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MODES AND MOMENTS OF UNIMODAL DISTRIBUTIONS

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Summary

For a unimodal distribution relations of its mode a with its absolute moment β_p and central absolute moment γ_p of order p are considered. The best constants A_p and B_p are given for the inequalities $|a|\leqq A_p\beta_p^{1/p}$ ($p>0$) and $|a-m|\leqq B_p\gamma_p^{1/p}$ ($p\geqq 1$) where m is the mean. The results follow from discussion of more general moments.

1. Introduction

Let μ be a unimodal distribution with mode a and let β_p be its absolute moment of order $p>0$. It is shown in Sato [4] that there is a constant A_p such that

$$|a|\leqq A_p\beta_p^{1/p}\,. \tag{1.1}$$

When μ has finite mean m, the central absolute moment of order $p\geqq 1$ is denoted by γ_p. It is also shown in [4] that there is a constant B_p for $p>1$ such that

$$|a-m|\leqq B_p\gamma_p^{1/p}\,. \tag{1.2}$$

Here A_p and B_p are constants depending only on p. The latter is an extension of a result of Johnson and Rogers [3], who give (1.2) for $p=2$ and prove that $B_2=\sqrt{3}$ is the best constant. This result for $p=2$ is rediscovered by Vysochanskii and Petunin [5]. By monotonicity of $\gamma_p^{1/p}$ in p, the existence of B_p for some $p=p_0$ implies its existence for any $p\geqq p_0$. We can make a similar assertion for A_p by monotonicity of $\beta_p^{1/p}$ for $p>0$. But the case of small p is interesting, since there are many unimodal distributions that have absolute moments of order p only for small p. For example stable distributions of exponent α ($0<\alpha<2$) are unimodal and have absolute moments of order p only for $0<p<\alpha$.

Key words and phrases: Unimodal distribution, mode, moment.

In this paper we will present a new proof of (1.1) and (1.2) and give the best constants. We will show that

$$B_p=(p+1)^{1/p} \tag{1.3}$$

is the best constant in (1.2) for $p\geqq 1$ (now the case $p=1$ is included) and that the best constant in (1.1) for $p>0$ is the unique solution of the equation

$$x^{p+1}-(p+1)x-p=0 \tag{1.4}$$

for $x>1$. Thus $A_p>(p+1)^{1/p}$ for $p>0$; $A_2=2$, $A_1=1+\sqrt{2}$, and, approximately, $A_{1/2}=2.81451$.

Given a function g on the line, we call the integral $\int g(x)\mu(dx)$ the g-moment of μ, and $\int g(x-m)\mu(dx)$ the central g-moment of μ. We will give inequalities involving modes and g-moments of unimodal distributions. The bounds (1.1) and (1.2) are extended to more general moments.

2. Modes and g-moments

A distribution μ is called unimodal if there is a point a such that the distribution function of μ is convex on $(-\infty, a)$ and concave on (a, ∞). The point a is called a mode of μ. If μ is unimodal, then the set of modes of μ is either a one point set or a closed interval. Write the restriction of μ to an interval I as $\mu|_I$. A distribution μ is unimodal with mode a if and only if $\mu|_{(-\infty,a)}$ is absolutely continuous with nondecreasing density and $\mu|_{(a,\infty)}$ is absolutely continuous with nonincreasing density.

Let $g(x)$ be a nonnegative continuous function on the line such that $g(x)=g(-x)$ and $g(x)$ is increasing for $x>0$. The words *increase* and *decrease* are used in the strict sense.

THEOREM 2.1. *For every $a>0$, there is a unique point c satisfying $0<c<a$ such that, if μ is a unimodal distribution with mode a, then*

$$\int g(x)\mu(dx)\geqq(a+c)^{-1}\int_{-c}^{a} g(x)dx\ . \tag{2.1}$$

The point c is the unique point satisfying $0<c<a$ and

$$g(c)=(a+c)^{-1}\int_{-c}^{a} g(x)dx\ . \tag{2.2}$$

Equality holds in (2.1) *if and only if μ is the uniform distribution on* $[-c, a]$.

