the associative binary operation $*$:

$$\bar{F}(s_1*t_1, s_2*t_2)=\bar{F}(s_1, s_2)\bar{F}(t_1, t_2) \tag{3.2}$$

for all $s_1, s_2, t_1, t_2>e$.

PROPOSITION 2. *The (continuous) solution of* (3.2) *is*:

$$\bar{F}(s,t)=\exp(-\lambda_1 g(s)-\lambda_2 g(t)) \tag{3.3}$$

for some $\lambda_1>0$, $\lambda_2>0$.

PROOF. Combining (2.5) and (3.2) we have:

$$\bar{F}(g^{-1}(g(s_1)+g(t_1)),\, g^{-1}(g(s_2)+g(t_2)))=\bar{F}(s_1,s_2)\bar{F}(t_1,t_2)\,. \tag{3.4}$$

If $g(s_i)=u_i$, $g(t_i)=v_i$, $i=1,2$ and $\bar{F}(g^{-1}(\cdot),g^{-1}(\cdot))=H(\cdot\,,\cdot)$, then (3.4) becomes

$$H(u_1+v_1,\,u_2+v_2)=H(u_1,u_2)H(v_1,v_2)\,.$$

Whose solution is $H(u,v)=\exp(-\lambda_1 u-\lambda_2 v)$. Hence $\bar{F}(s,t)=\exp(-\lambda_1 g(s)-\lambda_2 g(t))$.

Remark 2. If (X,Y) has joint survival function like in (3.3), then X and Y are independent with marginal survival functions $\bar{F}_X(s)=\exp(-\lambda_1 g(s))$, $\bar{F}_Y(t)=\exp(-\lambda_2 g(t))$.

4. A Marshall-Olkin type class of distributions

A more fruitful way of extending the Lack of Memory Property to the bivariate case is to investigate the equation (see Marshall and Olkin [8])

$$\bar{F}(s_1+t,\,s_2+t)=\bar{F}(s_1,s_2)\bar{F}(t,t) \tag{4.1}$$

where $s_1, s_2, t>0$. If we assume that the marginals are exponential, then the unique solution of (4.1) among survival functions is

$$\bar{F}(s,t)=\exp(-\lambda_1 s-\lambda_2 t-\lambda_{12}\max(s,t)) \tag{4.2}$$

where $\lambda_1, \lambda_2, \lambda_{12}>0$.

This family has properties which have proved useful for reliability applications (see Basu and Block [2], Galambos and Kotz [5], Chap. 5).

Our main purpose is to consider (4.1) with the operation $*$. Let $\bar{F}$ be a bivariate survival function such that:

$$\text{a) } \bar{F}(s,e)=\bar{F}_1(s) \quad \text{and} \quad \text{b) } \bar{F}(e,s)=\bar{F}_2(s) \tag{4.3}$$

where $\bar{F}_1$ and $\bar{F}_2$ are the marginal univariate survival function, and e is the identity element w.r.t. the operation $*$.

THEOREM 1. *Let*

$$\bar{F}(s_1 * t,\, s_2 * t)=\bar{F}(s_1,s_2)\bar{F}(t,t) \tag{4.4}$$

and

$$\bar{F}_i(s * t)=\bar{F}_i(s)\bar{F}_i(t)\ , \qquad i=1,2 \tag{4.5}$$

where $s_1, s_2, s, t>e$. *Then the (continuous) solution of the equations* (4.4) *and* (4.5) *is*

$$\bar{F}(s, t)=\exp\left(-\lambda_1 g(s)-\lambda_2 g(t)-\lambda_{12} g(\max(s, t))\right) \tag{4.6}$$

with $\lambda_1, \lambda_2, \lambda_{12}>0$.

