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COMPETITORS OF THE WILCOXON SIGNED RANK TEST

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Summary

Distribution-free statistics are proposed for one-sample location test, and are compared with the Wilcoxon signed rank test. It is shown that one of the statistics is superior to the Wilcoxon test in terms of approximate Bahadur efficiency. And we compare that statistic with the Wilcoxon test from the viewpoint of asymptotic expansion of power function under contiguous alternatives.

1. Introduction

Let $X_1, X_2, \cdots, X_N$ be independently and identically distributed random variables with absolutely continuous distribution function $F(x-\theta)$, where the associated density function satisfies $f(x)=f(-x)$ for all x and θ is a location parameter. The problem is to test the null hypothesis $H_0: \theta=0$ against alternative $H_1: \theta>0$.

For this problem, many test statistics are proposed already; especially, the theory of the locally most powerful signed rank test has introduced a class of linear signed rank test statistics which includes the Wilcoxon signed rank test, the normal score test, the sign test and etc. (cf. Hájek and Šidák [7], p. 74).

In this paper we shall propose a class of distribution-free test statistics which does not belong to linear signed rank statistics. The class will be proposed from the viewpoint of consistent estimator of $\mathrm{pr}(X_1+X_2>0, X_1+X_3>0, \cdots, X_1+X_r>0)$ for $r=1, 2, \cdots, N$.

Let $\Psi(x)=1$, if $x>0$ and $=0$ otherwise. For testing $H_0: \theta=0$ against $H_1: \theta>0$, we propose the following statistic S_r

$$S_r=\binom{N}{r}^{-1} \sum_{1\leq i_1\leq i_2\leq\cdots\leq i_r\leq N} C_r(X_{i_1}, X_{i_2}, \cdots, X_{i_r})$$

where for $r\geqq 2$

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$$C_r(x_1, x_2, \cdots, x_r) = r^{-1} \sum_{i=1}^{r} \left\{ \prod_{j \neq i}^{r} \Psi(x_i + x_j) \right\}$$

and $C_1(x) = \Psi(x)$. When S_r is large we reject H_0.

Note that S_1 is the sign test statistic and S_2 is equivalent to the Wilcoxon signed rank test statistic. From another aspect, Kumazawa [8] has proposed the class of test statistics such that

$$T_N = N^{-1} \sum_{i=1}^{N} h\left(N^{-1} \sum_{j=1}^{N} \Psi(X_i + X_j)\right) ,$$

where $h(t)$ is a nondecreasing and right continuous function. If we take $h(t) = t^{r-1}$ for $r \geqq 2$, S_r is asymptotically equivalent to T_N.

Let R_i^+ be a rank of $|X_i|$ among $\{|X_1|, |X_2|, \cdots, |X_N|\}$ and sign $x = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ otherwise. Then under H_0, $\{R_1^+, R_2^+, \cdots, R_N^+\}$ and $\{\text{sign}\, X_1, \text{sign}\, X_2, \cdots, \text{sign}\, X_N\}$ are independent and their distributions do not depend on F (cf. Hájek and Šidák [7], p. 40). For $i, j = 1, 2, \cdots, N$, $X_i + X_j > 0$ is equivalent to $R_i^+ \,\text{sign}\, X_i + R_j^+ \,\text{sign}\, X_j > 0$ and therefore the distributions of $\{\Psi(X_i + X_j)\}$, $i, j = 1, 2, \cdots, N$ do not depend on F.

Then we have the following theorem.

THEOREM 1. *Under H_0, the distribution of S_r does not depend on F; that is S_r is a distribution-free statistic.*

In Section 2, we shall compare S_r $(r \geqq 2)$ with the Wilcoxon signed rank test $W = \sum_{1 \leqq i \leqq j \leqq N} \Psi(X_i + X_j)$ in terms of Pitman asymptotic relative efficiency (A.R.E.), and S_3 whose Pitman A.R.E. coincides with W will be compared in terms of approximate Bahadur A.R.E. Further, in Section 3 we shall compare S_3 with W by means of asymptotic expansion of power function under contiguous alternatives.

