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SOME SUFFICIENT CONDITIONS FOR THE E- AND MV-OPTIMALITY OF BLOCK DESIGNS HAVING BLOCKS OF UNEQUAL SIZE

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Summary

In this paper we consider experimental situations in which v treatments are to be tested in b blocks where b_i blocks contain k_i experimental units, $i=1,\cdots,p$, $k_1<k_2<\cdots<k_p$. The idea of a group divisible (GD) design is extended to that of a group divisible design with unequal block sizes (GDUB design) and then a number of results concerning the E- and MV-optimality of GD designs are generalized to the case of GDUB designs.

1. Introduction

In this paper we consider experimental situations in which v treatments are to be applied to experimental units occurring in b blocks where b_i blocks contain k_i experimental units, $i=1,\cdots,p$, $\sum_{i=1}^{p} b_i=b$ and $k_1<\cdots<k_p$. We shall use d to denote some block design which can be used in such a setting and $N(d)$ to denote the $v\times b$ incidence matrix of d whose entries $n_{ij}(d)$ are nonnegative integers indicating how many experimental units treatment i is applied to in block j. If $n_{ij}(d)=0$ or 1 for all $i=1,\cdots,v$, $j=1,\cdots,b$, then d is said to be a binary design. The i-th row sum and j-th column sum of $N(d)$ will be denoted by $r_i(d)$ and $k_j(d)$. The matrix $N(d)\cdot N(d)'$ ($N(d)'$ is the matrix transpose of $N(d)$) is referred to as the concurrence matrix of d and its entries are denoted by $\lambda_{ij}(d)$. The statistical model assumed here for analyzing the data obtained from a given design d is the additive two-way classification model. This model specifies that an observation Y_{stu} (the u-th observation on treatment s in block t) can be expressed as

$$Y_{stu}=\alpha_s+\beta_t+E_{stu}\,,\quad s=1,\cdots,v\,,\quad t=1,\cdots,b\,,\quad u=0,\cdots,n_{st}(d)\,,$$

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where α_s and β_t are constants representing the effects of treatment s and block t, respectively, and E_{stu} is a random variable having expectation zero. It is assumed that the variability of the observations obtained in a given block of d is proportional to the size of the block. Under this model, the coefficient matrix of the reduced normal equations for obtaining the generalized least squares estimates of the treatment effects can be written in matrix form as

$$C(d)=\text{diag}\,(r_1(d),\cdots,r_v(d))-N(d)\,\text{diag}\,[k_1(d)^{-1},\cdots,k_b(d)^{-1}]\cdot N(d)'\,, \tag{1.1}$$

where $\text{diag}\,(a_1,\cdots,a_n)$ denotes an $n\times n$ diagonal matrix. The $v\times v$ matrix $C(d)$ is usually referred to as the C-matrix of d and is known to be positive semi-definite with zero row sums.

It is well known that any estimable linear combination $l'\alpha=\sum_{i=1}^{v} l_i\alpha_i$ of the treatment effects must have $\sum_{i=1}^{v} l_i=0$, i.e., it must be a treatment contrast. A block design in which all treatment contrasts are estimable is said to be connected and it is known that a block design is connected if and only if its C-matrix has rank $v-1$. For the experimental situations we will be considering, connectedness is a desirable property for a block design to have, hence we will restrict our attention to designs possessing this property and use $D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$ to denote the class of all connected block designs having v treatments arranged in b blocks where b_s blocks are of size k_s, $s=1,\cdots,p$, $\sum_{s=1}^{p} b_s=b$ and $k_1<k_2<\cdots<k_p$. For a given design $d\in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$, we shall assume that the columns of $N(d)$ have been arranged so that columns 1 through b_1 correspond to the blocks of size k_1, columns b_1+1 through b_1+b_2 to the blocks of size k_2, etc. With this in mind we shall also let $N(d_s)$ denote the portion of $N(d)$ corresponding to the b_s blocks of size k_s, and let d_s denote the block design having $N(d_s)$ as its incidence matrix. Then, it follows from (1.1) that

$$N(d)=(N(d_1),\,N(d_2),\cdots,N(d_p))\,,$$

and

$$\begin{aligned} kC(d)&=\text{diag}\,[r_1(d)k,\cdots,r_v(d)k]-k\sum_{s=1}^{p}(1/k_s)N(d_s)N(d_s)' \\ &=k\sum_{s=1}^{p}\{\text{diag}\,[r_1(d_s),\cdots,r_v(d_s)]-(1/k_s)N(d_s)N(d_s)'\} \\ &=k\sum_{s=1}^{p}C(d_s) \end{aligned} \tag{1.2}$$

where $k=\prod_{s=1}^{p}k_s$, $r_i(d_s)$ is the i-th row sum of $N(d_s)$ and $C(d_s)$ is the C-matrix of d_s. We also note that if $C(d)=(c_{ij}(d))$, then

