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A NOTE ON TESTING FOR CONSTANT HAZARD AGAINST A CHANGE-POINT ALTERNATIVE

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Summary

The problem of testing for constant hazard against a change-point alternative is considered. It is shown that this problem is related to another one in quality control. Based on this relationship, a test is proposed. The main advantages of this test are its computational simplicity and the ready availability of small and large sample distribution theory.

1. Introduction

We consider the problem of testing for constant hazard against a change-point alternative. Let $T_1, T_2, \cdots, T_n$ be independent and identically distributed (iid) lifetimes of n subjects with common hazard function

$$h(t)=\lambda I_{[0,\tau]}(t)+\lambda(1-\xi)I_{(\tau,\infty)}(t) , \tag{1}$$

where I_S is the indicator function of set S, λ ($\lambda>0$) and ξ ($0\leqq\xi<1$) are two unknown constants, and τ is the unknown change-point where an abrupt change occurs in the hazard function. Matthews and Farewell [7] applied this model to the data of times-to-relapse after remission induction for patients with leukemia, where it is suspected that the relapse rate may be reduced after an unknown period of time τ. They examined, by simulation, the behavior of the likelihood ratio test for H_0: $\xi=0$ (no change-point) against H_1: $\xi>0$. It is clear that the classical asymptotic result for likelihood ratio statistics does not apply here because of the discontinuity present at τ. Matthews, Farewell and Pyke [8] proposed a test based on a score-statistic process and showed that its asymptotic behavior is related to the supremum of an Ornstein-Uhlenbeck process. However, the null distribution of this test depends

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on λ and two pre-assigned bounds τ_l and τ_u for the change-point τ.

This problem (called Problem A hereafter) is related to another one in quality control. Suppose that one observes n independent random variables $Y_1, Y_2, \cdots, Y_n$ with $Y_1, \cdots, Y_k \sim \text{iid}(E(\lambda))$ and $Y_{k+1}, \cdots, Y_n \sim \text{iid}(E(\lambda(1-\xi)))$ where k is the unknown change-point, $E(\lambda)$ denotes the exponential distribution with failure rate λ. The problem is to test for H_0': $\xi=0$ (no change-point) against H_1': $\xi>0$. We will refer to this problem as Problem B. References to different variations of Problem B may be found in three review papers by Hinkley, Chapman and Runger [4], Shaban [9] and Zacks [10].

The relationship between these two problems is given in Section 2. Based on this relationship, a test for Problem A is proposed in Section 3. The main advantages of this test are its computational simplicity and the ready availability of small and large sample distribution theory. In Section 4, some efficiency results are presented.

2. Relationship between Problems A and B

Let $\rho=1-\exp(-\lambda\tau)=\mathrm{P}(T_1\leqq\tau)$ and $\theta=\lambda(1-\xi)$. Under H_0, $\lambda=\theta$ while τ and ρ are not uniquely defined. Denote the order statistics of $T_1, \cdots, T_n$ by $T_{(i)}$, $i=1, \cdots, n$. Setting $T_{(0)}=0$, define the normalized spacings by

$$D_i=(n-i+1)(T_{(i)}-T_{(i-1)}), \qquad i=1, \cdots, n. \tag{2}$$

It is well known that the normalized spacings are iid $(E(\lambda))$ under H_0; i.e., $\mathcal{L}(D_1, \cdots, D_n|H_0)=\mathcal{L}(Y_1, \cdots, Y_n|H_0')$ where $\mathcal{L}(X)$ denotes the distribution of random vector X.