2. Asymptotic relative efficiencies

We shall compare the new class with the Wilcoxon signed rank test in terms of Pitman and approximate Bahadur asymptotic relative efficiencies. For the purpose of the interchangeability of integral and differential, we assume the following condition.

CONDITION 1. Density function $f(x)$ is bounded and continuous almost everywhere.

Let us define U-statistic U_r

$$U_r = \binom{N}{r}^{-1} \sum_{1 \leqq i_1 < i_2 < \cdots < i_r \leqq N} C_r(X_{i_1}, X_{i_2}, \cdots, X_{i_r}) .$$

From the definition of S_r, it is easy to see that $\sqrt{N}(S_r-U_r)$ converges to 0 in probability. Then Pitman A.R.E. of S_r is equal to that of U_r.

In the sequel we denote by $\mathrm{E}_0(\cdot)$ and $\mathrm{E}_\theta(\cdot)$ respectively, the expectations under H_0 and H_1. Variances are similarly denoted by $\mathrm{V}_0(\cdot)$ and $\mathrm{V}_\theta(\cdot)$. Then from the theory of U-statistics (cf. Serfling [12], Chap. 5), we have

$$\mathrm{E}_\theta(U_r)=\mathrm{E}_\theta[C_r(X_1, X_2, \cdots, X_r)]=\int (F(x+2\theta))^{r-1}f(x)dx$$

and

$$\begin{aligned}\mathrm{V}_0(U_r)&=\frac{r^2}{N}\{\mathrm{E}_0[C_r(X_1, X_2, \cdots, X_r)C_r(X_1, X_{r+1}, \cdots, X_{2r-1})]\\&\quad-(\mathrm{E}_0[C_r(X_1, X_2, \cdots, X_r)])^2\}+O(N^{-2})\\&=\frac{1}{N}\left[\frac{2}{2r-1}-\frac{2((r-1)!)^2}{(2r-1)!}\right]+O(N^{-2})\,.\end{aligned}$$

Since C_r is a bounded kernel, $[U_r-\mathrm{E}_\theta(U_r)]/\sqrt{\mathrm{V}_\theta(U_r)}$ has limiting normal distribution with mean 0 and variance 1.

Furthermore, in the same way as Lemma 3.4 of Mehra and Sarangi [10], under Condition 1 we have

$$\frac{d}{d\theta}\int (F(x+2\theta))^{r-1}f(x)dx=2(r-1)\int (F(x+2\theta))^{r-2}f(x+2\theta)f(x)dx\,.$$

From the above discussion we can establish that the Noether's [11] regularity conditions are satisfied and Pitman A.R.E. of S_r $(r\geqq 2)$ with respect to the Wilcoxon signed rank test W is given by (cf. Serfling [12], p. 318)

$$e_p(S_r\,|\,W)=\left(\frac{\text{efficacy of } S_r}{\text{efficacy of } W}\right)^2=\frac{(r-1)^2\left\{\int (F(x))^{r-2}f^2(x)dx\right\}^2}{6\left\{\int f^2(x)dx\right\}^2\left\{\frac{1}{2r-1}-\frac{((r-1)!)^2}{(2r-1)!}\right\}}\,.$$

This A.R.E. coincides the result obtained by Kumazawa [8] who has obtained it by another method.

Since S_2 is equivalent to W, $e_p(S_2\,|\,W)=1$ for all distributions. Furthermore, it is shown that $e_p(S_3\,|\,W)=1$. Then we cannot see the difference between S_3 and W in terms of Pitman A.R.E. Thus we shall compare S_3 with the Wilcoxon test W by approximate Bahadur A.R.E.

In 1960, Bahadur [4] proposed two measures of the asymptotic performance of tests: the one is approximate Bahadur A.R.E., based on the limiting distributions of test statistics; and the other is exact Bahadur A.R.E., based on the limiting forms of the probabilities of large deviations of the statistics from their asymptotic means.