(i) $kc_{ii}(d)=k\sum_{s=1}^{p}(r_i(d_s)k_s-\lambda_{ii}(d_s))/k_s\,,$ for $i=1,\cdots,v\,,$

(1.3) and

(ii) $-kc_{ij}(d)=k\sum_{s=1}^{p}\lambda_{ij}(d_s)/k_s\,,$ for $i,j=1,\cdots,v\,,\ i\neq j\,,$

where $\lambda_{ij}(d_s)$ is the (i, j)-th entry of $N(d_s)N(d_s)'$.

The types of designs of primary concern in this paper will be referred to as group divisible designs with unequal block sizes (GDUB designs).

DEFINITION 1.4. Let $d\in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$ have C-matrix $C(d)=(c_{ij}(d))$ and let $k=\prod_{s=1}^{p}k_s$. Then d is called a GDUB design with parameters m, n, $\lambda_1(d)$ and $\lambda_2(d)$ if the treatments $1,2,\cdots,v$ can be divided into m mutually disjoint groups $V_1,\cdots,V_m$ of size $n=v/m$ such that;

(i) if $i,j\in V_s$, $i\neq j$, $kc_{ij}(d)=-\lambda_1(d)$ for some constant $\lambda_1(d)$
(ii) if $i\in V_s$, $j\in V_t$, $s\neq t$, then $kc_{ij}(d)=-\lambda_2(d)$ for some constant $\lambda_2(d)$.

Comment 1.5. The reader should note that the class of GDUB designs defined in Definition 1.4 contains many of the well-known standard designs as special cases. In particular we note the following special cases;

(i) Upon taking $\lambda_1(d)=\lambda_2(d)=\lambda(d)$ in Definition 1.4, the definition of a GDUB design reduces to that of a variance balanced design.
(ii) Upon taking $p=1$ in Definition 1.4 and requiring that d be binary, then the definition of a GDUB design reduces to that of a standard group divisible (GD) design.
(iii) Upon taking $p=1$, $\lambda_1(d)=\lambda_2(d)=\lambda(d)$ and requiring that d be binary in Definition 1.4, the definition of a GDUB design reduces to that of a balanced incomplete block (BIB) design.

For further information concerning and specific definitions for variance balanced, GD and BIB designs, the reader is referred to Raghavarao [13]. Some further facts concerning GDUB designs are given in Section 2.

In this paper we consider the determination and construction of E- and MV-optimal block designs in classes $D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$.

DEFINITION 1.6. For $d\in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$, let

$$0=\mu_0(d)<\mu_1(d)\leqq\cdots\leqq\mu_{v-1}(d)$$

denote the eigenvalues of $C(d)$. A design d^* is said to be E-optimal in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ if for any other design $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$,

$$\mu_1(d^*) \geqq \mu_1(d) .$$

Ehrenfeld [6] proved that if $d^* \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ is E-optimal, then for any other design $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$,

$$\max_{\substack{l'J_{v1}=0 \\ l'l=1}} \mathrm{var}_{d^*}(l'\hat{\alpha}) \leqq \max_{\substack{l'J_{v1}=0 \\ l'l=1}} \mathrm{var}_d(l'\hat{\alpha})$$

where J_{mn} denotes an $m \times n$ matrix of ones and $\mathrm{var}_d(l'\hat{\alpha})$ denotes the variance of the least squares estimator $l'\hat{\alpha}$ for the estimable function $l'\alpha$ under d.

DEFINITION 1.7. A design $d^* \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ is said to be MV-optimal if for any other $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$,

$$\max_{i \neq j} \mathrm{var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) \leqq \max_{i \neq j} \mathrm{var}_d(\hat{\alpha}_i - \hat{\alpha}_j) .$$

A number of results are already known concerning the E- and MV-optimality of block designs in classes $D(v; b_1; k_1)$, i.e., see Cheng [1], Constantine [4], [5], Jacroux [7]–[11] and Takeuchi [14], [15]. However, the only result known to the author about the E- and MV-optimality of block designs in classes $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ follows from a result proven by Kiefer [12] and can be stated in the following way.