To see the relationship between the two problems under the alternative hypotheses, we present a useful representation of T_i in the general case of $\lambda\neq\theta$. Let $V_1, \cdots, V_n$ be iid $(E(1))$. Thus,

$$T_i=\begin{cases} \lambda^{-1}V_i, & V_i\leqq\lambda\tau \\ \theta^{-1}(V_i-\lambda\tau)+\tau, & \text{otherwise} \end{cases} \tag{3}$$

are iid with common hazard function (1). Again, denote the order statistics of $V_1, \cdots, V_n$ by $V_{(i)}$, $i=1, \cdots, n$. Let K be the number of T_i not exceeding τ. Setting $V_{(0)}=0$, we have $D_{K+1}=(n-K)[\theta^{-1}(V_{(K+1)}-\lambda\tau)-\lambda^{-1}(V_{(K)}-\lambda\tau)]$ and

$$D_i=\begin{cases} \lambda^{-1}(n-i+1)(V_{(i)}-V_{(i-1)}), & i=1, \cdots, K \\ \theta^{-1}(n-i+1)(V_{(i)}-V_{(i-1)}), & i=K+2, \cdots, n. \end{cases} \tag{4}$$

If K were a constant, we would have $D_1, \cdots, D_K \sim \text{iid}(E(\lambda))$ and D_{K+2},

$\cdots, D_n \sim \text{iid}\,(E(\theta))$. This suggests that $D_1, \cdots, D_n$ in Problem A play the role of $Y_1, \cdots, Y_n$ in Problem B, and K in Problem A correspond to k in Problem B. More precisely, since K is binomial $(n, p=\rho=\mathrm{P}(T_1 \leqq \tau))$, it follows from the central limit theorem that $\mathrm{P}(|K/n-\rho|>C_n n^{-1/2}) \to 0$ as $n \to \infty$, if $C_n \to \infty$ (e.g., $C_n = \log n$). So, with probability approaching 1 as $n \to \infty$,

$$D_i = \begin{cases} \lambda^{-1}(n-i+1)(V_{(i)}-V_{(i-1)}) , & i=1, \cdots, [n\rho - C_n n^{1/2}] \\ \theta^{-1}(n-i+1)(V_{(i)}-V_{(i-1)}) , & i=[n\rho + C_n n^{1/2}], \cdots, n . \end{cases}$$

Loosely speaking, $\{D_i: 1 \leqq i \leqq [n\rho - C_n n^{1/2}]\}$ are asymptotically iid $(E(\lambda))$ while $\{D_i: [n\rho + C_n n^{1/2}] \leqq i \leqq n\}$ are asymptotically iid $(E(\theta))$. Moreover, it is not too difficult to show that for any fixed integer $m>0$, $D_{K-m}, \cdots, D_K$ are asymptotically iid $(E(\lambda))$ and $D_{K+2}, \cdots, D_{K+m}$ are asymptotically iid $(E(\theta))$. (In fact, conditioning on $K=r$, $T_{(r+1)}-\tau, \cdots, T_{(n)}-\tau$ are the order statistics of $(n-r)$ iid $(E(\theta))$ random variables and hence $D_{r+2}, \cdots, D_n \sim \text{iid}\,(E(\theta))$.)

3. A test for Problem A

In Problem B, when λ is known, Kander and Zacks [6] proposed, based on a Bayesian approach with a uniform prior on the change-point k, to reject H_0' if $\lambda \sum_{i=1}^{n} (i-1) Y_i$ is large. A natural generalization of this test to the case of λ unknown is to reject H_0' if $\sum_{i=1}^{n} (i-1) Y_i \Big/ \sum_{i=1}^{n} Y_i$ is large (cf. (2.4) of Hsu [5]). Based on the relationship in Section 2, we therefore propose to reject H_0 if

$$(5) \qquad S_n = \frac{1}{2n}\left[\sum_{i=1}^{n} (i-1) D_i \Big/ \sum_{i=1}^{n} D_i + n + 1\right] = \sum_{i=1}^{n} \left(\frac{i}{n}\right) T_{(i)} \Big/ \sum_{i=1}^{n} T_{(i)}$$

is large.