Though the exact Bahadur A.R.E. is desirable, which has been pointed out by several authors (cf. Abrahamson [1] and Bahadur [4], [5]), we could not unfortunately obtain the limiting form of the probability of large deviation of S_3. But we shall obtain the approximate Bahadur A.R.E. of S_3 relative to the Wilcoxon signed rank test W.

Since $\sqrt{N}(S_3-U_3)$ converges to 0 in probability, $[S_3-\mathrm{E}_0(S_3)]/\sqrt{\mathrm{V}_0(S_3)}$ has limiting normal distribution with mean 0 and variance 1 under H_0. And it is known that $[W-\mathrm{E}_0(W)]/\sqrt{\mathrm{V}_0(W)}$ has the same limiting normal distribution under H_0.

In the sequel we denote by pr_θ the probability under H_1. Then we can easily establish that for any $\varepsilon>0$

$$\lim_{N\to\infty} \mathrm{pr}_\theta\left(\left|\frac{S_3-\mathrm{E}_0(S_3)}{\sqrt{N\,\mathrm{V}_0(S_3)}}-\mu_1(\theta)\right|>\varepsilon\right)=0$$

and

$$\lim_{N\to\infty} \mathrm{pr}_\theta\left(\left|\frac{W-\mathrm{E}_0(W)}{\sqrt{N\,\mathrm{V}_0(W)}}-\mu_2(\theta)\right|>\varepsilon\right)=0\,,$$

where

$$\mu_1(\theta)=\sqrt{3}\left[\mathrm{pr}_\theta\,(X_1+X_2>0,\ X_1+X_3>0)-\frac{1}{3}\right]$$

and

$$\mu_2(\theta)=\sqrt{3}\left[\mathrm{pr}_\theta\,(X_1+X_2>0)-\frac{1}{2}\right]\,.$$

From the above discussion and Bahadur [4], pp. 276–278, we can obtain the approximate Bahadur A.R.E. of S_3 relative to W;

$$e_B(S_3\,|\,W;\theta)=\left(\frac{\mu_1(\theta)}{\mu_2(\theta)}\right)^2=\frac{\{\mathrm{pr}_\theta\,(X_1+X_2>0,\ X_1+X_3>0)-1/3\}^2}{\{\mathrm{pr}_\theta\,(X_1+X_2>0)-1/2\}^2}\,.$$

For this A.R.E., we get the following remarkable theorem.

THEOREM 2. *For any F whose density $f(x)$ satisfies Condition* 1, *and any $\theta>0$,*

$$e_B(S_3\,|\,W;\theta)>1\,.$$

PROOF. For $\theta>0$, we have

$$\mathrm{pr}_\theta\,(X_1+X_2>0,\ X_1+X_3>0)-\frac{1}{3}=\int\{F(x+2\theta)\}^2 f(x)dx-\frac{1}{3}>0$$

and

$$\mathrm{pr}_\theta\,(X_1+X_2>0)-\frac{1}{2}=\int F(x+2\theta)f(x)dx-\frac{1}{2}>0\,.$$

Let

$$m(\theta)=\int \{F(x+2\theta)\}^2 f(x)dx-\int F(x+2\theta)f(x)dx+\frac{1}{6} .$$

Then from Condition 1, we get

$$\frac{dm(\theta)}{d\theta}=4\int F(x+2\theta)f(x+2\theta)f(x)dx-2\int f(x+2\theta)f(x)dx .$$

Because of symmetry of F, we find that

$$\int F(x+2\theta)f(x+2\theta)f(x)dx+\int F(-x-2\theta)f(x+2\theta)f(x)dx$$
$$=\int f(x+2\theta)f(x)dx .$$

Letting $t=-x-2\theta$, we have

$$\int F(-x-2\theta)f(x+2\theta)f(x)dx=\int F(t)f(-t)f(-t-2\theta)dt$$
$$=\int F(x)f(x)f(x+2\theta)dx .$$