THEOREM 1.8. *If $d^* \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ is variance balanced and has $C(d^*)$ with maximal trace among all designs in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$, then d^* is both E- and MV-optimal in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.*

In this paper we further consider the determination and construction of E- and MV-optimal designs in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$. In Sections 3 and 4, several well-known results concerning the E- and MV-optimality of certain types of GD designs are extended to the case of GDUB designs in classes $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.

2. Preliminary facts and lemmas

In this section, we derive several preliminary facts concerning arbitrary designs in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ and GDUB designs in particular.

LEMMA 2.1. *Let $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ be an arbitrary design with C-matrix $C(d) = (c_{ij}(d))$, and let $k = \prod_{s=1}^{p} k_s$. Then the following* *facts hold for* d.

(i) *For any* $i=1,\cdots,v$,

$$c_{ii}(d)\leqq\sum_{s=1}^{p} r_i(d_s)(k_s-1)/k_s\,,$$

with equality if and only if d *is a binary design.*

(ii) *If we let* $\bar{c}=\sum_{s=1}^{p} b_s k(k_s-1)/v$, *then*

$$\sum_{i=1}^{v} c_{ii}(d)\leqq\sum_{i=1}^{v}\sum_{s=1}^{p} r_i(d_s)(k_s-1)/k_s=\sum_{s=1}^{p} b_s(k_s-1)=v\bar{c}/k$$

with equality if and only if d *is binary.*

(iii) *If we let* $\bar{c}$ *be defined as in* (ii), *then*

$$\min_{1\leqq i\leqq v} kc_{ii}(d)\leqq\bar{c}\,.$$

PROOF. We shall only prove (i) since the proofs of (ii) and (iii) are similar.

(i) Simply observe that

$$\begin{aligned}c_{ii}(d)&=\sum_{s=1}^{p} c_{ii}(d_s)=\sum_{s=1}^{p}\{(r_i(d_s)k_s-\lambda_{ii}(d_s))/k_s\}\\&=\sum_{s=1}^{p}\left\{r_i(d_s)k_s-\left(\sum_{j=1}^{b_s} n_{ij}(d_s)^2\right)\Big/k_s\right\}\\&\leqq\sum_{s=1}^{p}\left\{r_1(d_s)k_s-\left(\sum_{j=1}^{b_s} n_{ij}(d_s)\right)\Big/k_s\right\}\\&=\sum_{s=1}^{p}\{r_i(d_s)(k_s-1)/k_s\}\end{aligned}$$

where the inequality follows because all the $n_{ij}(d_s)$ are nonnegative integers and the last equality follows because $\sum_{i=1}^{v} r_i(d)=\sum_{j=1}^{b} k_j(d)$. Clearly we will have equality in the above expression if and only if $n_{ij}(d_s)=0$ or 1 for all $i=1,\cdots,v$, $j=1,\cdots,b_s$ and $s=1,\cdots,p$.

For the remainder of this section, we shall be considering GDUB designs. In all further discussion of GDUB designs, it will be assumed that the treatments $1,2,\cdots,v$ have been labeled so that treatments $1,\cdots,n$ form the first group, treatments $n+1,\cdots,2n$ the second group, etc.

Our next lemma contains some basic facts about GDUB designs which are used in the next section. Since these facts are easily proven, the lemma is stated without proof.

LEMMA 2.2. *Suppose* $d\in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$ *is a GDUB design with C-matrix* $C(d)=(c_{ij}(d))$, *parameters* m, n, $\lambda_1(d)$, $\lambda_2(d)$ *and let*

$k=\prod_{s=1}^{p} k_s$. *Then the following facts hold for d;*

(i) $\lambda_2(d)>0$, i.e., $\lambda_2(d)=1, 2, 3, \cdots$,

(ii) $\lambda_1(d)=\sum_{s=1}^{p} \lambda_{12}(d_s)\cdot(k/k_s)$,

(iii) $\lambda_2(d)=\sum_{s=1}^{p} \lambda_{1,n+1}(d_s)\cdot(k/k_s)$.