Define $Z_i = (n-i+1)(V_{(i)} - V_{(i-1)})$, $i=1, \cdots, n$, which iid are $(E(1))$. Under H_0, from (3),

$$S_n = 1 - \sum_{i=1}^{n-1} \left(\frac{n-i}{2n}\right) Z_i \Big/ \sum_{i=1}^{n} Z_i ,$$

and by (5.5.4) on page 103 of David [3], S_n has pdf

$$(6) \qquad f_{S_n}(s \mid H_0) = \sum_{i=1}^{n-1} w_i f_i(1-s) ,$$

where $d_i = (n-i)/2n$, $i=1, \cdots, n-1$, $w_i = d_i^{n-2} / \prod_{h \neq i} (d_i - d_h)$, $i=1, \cdots, n-1$ and

$$f_i(y)=\begin{cases}(n-1)d_i^{-1}(1-y/d_i)^{n-2}, & 0\leqq y\leqq d_i\\ 0, & \text{otherwise}.\end{cases}$$

While formula (6) provides a way to compute the exact levels of significance, numerical results show that the asymptotic normal approximation appears quite accurate for $n\geqq 10$. See Table 1 of Hsu [5].

In what follows, we will study the asymptotic behavior of S_n. Let $H_1(\lambda, \xi, \tau)$ denote the alternative hypothesis with parameter values λ, ξ and τ. Part (i) of the following proposition provides an asymptotic test procedure while part (ii) is used to derive efficiency results in Section 4.

PROPOSITION 1.
(i) $\mathcal{L}((48n)^{1/2}(S_n-3/4)\,|\,H_0)\xrightarrow{\mathcal{D}} N(0,1)$.
(ii) *For* $\xi_n=(c+o(1))n^{-1/2}$, $c>0$,

$$\mathcal{L}((48n)^{1/2}(S_n-3/4)\,|\,H_1(\lambda,\xi_n,\tau))\xrightarrow{\mathcal{D}} N(\sqrt{3}\,c\rho(1-\rho),1),$$

where $\rho=1-\exp(-\lambda\tau)$.

PROOF. Part (i) is essentially a special case of part (ii). For (ii), since $\mathcal{L}(S_n\,|\,H_1(\lambda,\xi,\tau))=\mathcal{L}(S_n\,|\,H_1(1,\xi,\lambda\tau))$, λ may be assumed equal to 1 without loss of generality. To prove (ii) with $\lambda=1$, $\xi_n=(c+o(1))n^{-1/2}$ and $\theta_n=1-\xi_n$, let K again be the number of $T_i\leqq\tau$, and note that from (3),

$$(7)\quad \begin{aligned}\sum_{i=1}^n\frac{i}{n}T_{(i)}&=\sum_{i=1}^n\frac{i}{n}V_{(i)}+(\theta_n^{-1}-1)\sum_{i=K+1}^n\frac{i}{n}(V_{(i)}-\tau)\\ &=\sum_{i=1}^n\frac{n+i}{2n}Z_i+(\theta_n^{-1}-1)(n-K)(3n+K+1)/(4n)\\ &\quad+(\theta_n^{-1}-1)\Big[\sum_{i=K+1}^n\frac{i}{n}(V_{(i)}-\tau)-(n-K)(3n+K+1)/(4n)\Big],\end{aligned}$$

$$(8)\quad \begin{aligned}\sum_{i=1}^n T_{(i)}&=\sum_{i=1}^n V_{(i)}+(\theta_n^{-1}-1)\sum_{i=K+1}^n(V_{(i)}-\tau)\\ &=\sum_{i=1}^n Z_i+(\theta_n^{-1}-1)(n-K)+(\theta_n^{-1}-1)\Big[\sum_{i=K+1}^n(V_{(i)}-\tau)-(n-K)\Big].\end{aligned}$$

To complete the proof, we need the following lemma.