PROOF. Denote the Lebesgue measure on the line by λ. Let $a>0$ and let μ be a unimodal distribution with mode a. We estimate the g-moment of μ from below in three steps.

Step 1. Let $\alpha=\mu(-\infty, -a]+\mu[a, \infty)$ and let

$$\mu_1=\mu|_{(-a,a)}+\alpha a^{-1}\lambda|_{(0,a)} .$$

Obviously, μ_1 is a unimodal distribution with mode a. Since g is even and increasing on the positive line, the g-moment of μ_1 is smaller than or equal to that of μ. If $\mu_1\neq\mu$, then they are not equal.

Step 2. Let $\beta=\mu_1(0, a)$ and let

$$\mu_2=\mu_1|_{(-a,0)}+\beta a^{-1}\lambda|_{(0,a)} .$$

Again this is unimodal with mode a. Let $f_1(x)$ be the nondecreasing density of μ_1 on $(-a, a)$. If f_1 is flat on $(0, a)$, then $\mu_2=\mu_1$. If $f_1(0+)<f_1(a-)$, then, noting $f_1(0+)<\beta a^{-1}<f_1(a-)$ and choosing $0<a'<a$ that satisfies $f_1(a'-)\leqq\beta a^{-1}\leqq f_1(a'+)$, we have

$$\int_0^{a'} (\beta a^{-1}-f_1(x))dx=\int_{a'}^a (f_1(x)-\beta a^{-1})dx$$

and

$$\int_0^{a'} g(x)(\beta a^{-1}-f_1(x))dx<\int_{a'}^a g(x)(f_1(x)-\beta a^{-1})dx ,$$

which implies that the g-moment of μ_2 is smaller than that of μ_1.

Step 3. Let $\gamma=\mu_2(-a, 0)$ and $b=\gamma\beta^{-1}a$. Then, $0\leqq b\leqq a$. Let $\mu_3=\beta a^{-1}\lambda|_{(-b,a)}$, the uniform distribution on $[-b, a]$. Now

$$\int g(x)\mu_3(dx)\leqq\int g(x)\mu_2(dx) , \tag{2.3}$$

because, letting $f_2(x)$ be the nondecreasing density of μ_2 on $(-a, a)$, we have

$$\int_{-b}^0 (\beta a^{-1}-f_2(x))dx=\int_{-a}^{-b} f_2(x)dx$$

and

$$\int_{-b}^0 g(x)(\beta a^{-1}-f_2(x))dx\leqq\int_{-a}^{-b} g(x)f_2(x)dx .$$

Strict inequality holds in (2.3) if $\mu_3\neq\mu_2$.

Now define, for fixed a,

$$\varphi(b)=(a+b)^{-1}\int_{-b}^{a} g(x)dx\ .$$

The above three steps show that

$$\int g(x)\mu(dx)\geqq\varphi(b)$$

for some b in $[0, a]$. Here b depends on μ. Strict inequality holds unless μ is the uniform distribution on $[-b', a]$ for some $b'\in[0, a]$. As b moves on $[0, a]$, the function $\varphi(b)$ takes the minimum at a unique point c. In fact, $\varphi'(b)=(a+b)^{-2}\psi(b)$ where

$$\psi(b)=(a+b)g(-b)-\int_{-b}^{a} g(x)dx=\int_{-b}^{b}(g(b)-g(x))dx-\int_{b}^{a}(g(x)-g(b))dx\ ,$$

and $\psi(b)$ is a continuous increasing function with $\psi(0)<0$ and $\psi(a)>0$. The point c is the unique point such that $0<c<a$ and $\psi(c)=0$, which is equivalent to (2.2). The proof is complete.