(iv) *For* $i=1, 2, \cdots, v$, *and some constant* $\bar{c}$,

$$\begin{aligned} kc_{ii}(d)=\bar{c} &= (n-1)\lambda_1(d)+(m-1)n\lambda_2(d) \\ &= (v-1)\lambda_1(d)+(m-1)n(\lambda_2(d)-\lambda_1(d)) \\ &= (v-1)\lambda_2(d)+(n-1)(\lambda_1(d)-\lambda_2(d)) , \end{aligned}$$

(v) $v\bar{c} \leqq k\sum_{s=1}^{p} b_s(k_s-1)$, *with equality if and only if d is binary.*

(vi) *If* $|\lambda_1(d)-\lambda_2(d)|\leqq 1$, *then* $\min\{\lambda_1(d), \lambda_2(d)\}=[\bar{c}/(v-1)]$ *where* $[\cdot]$ *denotes the greatest integer function.*

(vii) *The eigenvalues of* $C(d)$ *are*

0 *occurring with multiplicity* 1,

$(\bar{c}+\lambda_1(d))/k$ *occurring with multiplicity* $m(n-1)$, *and*

$v\lambda_2(d)/k$ *occurring with multiplicity* $m-1$.

(viii) *If* $\lambda_1(d)\leqq\lambda_2(d)$, *then* $\mu_1(d)=(\bar{c}+\lambda_1(d))/k$, *and if* $\lambda_1(d)\lambda_2(d)$, *then* $\mu_1>(d)=v\lambda_2(d)/k$.

(ix) *If treatments i and j occur in the same group*

$$\mathrm{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)=2k\sigma^2/(\bar{c}+\lambda_1(d))$$

and if treatments i and j occur in different groups,

$$\mathrm{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)=(2k\sigma^2/(\bar{c}+\lambda_1(d)))(1-(\lambda_2(d)-\lambda_1(d))/(mn\lambda_2(d))) .$$

(x) *If* $\lambda_1(d)\leqq\lambda_2(d)$, *then*

$$\max_{i\neq j} \mathrm{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)=2k\sigma^2/(\bar{c}+\lambda_1(d)) ,$$

and if $\lambda_1(d)>\lambda_2(d)$, *then*

$$\max_{i\neq j} \mathrm{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)=(2k\sigma^2/(\bar{c}+\lambda_1(d)))(1-(\lambda_2(d)-\lambda_1(d))/(mn\lambda_2(d))) .$$

3. Results on the E-optimality of GDUB designs

In this section, we give our main results concerning the E-optimality of GDUB designs in classes $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$. For convenience, we shall assume throughout the sequel that

$$k_s \leqq v \quad \text{for all} \quad s=1, 2, \cdots, p.$$

We begin by stating a result which can be proven using techniques analogous to those used in Constantine [4] or Jacroux [9].

LEMMA 3.1. *Suppose* $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ *has C-matrix* $C(d)=(c_{ij}(d))$.

(i) *For any* $i \neq j$,

$$\mu_1(d) \leqq (c_{ii}(d)+c_{jj}(d)-2c_{ij}(d))/2.$$

(ii) *If M is some proper subset of m treatments out of* $1, 2, \cdots, v$, *then*

$$\mu_1(d) \leqq (v/(m(v-m)))(\sum_{i \in M} c_{ii}(d)+\sum_{i \in M} \sum_{\substack{j \in M \\ j \neq i}} c_{ij}(d)).$$

Our first theorem concerning the E-optimality of GDUB designs is a natural extension of a result proved by Takeuchi [14] for GD designs.

THEOREM 3.2. *For* $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$, *let z denote the greatest common divisor for* $k/k_1, \cdots, k/k_p$ *and suppose* $k/k_s = zz_s$, $s=1, \cdots, p$. *If* $d^* \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ *is a GDUB design which is binary and has* $\lambda_2(d^*)=\lambda_1(d^*)+z$, *then* d^* *is E-optimal in* $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.