PROOF. Let in (4.4) be $s_1=s_2=s$. Then

$$\bar{F}(s * t, s * t)=\bar{F}(s, s)\bar{F}(t, t)\ .$$

Hence, $\bar{F}(s, s)=\exp(-\delta g(s))$ $(\delta>0)$. Therefore (4.3) yields

$$\bar{F}(s * t, e * t)=\bar{F}_1(s)\bar{F}(t, t)=\exp\left(-\theta_1 g(s)-\delta g(t)\right)$$

by Proposition 1. Setting $s * t=u$ (hence $g(s)=g(u)-g(t)$) we have

$$\bar{F}(u, t)=\exp\left(-\theta_1(g(u)-g(t))-\delta g(t)\right)\ , \qquad u\geqq t\ .$$

Analogously,

$$\bar{F}(u, t)=\exp\left(-\theta_2(g(t)-g(u))-\delta g(u)\right)\ , \qquad u\leqq t\ .$$

If $\lambda_1=\delta-\theta_2$, $\lambda_2=\delta-\theta_1$, $\lambda_{12}=\theta_1+\theta_2-\delta$, this may be written as

$$\bar{F}(u, t)=\exp\left(-\lambda_1 g(u)-\lambda_2 g(t)-\lambda_{12} g(\max(u, t))\right)\ .$$

In order that it be a bivariate survival function, $\bar{F}$ must be nonincreasing in each variable. This implies $\lambda_1>0$, $\lambda_2>0$. Moreover $\bar{F}$ must satisfy the following:

$$\bar{F}(u_1, t_1)+\bar{F}(u_2, t_2)-\bar{F}(u_1, t_2)-\bar{F}(u_2, t_1)\geqq 0$$

this implies $\lambda_{12}>0$.

Remark 3. In the proof of Proposition 2 and Theorem 1 we have assumed that the associative binary operation $*$ is always represented as $g^{-1}(g(x)+g(y))$. Actually as we said, any function $h=\alpha g$ $(\alpha\neq 0)$ represents the same operation, and even if we choose two different representations for the operation $*$ in the same functional equation, the result does not change.

Different particularizations of operation $*$ lead to different bivariate survival functions.

Example 4. When $x * y=x+y$, $g(x)=x$; then

$$\bar{F}(x, y)=\exp(-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x, y)) .$$

This is the Marshall-Olkin distribution (4.2).

Example 5. When $x*y=xy$, $g(x)=\log x$; then

$$\bar{F}(x, y)=x^{-\lambda_1}y^{-\lambda_2}(\max(x, y))^{-\lambda_{12}} .$$

This is a bivariate Pareto distribution over the set $(1, +\infty)\times(1, +\infty)$.

Example 6. When $x*y=(x^{\alpha}+y^{\alpha})^{1/\alpha}$, $g(x)=x^{\alpha}$; then

$$\bar{F}(x, y)=\exp(-\lambda_1 x^{\alpha}-\lambda_2 x^{\alpha}-\lambda_{12}\max(x^{\alpha}, y^{\alpha})) .$$

This is a bivariate Weibull distribution (bivariate Weibull distribution was introduced briefly by Marshall and Olkin [8] and was studied in detail by Moeschberger [9]).

5. Some properties

The class of distributions defined in Section 4 inherits some properties of the Marshall-Olkin distribution. The survival function (4.6) is a mixture of two components:

$$\bar{F}(x, y)=\alpha\bar{F}_a(x, y)+(1-\alpha)\bar{F}_s(x, y) , \qquad 0\leqq\alpha\leqq 1$$

where $\bar{F}_a$ is absolutely continuous w.r.t. Lebesgue measure and $\bar{F}_s$ concentrates its mass on the line $x=y$. The weights of the mixture depend on the parameters λ_1, λ_2, λ_{12}:

$$\alpha=(\lambda_1+\lambda_2)/(\lambda_1+\lambda_2+\lambda_{12})$$

and

$$\bar{F}_s=\exp(-(\lambda_1+\lambda_2+\lambda_{12})g(\max(x, y))) .$$

If g is differentiable, then the density of the absolutely continuous part has the following form:

$$f_a(x, y)=(1/\alpha)(\partial\bar{F}(x, y)/\partial x\partial y)$$
$$=\begin{cases} [(\lambda_1+\lambda_2+\lambda_{12})/(\lambda_1+\lambda_2)](\lambda_1+\lambda_{12})\lambda_2 g'(x)g'(y)\bar{F}(x, y) , & x>y \\ [(\lambda_1+\lambda_2+\lambda_{12})/(\lambda_1+\lambda_2)](\lambda_2+\lambda_{12})\lambda_1 g'(x)g'(y)\bar{F}(x, y) , & x<y . \end{cases}$$

It is of interest to note that

$$\bar{F}_1(x)=\bar{F}(x, e)=\exp(-(\lambda_1+\lambda_{12})g(x))$$
$$\bar{F}_2(y)=\bar{F}(e, y)=\exp(-(\lambda_2+\lambda_{12})g(y)) .$$

Also it can be seen from (4.6) that a necessary and sufficient condition for X and Y to be independent is that $\lambda_{12}=0$.