Remark 2.1. Define $k(a)$ for $a>0$ by $k(a)=c$ in Theorem 2.1 and $k(0)=0$. We see that, if μ is unimodal with mode a, then

$$\int g(x)\mu(dx)\geqq g(k(|a|))\ .$$

Let $M=\sup g(x)\leqq\infty$. The equation (2.2) shows that

$$\int_{-c}^{c}(g(c)-g(x))dx=\int_{c}^{a}(g(x)-g(c))dx\ .$$

As a increases, c must increase in order to satisfy this identity. That is, $k(x)$ is increasing in x. Now it is easy to see that $k(x)$ is a continuous increasing function from $[0, \infty)$ onto itself. Hence $g(k(x))$ is a continuous increasing function from $[0, \infty)$ onto $[g(0), M)$. Let $x=h_1(y)$ be the inverse function of $y=g(k(x))$. If μ is unimodal with mode a, then

$$|a|\leqq h_1\left(\int g(x)\mu(dx)\right)\ .$$

For every $y\geqq g(0)$, $h_1(y)$ is the supremum of modes taken over all unimodal distributions that have g-moment y. In fact, for $x=h_1(y)$, the uniform distribution on $[-k(x), x]$ has g-moment y.

3. Modes and absolute moments of order $p>0$

The preceding theorem has the following consequence.

THEOREM 3.1. *For $p>0$, let A_p be the unique solution of the equa-*

tion (1.4) *in* $(1, \infty)$. *If* μ *is unimodal with mode* a, *then*

$$|a| \leqq A_p \beta_p^{1/p}, \tag{3.1}$$

where $\beta_p = \int |x|^p \mu(dx)$. *Equality holds in* (3.1) *if and only if* μ *and* a *satisfy one of the following*:
(i) $a=0$ *and* μ *is the* δ*-distribution at* 0;
(ii) $a>0$ *and* μ *is the uniform distribution on* $[-a/A_p, a]$;
(iii) $a<0$ *and* μ *is the uniform distribution on* $[a, -a/A_p]$.

PROOF. Let $a>0$. It is enough to prove the theorem in this case. Let $g(x)=|x|^p$ in Theorem 2.1. Then

$$\int |x|^p \mu(dx) \geqq c^p,$$

where c is the unique solution of the equation

$$(a+c)c^p-(p+1)^{-1}(a^{p+1}+c^{p+1})=0$$

for $0<c<a$. We see that $1/A_p$ is the value of c for $a=1$. The value of c for general $a>0$ is $c=a/A_p$. Hence we obtain (3.1). Equality holds in (3.1) if and only if (ii) holds, as Theorem 2.1 says. The proof is complete.

Remark 3.1. Let A be the unique positive solution of the equation

$$x \log x - x - 1 = 0. \tag{3.2}$$

The constant A_p decreases as p increases, and

$$\lim_{p \downarrow 0} A_p = A, \qquad \lim_{p \uparrow \infty} A_p = 1. \tag{3.3}$$

It is easily seen from (3.2) that $e<A<2e$. An approximate value is $A=3.59112$.

In fact, let $a>0$ and let $0<p<p'$. We have

$$|a| \leqq A_p \beta_p^{1/p} < A_p \beta_{p'}^{1/p'}$$

unless μ is concentrated at a (see Hardy et al. [2], p. 157). Choose μ to be the uniform distribution on $[-a/A_{p'}, a]$. Then $a=A_{p'}\beta_{p'}^{1/p'}$. Hence we have $A_{p'}<A_p$. If $\lim_{p\uparrow\infty} A_p>1$, then $1-(p+1)A_p^{-p}-pA_p^{-p-1}=0$ leads to a contradiction. Therefore A_p tends to 1 as $p \uparrow \infty$. If we fix $x>1$ and let p decrease to 0, then

$$x^{p+1}-(p+1)x-p=p(x \log x - x - 1)+O(p^2).$$

Hence, if $1<x<A$, then $x^{p+1}-(p+1)x-p$ is negative for small p; if

$x>A$, then it is positive for small p. This shows that $\lim_{p\downarrow 0} A_p=A$.