PROOF. For any $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ and any $1 \leqq i \leqq v$, observe that

$$\begin{aligned} kc_{ii}(d) &= \sum_{s=1}^{p} \{(r_i(d_s)kk_s - \lambda_{ii}(d_s)k)/k_s\} \\ &= z \sum_{s=1}^{p} \{r_1(d_s)z_s k_s - \lambda_{ii}(d_s)z_s\}, \end{aligned}$$

and for all $i \neq j$

$$-kc_{ij}(d) = z \sum_{s=1}^{p} \lambda_{ij}(d_s)z_s.$$

Hence every entry of $C(d)$ can be written as a multiple of z and corresponding to each $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ we have a matrix $\tilde{C}(d) =(1/z)C(d)$. Clearly the matrix $k\tilde{C}(d^*)$ corresponding to the GDUB design d^* satisfying the conditions of the theorem has integral off-diagonal entries $\tilde{\lambda}_1(d^*)=\lambda_1(d^*)/z$ and $\tilde{\lambda}_2(d^*)=\lambda_2(d^*)/z$ such that $\tilde{\lambda}_2(d^*)=\tilde{\lambda}_1(d^*)+1$. Also it should be observed that for any $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$, the eigenvalues of $\tilde{C}(d)$, denoted by

$$0=\tilde{\mu}_0(d)<\tilde{\mu}_1(d) \leqq \cdots \leqq \tilde{\mu}_{v-1}(d),$$

are related to those of $C(d)$ by

$$\tilde{\mu}_i(d)=\mu_i(d)/z\,, \qquad \text{for} \quad i=1, 2, \cdots, v-1\,.$$

Hence, to prove the theorem, it is enough to show that for any $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$,

$$\tilde{\mu}_1(d) \leqq \tilde{\mu}_1(d^*)\,.$$

From these arguments, for purposes of proving the theorem, we can assume without loss of generality that $z=1$, i.e., that $k/k_1, \cdots, k/k_p$ are setwise relatively prime. We shall assume this to be the case for the remainder of this proof.

So let $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ be arbitrary with C-matrix $C(d) =(c_{ij}(d))$. We now consider several cases for $C(d)$.

Case 1. Suppose $kc_{ii}(d)<\bar{c}$ for some i. Without loss of generality, we may assume

$$kc_{11}(d) \leqq \bar{c}-1\,.$$

By Lemma 3.1 (i), for each $j \geqq 2$,

$$\mu_1(d) \leqq (c_{11}(d)+c_{jj}(d)-2c_{1j}(d))/2\,,$$

and by Lemma 2.2 (viii),

$$\mu_1(d^*)=(\bar{c}+\lambda_1(d^*))/k\,.$$

Thus for d to have $\mu_1(d)>\mu_1(d^*)$, it must have for each $j \geqq 2$,

$$\bar{c}+\lambda_1(d^*)<k(c_{11}(d)+c_{jj}(d)-2c_{1j}(d))/2\,,$$

which implies that for each $j \geqq 2$, d must have

$$\bar{c}+\lambda_1(d^*)+(1/2) \leqq k(c_{11}(d)+c_{jj}(d)-2c_{1j}(d))/2\,.$$

Now, using Lemma 2.1 (ii), Lemma 2.2 (iv) and the above arguments, it also follows that d must have

$$\begin{aligned}(v-1)(\bar{c}+\lambda_1(d^*)+(1/2)) &= v\bar{c}-(v/2)+(2n-1)/2 \\ &\leqq (k/2)\sum_{j=2}^{v}(c_{11}(d)+c_{jj}(d)-2c_{1j}(d)) \\ &=(k/2)\left\{(v-1)c_{11}(d)+\sum_{j=2}^{v}c_{jj}(d)-2\sum_{j=2}^{v}c_{1j}(d)\right\} \\ &=(1/2)\left\{vkc_{11}(d)+k\sum_{j=1}^{v}c_{jj}(d)\right\} \\ &\leqq (1/2)\{v(\bar{c}-1)+v\bar{c}\} \\ &=v\bar{c}-(v/2)\,,\end{aligned}$$

which is a contradiction since $n \geqq 2$. Thus, in this case, d must have $\mu_1(d) \leqq \mu_1(d^*)$.

Case 2. Suppose $kc_{ii}(d) \geqq \bar{c}$ for $i=1, \cdots, v$. If $kc_{ii}(d) \geqq \bar{c}$ for $i=1, \cdots, v$, then it follows that

$$kc_{ii}(d) = \bar{c} \qquad \text{for} \quad i=1, \cdots, v$$

since by Lemma 2.1 (ii), $k \sum_{i=1}^{v} c_{ii}(d) \leqq v\bar{c}$. Now, for d to have $\mu_1(d) > \mu_1(d^*)$, it must have by Lemma 3.1 (i), for all $i \neq j$,

$$k(c_{ii}(d) + c_{jj}(d) - 2c_{ij}(d))/2 = k\bar{c} - kc_{ij}(d) \geqq \bar{c} + \lambda_1(d^*) + 1 .$$