LEMMA 2.

$$n^{-1/2}\Big[\sum_{i=K+1}^n\frac{i}{n}(V_{(i)}-\tau)-(n-K)(3n+K+1)/(4n)\Big]=O_p(1),$$

$$n^{-1/2}\Big[\sum_{i=K+1}^{n}(V_{(i)}-\tau)-(n-K)\Big]=O_p(1)\ .$$

PROOF. Conditioning on $K=r$, $V_{(K+1)}-\tau,\cdots,V_{(n)}-\tau$ are the order statistics of $(n-r)$ iid $(E(1))$ random variables, and so

$$\mathcal{L}\Big(\sum_{i=K+1}^{n}\frac{i}{n}(V_{(i)}-\tau)-(n-K)(3n+K+1)/(4n)\,|\,K=r\Big)$$
$$=\mathcal{L}\Big(\sum_{i=1}^{n-r}\frac{n+r+i}{2n}(Z_i-1)\Big)\ ,$$
$$\mathcal{L}\Big(\sum_{i=K+1}^{n}(V_{(i)}-\tau)-(n-K)\,|\,K=r\Big)=\mathcal{L}\Big(\sum_{i=1}^{n-r}(Z_i-1)\Big)\ .$$

From Donsker's invariance principle (Billingsley [1]),

$$\sup\left\{\left|n^{-1/2}\sum_{i=1}^{r}(Z_i-1)\right|:\ 1\leqq r\leqq n\right\}=O_p(1)\ ,$$

and so

$$\sup_{0\leq r\leq n-1}\left|n^{-1/2}\sum_{i=1}^{n-r}\frac{n-r-i}{n}(Z_i-1)\right|$$
$$=\sup_{0\leq r\leq n-1}n^{-1}\left|\sum_{j=1}^{n-r-1}n^{-1/2}\sum_{i=1}^{j}(Z_i-1)\right|\leqq\sup_{1\leq r\leq n}\left|n^{-1/2}\sum_{i=1}^{r}(Z_i-1)\right|=O_p(1)\ .$$

Hence,

$$\sup_{0\leq r\leq n-1}\left|n^{-1/2}\sum_{i=1}^{n-r}\frac{n+r+i}{2n}(Z_i-1)\right|$$
$$=\sup_{0\leq r\leq n-1}\left|n^{-1/2}\sum_{i=1}^{n-r}\frac{2n-(n-r-i)}{2n}(Z_i-1)\right|$$
$$\leqq\sup_{1\leq r\leq n-1}\left|n^{-1/2}\sum_{i=1}^{n-r}(Z_i-1)\right|+\sup_{0\leq r\leq n-1}\left|n^{-1/2}\sum_{i=1}^{n-r}\frac{n-r-i}{2n}(Z_i-1)\right|=O_p(1)\ ,$$

which implies the lemma.

CONTINUATION OF THE PROOF OF PROPOSITION 1. From (7), (8) and Lemma 2, and using $K/n=\rho+o_p(1)$,

$$n^{-1}\sum_{i=1}^{n}\frac{i}{n}T_{(i)}=n^{-1}\sum_{i=1}^{n}\frac{n+i}{2n}+n^{-1}\sum_{i=1}^{n}\frac{n+i}{2n}(Z_i-1)$$
$$+n^{-1/2}c(1-\rho)(3+\rho)/4+o_p(n^{-1/2})$$
$$=3/4+n^{-1}\sum_{i=1}^{n}\frac{n+i}{2n}(Z_i-1)+n^{-1/2}c(1-\rho)(3+\rho)/4+o_p(n^{-1/2})\ ,$$

$$n^{-1}\sum_{i=1}^{n}T_{(i)}=1+n^{-1}\sum_{i=1}^{n}(Z_i-1)+n^{-1/2}c(1-\rho)+o_p(n^{-1/2})\ .$$

So,

$$S_n=3/4+n^{-1}\sum_{i=1}^{n}\frac{2i-n}{4n}(Z_i-1)+n^{-1/2}c\rho(1-\rho)/4+o_p(n^{-1/2}) .$$

The proposition follows by applying Lemma 1 of Chernoff, Gastwirth and Johns [2] and $n^{-1}\sum_{i=1}^{n}\left(\frac{2i-n}{4n}\right)^2\to 1/48$.