Then

$$\int f(x+2\theta)f(x)dx=\int \{F(x+2\theta)+F(x)\}f(x+2\theta)f(x)dx .$$

Since $F(x+2\theta)>F(x)$ for $\theta>0$, we have

$$2\int F(x+2\theta)f(x+2\theta)f(x)dx>\int f(x+2\theta)f(x)dx .$$

Hence $dm(\theta)/d\theta>0$ for $\theta>0$. While $m(0)=0$. Therefore we get $m(\theta)>0$ for $\theta>0$. Thus we have the desired result.

Similarly we can show that $e_B(S_3 | W; \theta)$ is monotone increasing with respect to θ. And it is easy to see that

$$\lim_{\theta\to\infty} e_B(S_3 | W; \theta)=\frac{16}{9} .$$

Theorem 2 means that S_3 is uniformly superior to W in terms of approximate Bahadur A.R.E. On the other hand, the theory of locally most powerful signed rank test insists that the Wilcoxon signed rank test is locally best for logistic distribution which satisfies Condition 1. This contrast may come from the difference of two criterions. Eplett [6] and Araki [3] proved the inadmissibility of linear rank and signed rank tests in terms of exact Bahadur A.R.E. S_3 is an example of test statistic which dominates Wilcoxon test W in terms of approximate Bahadur A.R.E.

For logistic distribution, we have

$$e_B(S_3 \mid W;\theta)=\left(\frac{2(1+2\exp(6\theta)-12\theta\exp(4\theta)+3\exp(4\theta)-6\exp(2\theta)}{3(\exp(2\theta)-1)(\exp(4\theta)-4\theta\exp(2\theta)-1)}\right)^2 .$$

Some of values of $e_B(S_3 \mid W;\theta)$ are listed in Table 1.

Table 1. Values of $e_B(S_3|W;\theta)$ for logistic distribution.

θ	0.01	0.02	0.05	0.1	0.15	0.2	0.5
$e_B(S_3\|W;\theta)$	1.004	1.008	1.020	1.040	1.061	1.081	1.204
θ	1.0	1.5	2.0	4.0	5.0	∞	
$e_B(S_3\|W;\theta)$	1.395	1.546	1.649	1.771	1.777	1.778	

Furthermore for logistic distribution, we have carried out simulations when $N=8$ and $N=10$ by generating 5,000 sets of logistic random digits and by estimating powers of randomized tests S_3 and W of size 0.05. Table 2 lists the results. Table 2 shows that W is more powerful than S_3 in the neighborhood of origin $\theta=0$, and that S_3 is more powerful in the case of large value of θ.

Table 2. Estimated powers of S_3 and W of size 0.05.

$N=8$

θ	0.05	0.15	0.5	1.0	2.0
Power of S_3	0.0569	0.0751	0.1855	0.4327	0.8718
Power of W	0.0573	0.0755	0.1860	0.4315	0.8676

$N=10$

θ	0.05	0.15	0.5	1.0	2.0
Power of S_3	0.0629	0.0867	0.2124	0.5206	0.9376
Power of W	0.0636	0.0867	0.2123	0.5194	0.9364

On the other hand, since the number of the possible values which S_3 takes is larger than that of the Wilcoxon test W, the averaged magnitude of the jumps of cumulative distribution function of S_3 is smaller than that of W under H_0. Then in small sample case, significance probability of S_3 is many times smaller than that of W. Therefore S_3 can analyze significance probability more closely than W.

It is important and interesting to obtain the exact Bahadur A.R.E. in the future.

3. Asymptotic expansion of power function

In this section we consider the asymptotic expansions of power functions of S_3 and W under contiguous alternatives $\theta=\delta/\sqrt{N}$ $(\delta>0)$. For one-sample problem, Albers, Bickel and van Zwet [2] have already obtained the expansions of linear signed rank statistics. Though W belongs to the linear signed rank statistics, S_3 does not. Then applying the Edgeworth expansion for U-statistics, which is due to Maesono [9], we shall obtain the expansion of S_3.