But then, using Lemma 2.1 (ii), Lemma 2.2 (iv) and the inequality given above, this implies that d must have

$$\begin{aligned}(v-1)(\bar{c} + \lambda_1(d^*) + 1) &= v\bar{c} + (n-1) \\ &\leqq (k/2) \sum_{j=2}^{v} (c_{11}(d) + c_{jj}(d) - 2c_{1j}(d)) \\ &= \sum_{j=2}^{v} (\bar{c} - kc_{1j}(d)) = (v-1)\bar{c} - k \sum_{j=2}^{2} c_{1j}(d) = v\bar{c} ,\end{aligned}$$

which is a contradiction. Thus d cannot have $\mu_1(d) > \mu_1(d^*)$, and the result follows.

Example 3.3. Consider the class of designs $D(6; 4, 3; 3, 4)$, and let d_1 and d_2 be the GD designs SR18 and S1 given in Clatworthy [3], i.e., d_1 is a GD design having parameters $v=6$, $b_1=4$, $k_1=3$, $r(d_1) \equiv b_1k_1/v = 2$, $m=3$, $n=2$, $\lambda_1(d_1)=0$ and $\lambda_2(d_1)=1$, whereas d_2 is a GD design having parameters $v=6$, $b_2=3$, $k_2=4$, $r(d_2) \equiv b_2k_2/v=2$, $m=3$, $n=2$, $\lambda_1(d_2)=2$ and $\lambda_2(d_2)=1$. Now let d^* be the design having $N(d^*)=(N(d_1), N(d_2))$ and observe that $z=1$ is the greatest common divisor for $k_1=3$ and $k_2=4$ and that d^* is a binary GDUB design in $D(6; 4, 3; 3, 4)$ having parameters $m=3$, $n=2$ and

$$\lambda_1(d^*) = k_2\lambda_1(d_1) + k_1\lambda_1(d_2) = 4 \cdot 0 + 3 \cdot 2 = 6$$

$$\lambda_2(d^*) = k_2\lambda_2(d_1) + k_1\lambda_2(d_2) = 4 \cdot 1 + 3 \cdot 1 = 7 .$$

Since $\lambda_2(d^*) = \lambda_1(d^*) + z$, we have by Theorem 3.2 that d^* is E-optimal in $D(6; 4, 3; 3, 4)$.

The next two theorems are extensions of results proven in Cheng [2] and Jacroux [11]. Since the proofs of these theorems are quite similar to the proof of Theorem 3.2, the results are stated without proof.

THEOREM 3.4. *For $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$, let z and z_s be as de-*

fined in Theorem 3.2 *for* $s=1,\cdots,p$. *If* $d^* \in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$ *is a GDUB design which is binary and has* $m=v/2$, $n=2$ *and* $\lambda_2(d^*)=\lambda_1(d^*)-z$, *then* d^* *is E-optimal in* $D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$.

THEOREM 3.5. *For* $D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$, *let* z *and* z_s *be as defined in Theorem* 3.2 *for* $s=1,\cdots,p$. *If* $d^* \in D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$ *is a GDUB design which is binary and has* $m=2$, $n=v/2$ *and* $\lambda_2(d^*)=\lambda_1(d^*)+2z$, *then* d^* *is E-optimal in* $D(v;b_1,\cdots,b_p;k_1,\cdots,k_p)$.

We now give two examples illustrating the usage of these last two theorems.

Example 3.6. Consider the class of designs $D(10;40,25;2,4)$, and let d_1 and d_2 be the GD designs R36 and R108 given in Clatworthy [3], i.e., d_1 is a GD design having parameters $v=10$, $b_1=40$, $k_1=2$, $r(d_1)\equiv b_1k_1/v=8$, $m=5$, $n=2$, $\lambda_1(d_1)=0$, $\lambda_2(d_1)=1$ and d_2 is a GD design having parameters $b_2=25$, $k_2=4$, $r(d_2)\equiv b_2k_2/v=10$, $m=5$, $n=2$, $\lambda_1(d_2)=6$ and $\lambda_2(d_2)=3$. Now let d^* be that design having $N(d^*)=(N(d_1),N(d_2))$ and observe that $z=2$ is the greatest common divisor for $k_1=2$ and $k_2=4$ and that d^* is a binary GDUB design in $D(10;40,25;2,4)$ having parameters $m=5$, $n=2$ and

$$\lambda_1(d^*)=k_2\lambda_1(d_1)+k_1\lambda_1(d_2)=4\cdot 0+2\cdot 6=12$$

$$\lambda_2(d^*)=k_2\lambda_2(d_1)+k_1\lambda_2(d_2)=4\cdot 1+2\cdot 3=10\,.$$

Since $\lambda_2(d^*)=\lambda_1(d^*)-z$, we have by Theorem 3.4 that d^* is E-optimal in $D(10;40,25;34)$.