Remark 4. If (X_1, X_2) is distributed as in (4.6), then the survival function of $\min(X_1, X_2)$ has the following form:

$$\mathrm{P}(\min(X_1, X_2)>s)=\mathrm{P}(X_1>s, X_2>s)=\exp(-(\lambda_1+\lambda_2+\lambda_{12})g(s))$$

therefore the distribution of $\min(X_1, X_2)$ has the same form (with different parameter) as the marginal distributions of X_1 and X_2.

Next theorem provides a different characterization of distribution (4.6).

THEOREM 2. *Let (X_1, X_2) be distributed according to F. Then F is as in* (4.6) *iff there exist independent random variables U_1, U_2, U_{12}, whose marginal distributions satisfy* (2.7), *such that* $X_1=\min(U_1, U_{12})$, $X_2=\min(U_2, U_{12})$.

PROOF.

$$\begin{aligned}\mathrm{P}(X_1>x, X_2>y)&=\mathrm{P}(U_1>x, U_{12}>x, U_2>y, U_{12}>y)\\&=\mathrm{P}(U_1>x)\mathrm{P}(U_2>y)\mathrm{P}(U_{12}>\max(x, y))\end{aligned}$$

by independence of U_1, U_2, U_{12}

$$=\exp(-\lambda_1 g(x)-\lambda_2 g(y)-\lambda_{12} g(\max(x, y)))$$

by Proposition 1. The "only if" part is obvious.

6. Characterization with two different associative operations

Equation (3.2) can be generalized considering two different associative operations $*$ and $\circ$.

PROPOSITION 3. *Let* $\bar{F}(s_1 * t_1, s_2 \circ t_2)=\bar{F}(s_1, s_2)\bar{F}(t_1, t_2)$. *Then*

$$\bar{F}(s, t)=\exp(-\lambda_1 g(s)-\lambda_2 h(t)) .$$

PROOF. The proof is obvious and descendes immediately from the representations $x * y=g^{-1}(g(x)+g(y))$, $x \circ y=h^{-1}(h(x)+h(y))$.

Now we investigate the possibility of generalizing also (4.4) by using two different associative operations. In order to solve this problem, we need the following:

LEMMA. *Let*

(6.1) a) $\bar{F}(x * t, \phi(y * t))=\bar{F}(x, \phi(y))\bar{F}(t, \phi(t))$

with ϕ continuous and strictly increasing

b) $\bar{F}_1(x*t)=\bar{F}_1(x)\bar{F}_1(t)$

with $\bar{F}_1(x)=\bar{F}(x,\ \psi(e))$

c) $\bar{F}_2(\psi(y*t))=\bar{F}_2(\psi(y))\bar{F}_2(\psi(t))$

with $\bar{F}_2(y)=\bar{F}(e,\ y)$.

Then

$$\bar{F}(x,\ y)=\exp\left(-\lambda_1 g(x)-\lambda_2 g(\psi^{-1}(y))-\lambda_{12}\max\left(g(x),\ g(\psi^{-1}(y))\right)\right)\ . \tag{6.2}$$

PROOF. Combining (2.5) and (6.1) we have

$$\bar{F}(g^{-1}(g(x)+g(t)),\ \psi(g^{-1}(g(y)+g(t))))=\bar{F}(x,\ \psi(y))\bar{F}(t,\ \psi(t))\ .$$

Setting $g(x)=v$, $g(t)=u$, $g(y)=w$, we have

$$\bar{F}(g^{-1}(v+u),\ \psi(g^{-1}(w+u)))=\bar{F}(g^{-1}(v),\ \psi(g^{-1}(w)))\bar{F}(g^{-1}(u),\ \psi(g^{-1}(u)))\ .$$