Let us define the following notations:

$$k(x, y, z)=\frac{1}{3}\{\Psi(x+y)\Psi(x+z)+\Psi(x+y)\Psi(y+z)+\Psi(x+z)\Psi(y+z)\} \\ -\mathrm{E}_\theta\Psi(X_1+X_2)\Psi(X_1+X_3)$$

$$g_1(x\,;\theta)=\mathrm{E}_\theta\{k(X_1, X_2, X_3)\,|\,X_1=x\}$$

$$g_2(x, y\,;\theta)=\mathrm{E}_\theta\{k(X_1, X_2, X_3)\,|\,X_1=x, X_2=y\}$$

$$\xi_1^2(\theta)=\mathrm{E}_\theta g_1^2(X_1\,;\theta)$$

$$\kappa_3(\theta)=\xi_1^{-3}(\theta)\{\mathrm{E}_\theta g_1^3(X_1\,;\theta)-6\mathrm{E}_\theta g_1(X_1\,;\theta)g_1(X_2\,;\theta)g_2(X_1, X_2\,;\theta)\}$$

$$R(x\,;\theta)=\Phi(x)-\phi(x)\frac{\kappa_3(\theta)}{6N^{1/2}}(x^2-1)$$

where $\Phi(x)$ and $\phi(x)$ denote the distribution function and the density of the standard normal distribution.

Then $S_3-\mathrm{E}_\theta(S_3)$ is approximated by

$$\binom{N}{3}^{-1}\sum_{1\leq i<j<m\leq N}k(X_i, X_j, X_m)\,.$$

From the definition of k, we find

$$g_1(x\,;\theta)=\frac{1}{3}\left\{(F(x+\theta))^2+2F(x+\theta)-2\int_{-\infty}^{x}F(t-\theta)f(t+\theta)dt\right\} \\ -\int(F(t+2\theta))^2f(t)dt\,.$$

Since $g_1(x\,;\theta)$ is monotone and continuous with respect to x,

$$\limsup_{|t|\to\infty}|\mathrm{E}(\exp\{itg_1(X_1\,;\theta)\})|<1\,.$$

Then from Maesono [9], we have the asymptotic expansion of the distribution of $[S_3-\mathrm{E}_\theta(S_3)]/\sqrt{\mathrm{V}_\theta(S_3)}$ with remainder term $o(N^{-1/2})$:

$$\mathrm{pr}_\theta\left\{\frac{S_3-\mathrm{E}_\theta(S_3)}{\sqrt{\mathrm{V}_\theta(S_3)}}\leq x\right\}=R(x\,;\theta)+o(N^{-1/2})\,. \tag{1}$$

Especially putting $\theta=0$, we obtain the approximation of significance point w_α which satisfies

$$\mathrm{pr}_0\left\{\frac{S_3-\mathrm{E}_0(S_3)}{\sqrt{\mathrm{V}_0(S_3)}}\geqq w_\alpha\right\}=\alpha+o(N^{-1/2}),$$

where pr_0 denotes the probability under H_0 and $0<\alpha<1$. Let u_α be the upper α-point of the standard normal distribution. Then from (1), we have

$$1-\Phi(u_\alpha)=1-\Phi(w_\alpha)+\phi(w_\alpha)\frac{\kappa_3(0)}{6N^{1/2}}(w_\alpha^2-1)+o(N^{-1/2}).$$

This implies that

$$w_\alpha=u_\alpha+\frac{\kappa_3(0)}{6N^{1/2}}(u_\alpha^2-1)+o(N^{-1/2}). \tag{2}$$

Let us assume the following condition.

CONDITION 2. Density function $f(x)$ has differentials $f'(x)$ and $f''(x)$ which satisfy

$$\int\{f'(x)\}^2dx<\infty \quad \text{and} \quad \int\{f''(x)\}^2dx<\infty .$$

Then we have the following theorem.