Example 3.7. Consider the class of designs $D(6;18,14;2,3)$ and let d_1 and d_2 be the GD designs SR7 and R45 given in Clatworthy [3], i.e., d_1 is a GD design having parameters $v=6$, $b_1=18$, $k_1=2$, $r(d_1)\equiv b_1k_1/v=6$, $m=2$, $n=3$, $\lambda_1(d_1)=0$ and $\lambda_2(d_1)=2$, whereas d_2 is a GD design having parameters $v=6$, $b_2=14$, $k_2=3$, $r(d_2)\equiv b_2k_2/v=7$, $m=2$, $n=3$, $\lambda_1(d_2)=4$ and $\lambda_2(d_2)=2$. Now let d^* be that design having $N(d^*)=(N(d_1),N(d_2))$ and observe that $z=1$ is the greatest common divisor for $k_1=2$ and $k_2=3$, and d^* is a binary GDUB design in $D(6;18,14;2,3)$ having parameters $m=2$, $n=3$ and

$$\lambda_1(d^*)=k_2\lambda_1(d_1)+k_1\lambda_1(d_2)=3\cdot 0+2\cdot 4=8$$

$$\lambda_2(d^*)=k_2\lambda_2(d_1)+k_1\lambda_2(d_2)=3\cdot 2+2\cdot 2=10\,.$$

Since $\lambda_2(d^*)=\lambda_1(d^*)+2z$, we have by Theorem 3.5 that d^* is E-optimal in $D(6;18,14;2,3)$.

4. Results on the MV-optimality of GDUB designs

In this section, we prove that some of the GDUB designs established in Section 3 as being E-optimal are also MV-optimalin $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.

We begin by stating a preliminary lemma which can be proven using techniques analogous to those used in Jacroux [10] and Takeuchi [14].

LEMMA 4.1. *Let $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ be arbitrary. Then for any $i \neq j$,*

$$\mathrm{var}_d(\hat{\alpha}_i - \hat{\alpha}_j) \geqq \sigma^2(c_{ii}(d) + c_{jj}(d) + 2c_{ij}(d))/(c_{ii}(d)c_{jj}(d) - c_{ij}(d)^2)$$
$$\geqq 4\sigma^2/(c_{ii}(d) + c_{jj}(d) - 2c_{ij}(d)) .$$

Our first theorem extends a result of Takeuchi [14] concerning the MV-optimality of GD designs having $\lambda_2(d) = \lambda_1(d) + 1$ and equal block size to corresponding GDUB designs in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.

THEOREM 4.2. *For $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$, let z denote the greatest common divisor for $k/k_1, \cdots, k/k_p$ and suppose $k/k_s = zz_s$ for $s = 1, \cdots, p$. If $d^* \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ is a GDUB design which is binary and has $\lambda_2(d^*) = \lambda_1(d^*) + z$, then d^* is MV-optimal in $D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$.*

PROOF. Using the same arguments as those used in the proof of Theorem 3.2, for purposes of proving the present result, we need only consider the case when $z = 1$. By Lemma 2.2 (ix),

$$\max_{i \neq j} \mathrm{var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) = 2\sigma^2 k/(\bar{c} + \lambda_1(d^*)) .$$

So let $d \in D(v; b_1, \cdots, b_p; k_1, \cdots, k_p)$ be arbitrary with C-matrix $C(d) = (c_{ij}(d))$. We now consider two cases for $C(d)$.