When $v=w$, then

$$\bar{F}(g^{-1}(v+u),\ \psi(g^{-1}(v+u)))=\bar{F}(g^{-1}(v),\ \psi(g^{-1}(v)))\bar{F}(g^{-1}(u),\ \psi(g^{-1}(u)))\ .$$

This is a Cauchy equation, whose solution is

$$\bar{F}(g^{-1}(v),\ \psi(g^{-1}(v)))=\exp(-\delta v)\ .$$

Since

$$\bar{F}_1(x)=\exp(-\theta_1 g(x))\ , \qquad \bar{F}_2(y)=\exp(-\theta_2 g(\psi^{-1}(y))$$

then, when $v=0$

$$\bar{F}(g^{-1}(u),\ \psi(g^{-1}(w+u)))=\bar{F}(g^{-1}(0),\ \psi(g^{-1}(w)))\exp(-\delta u)\ .$$

Whence

$$\begin{aligned}\bar{F}(t,\ \psi(y*t))&=\bar{F}(e,\ \psi(y))\exp(-\delta g(t))\\&=\bar{F}_2(\psi(y))\exp(-\delta g(t))\\&=\exp(-\theta_2 g(y))\exp(-\delta g(t))\ .\end{aligned}$$

This implies

$$\bar{F}(t,\ \psi(z))=\exp(-\theta_2(g(z)-g(t))-\delta g(t))\ , \qquad z\geqq t$$

that is

$$\bar{F}(t,\ s)=\exp(-\theta_2(g(\psi^{-1}(s))-g(t))-\delta g(t))\ , \qquad \psi^{-1}(s)\geqq t\ . \tag{6.3}$$

By applying the same argument, when $w=0$ we obtain

$$\bar{F}(t,\ s)=\exp(-\theta_1(g(t)-g(\psi^{-1}(s)))-\delta g(\psi^{-1}(s)))\ , \qquad \psi^{-1}(s)\leqq t\ . \tag{6.4}$$

If we combine (6.3) and (6.4) we obtain (6.2).

Remark 5. The survival function (6.2) is a mixture

$$\bar{F}(x, y)=\alpha\bar{F}_a(x, y)+(1-\alpha)\bar{F}_s(x, y) , \qquad 0\leqq\alpha\leqq1$$

where $\bar{F}_a$ is absolutely continuous w.r.t. Lebesgue measure, and $\bar{F}_s$ concentrates its mass on the curve $y=\psi(x)$.

The survival function (6.2) can be characterized by a lack-of-memory-property-type functional equation, with two different associative operations.

THEOREM 3. *Let*

(6.5) a) $\bar{F}(x*t, y\circ z)=\bar{F}(x, y)F(t, z)$, $\qquad e_g<x, t<g^{-1}(\infty)$, $e_h<y, z<h^{-1}(\infty)$

with $x*t=g^{-1}(g(x)+g(t))$, $y\circ z=h^{-1}(h(y)+h(z))$, $h(z)=g(t)$

b) $\bar{F}_1(x*t)=\bar{F}_1(x)\bar{F}_1(t)$

with $\bar{F}_1(x)=\bar{F}(x, e_h)$ *where* $y\circ e_h=y$

c) $\bar{F}_2(y\circ z)=\bar{F}_2(y)\bar{F}_2(z)$

with $\bar{F}_2(y)=\bar{F}(e_g, y)$ *where* $x*e_g=x$.

Then

$$\bar{F}(x, y)=\exp\left(-\lambda_1 g(x)-\lambda_2 h(y)-\lambda_{12}\max\left(g(x), h(y)\right)\right) . \tag{6.6}$$

PROOF. Put $y=h^{-1}(g(v))=\psi(v)$. Then (6.5) becomes

$$\bar{F}(x*t, \psi(g^{-1}(g(v)+g(t))))=\bar{F}(x, \psi(v))\bar{F}(t, \psi(t))$$

this is just (6.1). It is very easy to verify that conditions b) and c) of Theorem 3 are equivalent to conditions b) and c) of Lemma. The equivalence of (6.2) and (6.6) is immediate.