THEOREM 3. *Under Conditions* 1 *and* 2, *and for* $\theta=\delta/\sqrt{N}$ $(\delta>0)$, *we have*

$$\mathrm{pr}_\theta\left\{\frac{S_3-\mathrm{E}_0(S_3)}{\sqrt{\mathrm{V}_0(S_3)}}\geqq w_\alpha\right\}=1-\Phi(u_\alpha-a)+\frac{\phi(u_\alpha-a)}{N^{1/2}}\times\{P_1(f)\delta^2+P_2(f)u_\alpha\delta\}+o(N^{-1/2}), \tag{3}$$

where

$$a=2\sqrt{3}\,\delta\int f^2(x)dx,$$

$$P_1(f)=\frac{2\sqrt{3}}{5}\left(60\left\{\int(F(x)f(x))^2dx\right\}\left\{\int f^2(x)dx\right\}+5\int f^3(x)dx-24\left\{\int f^2(x)dx\right\}^2\right)$$

and

$$P_2(f)=\frac{6}{5}\left\{3\int f^2(x)dx-10\int(F(x)f(x))^2dx\right\}.$$

PROOF. See Appendix.

On the other hand, under some regularity conditions, Albers et al. [2] have obtained the asymptotic power of the Wilcoxon signed rank test W: that is

$$(4) \qquad 1-\Phi(u_\alpha-a)+o(N^{-1/2}) .$$

Note that the coincidence of $1-\Phi(u_\alpha-a)$ in (3) and (4) leads $e_p(S_3 | W)=1$.

Therefore we can compare S_3 with W by $P_1(f)\delta^2+P_2(f)u_\alpha\delta$: when $P_1(f)\delta^2+P_2(f)u_\alpha\delta$ is positive, S_3 is superior to W; when that value is negative, S_3 is inferior to W. The values of $P_1(f)$ and $P_2(f)$, and the sign of $P_1(f)\delta^2+P_2(f)u_\alpha\delta$ for normal, logistic, Cauchy and double-exponential distributions are listed in Table 3. It is not restrictive to assume $0<\alpha<1/2$, i.e. $u_\alpha>0$. Though the sign of $P_1(f)\delta^2+P_2(f)u_\alpha\delta$ for normal distribution depends on δ, S_3 is superior to W for Cauchy and double-exponential distributions which have heavy tails.

Table 3. Values of $P_1(f)$ and $P_2(f)$, and Sign of $P_1(f)\delta^2+P_2(f)u_\alpha\delta$.

Distribution	$P_1(f)$	$P_2(f)$	$P_1(f)\delta^2+P_2(f)u_\alpha\delta$
Normal	1.03×10^{-3}	-0.0138	$+$(δ large), $-$(δ small)
Logistic	0	0	0
Cauchy	8.08×10^{-3}	0.0331	$+$
Double-exponential	7.22×10^{-3}	0.025	$+$

Note: Though double-exponential distribution does not satisfy Condition 2, but the asymptotic power takes the same form.

For logistic distribution we had better compare S_3 with W by expanding the power with remainder term $o(N^{-1})$. Unfortunately however, we could not prove here the condition of the validity of Edgeworth expansion for U-statistics which has not been proved but described in Maesono [9].

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Appendix

Proof of Theorem 3

As the same way of Albers et al. [2] we first establish the following lemma.