4. Some efficiency results

To see the effect of the change-point τ on the performance of S_n, we consider the score test for H_0 against an alternative with known change-point. Let $H_1(\tau_0)=\{H_1(\lambda, \xi, \tau_0)\colon \lambda>0,\ 0<\xi<1\}$ be the alternative when τ is known equal to τ_0. Then, the score test for H_0 against $H_1(\tau_0)$ is (cf. (10) of Matthews et al. [8])

$$(9)\qquad \Delta_n(\tau_0)=\hat{\lambda}(1-\exp(-\hat{\lambda}\tau_0))^{-1/2}\exp(\hat{\lambda}\tau_0/2)n^{-1/2}\sum_{i=K+1}^{n}(T_{(i)}-\tau_0-\hat{\lambda}^{-1}) ,$$

where $\hat{\lambda}=n\Big/\sum_{i=1}^{n}T_i$ and K is the number of $T_i\leqq\tau_0$. Clearly $\Delta_n(\tau_0)$ is asymptotically standard normal under H_0. For $\xi_n=(c+o(1))n^{-1/2}$ and fixed λ and τ_0, using (3)

$$\hat{\lambda}^{-1}=\lambda^{-1}n^{-1}\sum_{i=1}^{n}V_i+\lambda^{-1}[(1-\xi_n)^{-1}-1]n^{-1}\sum_{i=K+1}^{n}(V_{(i)}-\lambda\tau_0) ,$$

$$\sum_{i=K+1}^{n}(T_{(i)}-\tau_0-\hat{\lambda}^{-1})=\lambda^{-1}(1-\xi_n)^{-1}\sum_{i=K+1}^{n}(V_{(i)}-\lambda\tau_0)-\hat{\lambda}^{-1}(n-K) .$$

Let $a(x)=\max(x-\lambda\tau_0, 0)$ and $b(x)=0$ or 1 according to whether $x\leqq\lambda\tau_0$ or not. Thus, $Ea(V_1)=\exp(-\lambda\tau_0)=1-\rho_0$, $Eb(V_1)=1-\rho_0$, and note that $\sum_{i=K+1}^{n}(V_{(i)}-\lambda\tau_0)=n(1-\rho_0)+\sum_{i=1}^{n}(a(V_i)-1+\rho_0)$, $n-K=n(1-\rho_0)+\sum_{i=1}^{n}(b(V_i)-1+\rho_0)$. So,

$$\hat{\lambda}^{-1}=\lambda^{-1}\left\{1+c(1-\rho_0)n^{-1/2}+n^{-1}\sum_{i=1}^{n}(V_i-1)+o_p(n^{-1/2})\right\}$$

and

$$\begin{aligned}n^{-1/2}&\sum_{i=K+1}^{n}(T_{(i)}-\tau_0-\hat{\lambda}^{-1})\\&=\lambda^{-1}\left\{c\rho_0(1-\rho_0)+n^{-1/2}\sum_{i=1}^{n}[a(V_i)-b(V_i)-(1-\rho_0)(V_i-1)]\right\}\\&\xrightarrow{\mathcal{D}}N(\lambda^{-1}c\rho_0(1-\rho_0),\ \lambda^{-2}\rho_0(1-\rho_0)) .\end{aligned}$$

Since $\hat{\lambda}(1-\exp(-\hat{\lambda}\tau_0))^{-1/2}\exp(\hat{\lambda}\tau_0/2)=\lambda\rho_0^{-1/2}(1-\rho_0)^{-1/2}+o_p(1)$,

$$\mathcal{L}(\Delta_n(\tau_0) \mid H_1(\lambda, \xi_n, \tau_0)) \xrightarrow{\mathcal{D}} N(c(\rho_0(1-\rho_0))^{1/2}, 1) . \tag{10}$$

Denote by ARE (τ_0) the Pitman asymptotic relative efficiency of S_n with respect to $\Delta_n(\tau_0)$ (against $H_1(\tau_0)$). From Proposition 1 and (10), ARE $(\tau_0)=3\rho_0(1-\rho_0)$. The ARE (τ_0) attains the maximum 3/4 at $\rho_0=1/2$ (i.e., τ_0=the median) and it decreases to 0 as ρ_0 tends to 0 or 1. So, as expected, the effect of the change-point τ_0 is rather significant when it occurs very early or late.

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