7. Copulae of the bivariate distributions functions

It is well known that any bivariate distribution function can be written as a function of its marginals, i.e. for any bivariate distribution function F, having marginals F_1 and F_2, there exists a function C_F such that:

$$F(x, y)=C_F(F_1(x), F_2(y)) . \tag{7.1}$$

If F is continuous, then C_F is unique. In such case

$$C_F(u, v)=\mathrm{P}(F_1(X)\leqq u,\ F_2(Y)\leqq v)\ . \tag{7.2}$$

Therefore C_F is a bivariate distribution on the unit square with uniform marginals. C_F is called copula (some references for the representation (7.1) are Sklar [13], Schweizer and Sklar [12], Chap. 6, Kimeldorf and Sampson [7] and Genest and MacKay [6]).

If we make use of (7.2), it is not difficult to see that the copula of the distribution characterized in (6.6) is the following:

$$\begin{aligned}C_F(u, v)=u+v-1+[&(1-u)^{\lambda_1/(\lambda_1+\lambda_{12})}(1-v)^{\lambda_2/(\lambda_2+\lambda_{12})}\\ &\cdot \min\left((1-u)^{\lambda_{12}/(\lambda_1+\lambda_{12})},\ (1-v)^{\lambda_{12}/(\lambda_2+\lambda_{12})}\right)]\ .\end{aligned}$$

As well as the distribution function in (6.6), this copula is a mixture of an absolutely continuous and a singular distributions. The singular part concentrates its mass on the curve:

$$u=1-(1-v)^{(\lambda_1+\lambda_{12})/(\lambda_2+\lambda_{12})} \qquad (\lambda_{12}\neq 0)\ .$$

It is interesting to note that this copula depends only on the parameters λ_1, λ_2, λ_{12}, and not on the functions g and h that characterize the associative operations in (6.5).

If $\lambda_{12}=0$ then $C_F(u, v)=uv$; this is the case of independence. If $\lambda_1=\lambda_2=0$ then $C_F(u, v)=\min(u, v)$, this is the maximum of the Fréchet class, when the whole mass is concentrated on the segment $u=v$, that is, the concordance of the two variables is maximum (see Scarsini [11]). All the other cases are intermediate, between independence and maximum concordance.

For the family of distributions (6.2) concordance is never negative. As usual in the literature, we have considered the copula of the distribution functions, that is the function that relates a distribution function to its marginals.

Obviously it is possible to show that there exists an analogous function for the survival function. Therefore

$$\bar{F}(x, y)=C_{\bar{F}}(\bar{F}_1(x),\ \bar{F}_2(y))\ .$$

If $\bar{F}$ is continuous, then

$$C_{\bar{F}}(u, v)=\Pr(\bar{F}_1(X)\leqq u,\ \bar{F}_2(Y)\leqq v)\ .$$

We call $C_{\bar{F}}$ survival copula. The survival copula of the (6.6) is:

$$C_{\bar{F}}(u, v)=u^{\lambda_1/(\lambda_1+\lambda_{12})}v^{\lambda_2/(\lambda_2+\lambda_{12})}\min\left(u^{\lambda_{12}/(\lambda_1+\lambda_{12})},\ v^{\lambda_{12}/(\lambda_2+\lambda_{12})}\right)\ .$$

8. Multivariate characterizations

The characterization provided in Section 4 can be generalized to the multivariate case.

THEOREM 4. *Let $N=\{1, 2, \cdots, n\}$ be the set of the first n integers, and let J_k be a subset of N with cardinality k. Let*

$$\bar{F}_N(x_1 * t, x_2 * t, \cdots, x_n * t)=\bar{F}_N(x_1, x_2, \cdots, x_n)\bar{F}_N(t, t, \cdots, t) \tag{8.1}$$

and

$$\bar{F}_{J_k}(x_{i_1} * t, \cdots, x_{i_n} * t)=\bar{F}_{J_k}(x_{i_1}, \cdots, x_{i_k})\bar{F}_{J_k}(t, t, \cdots, t)$$