Remark 3.2. If the integral $\int \log|x|\mu(dx)$ exists, the geometric mean g of μ is defined by

$$\mathrm{g}=\exp\int \log|x|\mu(dx)\ .$$

If μ is unimodal with mode a and $\int \log|x|\mu(dx)<\infty$, then

$$|a|\leqq A\mathrm{g}\ , \tag{3.4}$$

where A is given in Remark 3.1. In fact, if μ has finite β_p for some $p>0$, then $\beta_p^{1/p}$ tends to g as $p\downarrow 0$ (see [2], p. 156) and we have (3.4) from (3.1) and (3.3). If μ has infinite β_p for every $p>0$, then consider μ_n defined by

$$\mu_n=\mu|_{(-n,n)}+\alpha_n\delta_a\ , \qquad \alpha_n=\mu(-\infty, -n]+\mu[n, \infty)$$

and note that the geometric mean of μ_n tends to g as $n\to\infty$.

4. Modes and central g-moments

In this section let $g(x)$ be a nonnegative function such that $g(x)=g(-x)$ and $g(x)/x$ is nondecreasing in $x>0$.

THEOREM 4.1. *If μ is unimodal with mode a and has finite mean m and if $m\neq a$, then*

$$\int g(x-m)\mu(dx)\geqq 2^{-1}|a-m|^{-1}\int_{-|a-m|}^{|a-m|} g(x)dx\ . \tag{4.1}$$

Equality holds in (4.1) *if and only if μ is the uniform distribution on an interval with a chosen to be an endpoint of the interval.*

PROOF. Let μ be unimodal with mode a with mean m and $m\neq a$. By translation and reflection, we may assume $m=0$ and $a>0$. We estimate the g-moment of μ from below by the g-moment of another unimodal distribution with mode a and mean 0.

Step 1. Let $\alpha=\mu[a, \infty)$ and let

$$\mu_1=\mu|_{(-\infty,a)}+\alpha a^{-1}\lambda|_{(0,a)}\ .$$

Then μ_1 is unimodal with mode a. If $\mu_1\neq\mu$, then the g-moment of μ_1 is smaller than that of μ and the mean of μ_1 is negative.

Step 2. Let $\beta=\mu_1(0, a)$ and

$$\mu_2=\mu_1|_{(-\infty,0)}+\beta a^{-1}\lambda|_{(0,a)}\ .$$

As in Step 2 in the proof of Theorem 2.1, we see that μ_2 is unimodal with mode a and that, if $\mu_2\neq\mu_1$, then μ_2 has smaller g-moment and mean than μ_1.

Step 3. If $\mu_2=\mu$, then let $\mu_3=\mu$. Suppose that $\mu_2\neq\mu$. We see that $\mu_2|_{(-a,a)}$ has positive mean, since it does not have flat density. On the other hand, μ_2 has negative mean. So we can find $b>a$ such that

$$\mu_3=\mu_2|_{(-b,a)}+\mu_2(-\infty,\ -b)a^{-1}\lambda|_{(0,a)}$$

has zero mean. Obviously μ_3 is unimodal with mode a and its g-moment is smaller than that of μ_2.

Step 4. We have $\mu_3(0,a)\geqq 1/2$, since μ_3 is unimodal with mode a, concentrated on $(-\infty,a)$ and has zero mean. The case $\mu_3(0,a)=1/2$ occurs if and only if μ_3 is the uniform distribution on $[-a,a]$. Let $f_3(x)$ be the density of μ_3. We have $f_3(x)=\gamma$ on $(0,a)$ for some constant $\gamma\geqq(2a)^{-1}$. We claim that

$$\int_{-\infty}^{0} g(x)f_3(x)dx\geqq\gamma\int_{-a}^{0} g(x)dx\ . \tag{4.2}$$

This will imply

$$\int g(x)\mu_3(dx)\geqq\gamma\int_{-a}^{a} g(x)dx\geqq(2a)^{-1}\int_{-a}^{a} g(x)dx\ ,$$

from which (4.1) follows. First note that

$$\int_{-\infty}^{-a} xf_3(x)dx=\int_{-a}^{0} x(\gamma-f_3(x))dx\ .$$

Using this and increasingness of $g(x)/x$ in $x>0$, we have

$$\begin{aligned}\int_{-\infty}^{-a} g(x)f_3(x)dx&\geqq g(a)a^{-1}\int_{-\infty}^{-a}|x|f_3(x)dx\\&=g(a)a^{-1}\int_{-a}^{0}|x|(\gamma-f_3(x))dx\geqq\int_{-a}^{0} g(x)(\gamma-f_3(x))dx\ .\end{aligned}$$

Thus (4.2) follows. This proof shows that equality holds in (4.1) only if μ is the uniform distribution on an interval with a chosen to be an endpoint of the interval. As the converse statement for the equality is obvious, proof of the theorem is complete.