Case 3. Suppose $kc_{ii}(d) < \bar{c}$ for some $i = 1, \cdots, v$. Without loss of generality, assume

$$kc_{11}(d) \leqq \bar{c} - 1 .$$

By Lemma 4.1, for d to have

$$\max_{2 \leqq j \leqq v} \mathrm{var}_d(\hat{\alpha}_1 - \hat{\alpha}_j) \leqq \max_{i \neq j} \mathrm{var}_{d^*}(\hat{\alpha}_i - \hat{\alpha}_j) = 2\sigma^2 k/(\bar{c} + \lambda_1(d^*)) ,$$

it must have for each $j \geqq 2$,

$$4/(c_{11}(d) + c_{jj}(d) - 2c_{1j}(d)) < 2k/(\bar{c} + \lambda_1(d^*))$$

which implies that for each $j \geqq 2$, d must have

$$k(c_{11}(d)+c_{jj}(d)-2c_{1j}(d))/2 \geqq \bar{c}+\lambda_1(d^*)+1/2 .$$

But it was shown in the proof of Theorem 3.2 under Case 1 that this last inequality cannot hold for each $j \geqq 2$. Thus, in this case we have that d cannot be MV-better than d^*.

Case 4. Suppose $kc_{ii}(d) \geqq \bar{c}$ for $i=1, \cdots, v$. As in the proof of Theorem 3.2, $kc_{ii}(d) \geqq \bar{c}$ for $i=1, \cdots, v$ implies that $kc_{ii}(d)=\bar{c}$ for $i=1, \cdots, v$. Now, for $i=1$, since $kc_{1j}(d)$ is an integer for $j=2, \cdots, v$, $c_{11}(d) = -\sum_{j=2}^{v} c_{1j}(d)=\bar{c}k$ and since by Lemma 2.2 (vi), $\lambda_1(d^*)=[\bar{c}/(v-1)]$, it follows that for some $j \geqq 2$, $-kc_{1j}(d) \leqq \lambda_1(d^*)$. Hence for this value of j,

$$k(c_{11}(d)+c_{jj}(d)-2c_{1j}(d))/2=(2\bar{c}-2c_{1j})/2 \leqq \bar{c}+\lambda_1(d^*)$$

and by Lemma 4.1,

$$\mathrm{var}_d\,(\hat{\alpha}_1-\hat{\alpha}_j) \geqq 4\sigma^2/(c_{11}(d)+c_{jj}(d)-2c_{1j}(d)) \geqq 2k\sigma^2/(\bar{c}+\lambda_1(d^*)) .$$

Thus again d cannot be MV-better than d^* and the result follows.

Example 4.3. Consider the class of designs $D(6;4,3;3,4)$ and let d_1 and d_2 be the GD designs given in Example 3.3. Then the design d^* having $N(d^*)=(N(d_1), N(d_2))$ satisfies the conditions of both Theorems 3.2 and 4.2, hence d^* is both E- and MV-optimal in $D(6;4,3;3,4)$.

In the next theorem, we show that the GDUB designs proven in Theorem 3.4 to be E-optimal are also MV-optimal in $D(v;b_1, \cdots, b_p;k_1, \cdots, k_p)$. Since the proof of this theorem is similar to that of Theorem 4.2, the proof will be omitted.

THEOREM 4.4. *For $D(v;b_1, \cdots, b_p;k_1, \cdots, k_p)$, let z and z_s be as defined in Theorem 4.2 for $s=1, \cdots, p$. If $d^* \in D(v;b_1, \cdots, b_p;k_1, \cdots, k_p)$ is a GDUB design which is binary and has $m=v/2$, $n=2$ and $\lambda_2(d^*)=\lambda_1(d^*)-z$, then d^* is MV-optimal in $D(v;b_1, \cdots, b_p;k_1, \cdots, k_p)$.*

Example 4.5. Consider the class of designs $D(10;40,25;2,4)$ and let d_1 and d_2 be the binary GDUB designs given in Example 3.6. Then the design d^* having $N(d^*)=(N(d_1), N(d_2))$ satisfies the conditions of both Theorems 3.4 and 4.4, hence d^* is both E- and MV-optimal in $D(10;40,25;2,4)$.

Comment 4.6. A number of sufficient conditions for the E- and MV-optimality of GDUB designs in classes $D(v;b_1, \cdots, b_p;k_1, \cdots, k_p)$ have been obtained in this paper. However, methods of constructing GDUB designs which satisfy these sufficient conditions need to be developed. Some such methods have already been derived by the authors and are similar to the techniques illustrated in Examples 3.3, 3.6 and 3.7. A list of E- and MV-optimal GDUB designs having parameters in the practical range is also being tabulated. These results will be reported elsewhere.

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