for all subsets J_k of N, and all $k=1, 2, \cdots, n$, where $\bar{F}_{J_k}$ is the k-dimensional marginal of $\bar{F}_N$ relative to the variables with indices in J_k. Then

$$\begin{aligned}\bar{F}_N(x_1, x_2, \cdots, x_n)=\exp\{&-\sum_i \lambda_i g(x_i)-\sum_{i<j} \lambda_{ij} g(\max(x_i, x_j)) \\ &-\sum_{i<j<k} \lambda_{ijk} g(\max(x_i, x_j, x_k)), \cdots, \\ &-\lambda_{1,2,3,\ldots,n} g(\max(x_1, x_2, \cdots, x_n))\}\end{aligned} \tag{8.2}$$

where all the λ's are positive.

PROOF. The proof may be obtained by mathematical induction. Assume that the representation (8.2) holds for $(n-1)$-dimensional case. Then put in (8.1) $x_1=x_2=\cdots=x_n=x$. This gives

$$\bar{F}_N(x, x, \cdots, x)=\exp(-\delta_N g(x))$$

and

$$\bar{F}_N(x_1 * t, x_2 * t, \cdots, e * t)=\bar{F}_{N\setminus\{n\}}(x_1, x_2, \cdots, x_{n-1}) \exp(-\delta_N g(t)) .$$

Hence, setting $x_i * t=z_i$, $i=1, 2, \cdots, n$

$$\begin{aligned}&\bar{F}(z_1, z_2, \cdots, t) \\ &\quad=\exp\left\{-\sum_{i=1}^{n-1} \theta_i g(z_i)-\cdots-\theta_{1,2,\ldots,n-1} g(\max(z_1, z_2, \cdots, z_{n-1}))-\delta_N g(t)\right\}\end{aligned}$$

when $t<z_1, z_2, \cdots, z_{n-1}$. By repeating the same argument for the other variables we obtain the result.

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REFERENCES

[1] Aczél, J. (1966). *Lectures on Functional Equations and their Applications*, Academic Press, New York.

[2] Basu, A. P. and Block, H. W. (1975). On characterizing univariate and multivariate exponential distributions with applications, *Statistical Distributions in Scientific Work*, (eds. G. P. Patil, S. Kotz and J. Ord), **3**, D. Reidel, Dordrecht-Boston, 399–421.

[3] Castagnoli, E. (1978). Sulle operazioni associative tra variabili casuali, *Rivista di Matematica per le Scienze Economiche e Sociali*, **1**, 67-80.

[4] Castagnoli, E. and Muliere, P. (1984). Su una equazione funzionale e alcuni problemi di caratterizzazione, *Tech. Report No. 21*, Istituto di Matematica Finanziaria, Università di Parma.

[5] Galambos, J. and Kotz, S. (1978). *Characterizations of Probability Distributions*, Springer-Verlag, Berlin-New York.

[6] Genest, C. and MacKay, R. J. (1986). Copules archimédiennes et familles de lois bidimensionelles dont les marges sont données, *Canad. J. Statist.*, **14**, 145-159.

[7] Kimeldorf, G. and Sampson, A. R. (1978). Monotone dependence, *Ann. Statist.*, **6**, 895-903.

[8] Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution, *J. Amer. Statist. Ass.*, **62**, 30-49.

[9] Moeschberger, M. L. (1974). Life tests under dependent competing causes of failure, *Technometrics*, **16**, 39-47.

[10] Muliere, P. (1984). Una nota su operazioni associative, trasformate integrali e problemi di caratterizzazione in statistica, *Rivista di Matematica per le Scienze Economiche e Sociali*, **7**, 79-93.

[11] Scarsini, M. (1984). On measures of concordance, *Stochastica*, **8**, 201-218.

[12] Schweizer, B. and Sklar, A. (1983). *Probabilistic Metric Spaces*, North Holland, New York-Amsterdam.

[13] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges, *Publications de l'Institut de Statistique de l'Universitè de Paris*, **8**, 229-231.

[14] Wang, Y. H. (1976). A functional equation and its application to the characterization of the Weibull and stable distributions, *J. Appl. Probab.*, **13**, 385-391.