LEMMA. *Under the contiguous alternatives* $\theta=\delta/\sqrt{N}$ $(\delta>0)$,

$$\begin{aligned}\mathrm{pr}_\theta\left\{\frac{S_3-\mathrm{E}_0(S_3)}{\sqrt{\mathrm{V}_0(S_3)}}\leq x\right\} &=R\left\{\frac{\sqrt{\mathrm{V}_0(S_3)}}{\sqrt{\mathrm{V}_\theta(S_3)}}(x-u)\,;\,\theta\right\}+o(N^{-1/2})\\ &=\Phi(x-u)-\phi(x-u)\left\{\frac{\xi_1^2(\theta)-\xi_1^2(0)}{2\xi_1^2(0)}(x-u)+\frac{\kappa_3(0)}{6N^{1/2}}((x-u)^2-1)\right\}\\ &\quad+O\left(\left\{\frac{\mathrm{V}_\theta(S_3)-\mathrm{V}_0(S_3)}{\mathrm{V}_0(S_3)}\right\}^2\right)+o(N^{-1/2})\,,\end{aligned}$$

where

$$u=\frac{\mathrm{E}_\theta(S_3)-\mathrm{E}_0(S_3)}{\sqrt{\mathrm{V}_0(S_3)}}\,.$$

PROOF. Letting $\sigma_0^2=V_0(S_3)$ and $\sigma_\theta^2=V_\theta(S_3)$, from (1) we get

$$\mathrm{pr}_\theta\left\{\frac{S_3-E_0(S_3)}{\sigma_0}\leqq x\right\}=\mathrm{pr}_\theta\left\{\frac{S_3-E_\theta(S_3)}{\sigma_\theta}\leqq\frac{\sigma_0}{\sigma_\theta}\left(x+\frac{E_\theta(S_3)-E_0(S_3)}{\sigma_0}\right)\right\}$$
$$=R\left\{\frac{\sigma_0}{\sigma_\theta}(x-u);\theta\right\}+o(N^{-1/2}).$$

Further

$$R\left\{\frac{\sigma_0}{\sigma_\theta}(x-u);\theta\right\}$$
$$=R(x-u;\theta)+R'(x-u;\theta)\left(\frac{\sigma_0}{\sigma_\theta}-1\right)(x-u)+O\left(\left(\frac{\sigma_0}{\sigma_\theta}-1\right)^2\right),$$

where

$$R'(y;\theta)=\frac{d}{dy}R(y;\theta)=\phi(y)+\frac{\kappa_3(\theta)}{6N^{1/2}}\phi(y)(y^3-3y).$$

From equation (2.30) in Albers et al. [2], we have

$$\frac{\sigma_0}{\sigma_\theta}=1-\frac{1}{2}\frac{\sigma_\theta^2-\sigma_0^2}{\sigma_0^2}+\frac{3}{8}\left(\frac{\sigma_\theta^2-\sigma_0^2}{\sigma_0^2}\right)^2-\cdots.$$

Then

$$R\left\{\frac{\sigma_0}{\sigma_\theta}(x-u);\theta\right\}$$
$$=R(x-u;\theta)-\frac{1}{2}\frac{\sigma_\theta^2-\sigma_0^2}{\sigma_0^2}R'(x-u;\theta)(x-u)+O\left(\left\{\frac{\sigma_\theta^2-\sigma_0^2}{\sigma_0^2}\right\}^2\right).$$

Since $\sigma_\theta^2=(9/N)\xi_1^2(\theta)+O(N^{-2})$, we obtain

$$\frac{\sigma_\theta^2-\sigma_0^2}{\sigma_0^2}=\frac{\xi_1^2(\theta)-\xi_1^2(0)}{\xi_1^2(0)}+O(N^{-1}).$$

And because of $\theta=\delta/\sqrt{N}$, we get that $\kappa_3(\theta)=\kappa_3(0)+o(1)$. Then we find

$$R(y;\theta)=R(y;0)+o(N^{-1/2})$$

and

$$R'(y;\theta)=\phi(y)+\frac{\kappa_3(0)}{6N^{1/2}}\phi(y)(y^3-3y)+o(N^{-1/2}).$$

This completes the proof of lemma.