Remark 4.1. Define $l(x)$ by $l(x)=(2x)^{-1}\int_{-x}^{x} g(y)dy$ for $x>0$ and $l(0)=g(0+)$. Now, if μ is unimodal with mode a and has finite mean m, then

$$\int g(x-m)\mu(dx) \geqq l(|a-m|) .$$

Noting that $l(x)$ is a continuous increasing function from $[0, \infty)$ onto $[g(0+), \infty)$, let $x=h_2(y)$ be the inverse function of $y=l(x)$. Then

$$|a-m| \leqq h_2\left(\int g(x-m)\mu(dx)\right) .$$

If $y \geqq g(0+)$, then $h_2(y)$ is the supremum of modes taken over all unimodal distributions that have mean 0 and g-moment y. In fact, for $x=h_2(y)$, the uniform distribution on $[-x, x]$ has mean 0 and g-moment y.

Remark 4.2. The assumption of nondecreasingness of $g(x)/x$ in $x>0$ in Theorem 4.1 cannot be replaced by nondecreasingness of $g(x)$ in $x>0$. For example, let $g(x)=|x|^p$ with $0<p<1$ and choose $\mu(dx)=f(x)dx$ with $a=1$ and $m=0$ in the form $f(x)=\alpha c$ on $[-b, -1/2)$, α on $[-1/2, 1]$ and 0 outside of $[-b, 1]$ where $b>1/2$, $\alpha>0$, $0<c<1$. Then $\int g(x)\mu(dx)$ is smaller than $(p+1)^{-1}$ when b is sufficiently large.

5. Modes and central absolute moment of order $p \geqq 1$

We apply Theorem 4.1 to central absolute moments.

THEOREM 5.1. *Let $p \geqq 1$. If μ is unimodal with mode a and has finite mean m, then*

$$|a-m| \leqq (p+1)^{1/p}\gamma_p^{1/p} , \tag{5.1}$$

where $\gamma_p=\int |x-m|^p\mu(dx)$. Equality holds in (5.1) *if and only if μ is a δ-distribution or a uniform distribution on an interval with a chosen to be an endpoint of the interval.*

PROOF. If $a=m$, then (5.1) is trivial. If $a \neq m$, then, using Theorem 4.1 for $g(x)=|x|^p$, we get

$$\gamma_p \geqq (p+1)^{-1}|a-m|^p ,$$

that is (5.1). The statement about the case of the equality also follows from the theorem.

Remark 5.1. The coefficient $(p+1)^{1/p}$ in (5.1) decreases from 2 to 1 as p increases from 1 to ∞.

6. Modes and exponential moments

Let us consider exponential moments.

THEOREM 6.1. *Let $g(x)=e^{|x|}-1$. For $y\geqq 0$, let $h_1(y)$ be the supremum of modes taken over all unimodal distributions that have g-moment y, and let $h_2(y)$ be the supremum of modes taken over all unimodal distributions with g-moment y and mean 0. Then,*

$$h_1(y)=\log y+\log\log y+\log 2+(2^{-1}+o(1))(\log y)^{-1}\log\log y\ ,$$

$$h_2(y)=\log y+\log\log y+(1+o(1))(\log y)^{-1}\log\log y$$

as $y\to\infty$.

PROOF. By Remark 2.1, the function $x=h_1(y)$ for $y>0$ is given by $c=\log(y+1)$ and by the equation

$$(x+c-1)e^c-e^x+2=0$$

with the condition $x>c$. The function $x=h_2(y)$ is, according to Remark 4.1, the inverse function of $y=(e^x-x-1)/x$, $x>0$. Hence, by the method of asymptotic expansion (see Dieudonné [1], III. 8), we can prove that $h_1(y)$ and $h_2(y)$ behave as in the statement of the theorem.

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