Now we have

$$E_\theta(S_3)-E_0(S_3)=\int(F(x+2\theta))^2f(x)dx-\frac{1}{3}+O(N^{-1})$$

and

$$V_0(S_3)=\frac{9}{N}\xi_1^2(0)+O(N^{-2})=\frac{1}{3N}+O(N^{-2}) .$$

Conditions 1 and 2 ensure that

$$\frac{d^i}{d\theta^i}\int (F(x+2\theta))^2 f(x)dx=\int \frac{d^i}{d\theta^i}(F(x+2\theta))^2 f(x)dx , \qquad \text{for} \quad i=1, 2, 3 ,$$

$$\begin{aligned}\frac{d^j}{d\theta^j}\xi_1^2(\theta)=&\frac{d^j}{d\theta^j}\left(\frac{1}{9}\int \left\{(F(x+2\theta))^2+2F(x+2\theta)-2\int_{-\infty}^{x} F(t-2\theta)f(t)dt\right\}^2\right.\\ &\left.\cdot f(x)dx-\left\{\int (F(x+2\theta))^2 f(x)dx\right\}^2\right)\\ =&\frac{1}{9}\int \frac{d^j}{d\theta^j}\left\{(F(x+2\theta))^2+2F(x+2\theta)-2\int_{-\infty}^{x} F(t-2\theta)f(t)dt\right\}^2\\ &\cdot f(x)dx-\frac{d^j}{d\theta^j}\left\{\int (F(x+2\theta))^2 f(x)dx\right\}^2 , \qquad \text{for} \quad j=1, 2 ,\end{aligned}$$

and

$$\frac{d^j}{d\theta^j}\int_{-\infty}^{x} F(t-2\theta)f(t)dt=\int_{-\infty}^{x}\frac{d^j}{d\theta^j}F(t-2\theta)f(t)dt , \qquad \text{for} \quad j=1, 2 .$$

The proofs of the interchangeability are established by the same way as Lemma 3.4 in Mehra and Sarangi [10].

Then under Conditions 1 and 2, expanding with respect to θ, we have

$$u=2\sqrt{3}\,\delta\int f^2(x)dx+\frac{2\sqrt{3}\,\delta^2}{N^{1/2}}\int f^3(x)dx+o(N^{-1/2})$$

and

$$\xi_1^2(\theta)-\xi_1^2(0)=\xi_1^{2\prime}(0)\frac{\delta}{N^{1/2}}+o(N^{-1/2}) ,$$

where

$$\xi_1^{2\prime}(0)=\left.\frac{d\xi_1^2(\theta)}{d\theta}\right|_{\theta=0} .$$

Putting

$$b=\frac{2\sqrt{3}\,\delta^2}{N^{1/2}}\int f^3(x)dx ,$$

we find

$$\Phi(x-u)=\Phi(x-a)-b\phi(x-a)+o(N^{-1/2})$$

and

$$\phi(x-u)=\phi(x-a)+(x-a)\phi(x-a)b+o(N^{-1/2}) .$$

Hence

$$\text{(A1)} \quad R\left\{\frac{\sqrt{V_0(S_3)}}{\sqrt{V_\theta(S_3)}}(x-u);\,\theta\right\}=\Phi(x-a)-\phi(x-a)\left\{\frac{\xi_1^{2\prime}(0)}{2\xi_1^2(0)}(x-a)\frac{\delta}{N^{1/2}}\right.$$
$$\left.+\frac{\kappa_3(0)}{6N^{1/2}}((x-a)^2-1)+b\right\}+o(N^{-1/2}) .$$

From Conditions 1 and 2, and the equations (2) and (5) we have

$$R\left\{\frac{\sigma_0}{\sigma_\theta}(w_\alpha-u);\,\theta\right\}=\Phi(w_\alpha-a)+\frac{\phi(w_\alpha-a)}{N^{1/2}}\left\{\frac{\xi_1^{2\prime}(0)}{2\xi_1^2(0)}(w_\alpha-a)\delta\right.$$
$$\left.+\frac{\kappa_3(0)}{6}((w_\alpha-a)^2-1)+2\sqrt{3}\,\delta^2\int f^3(x)dx\right\}+o(N^{-1/2}) ,$$

$$\Phi(w_\alpha-a)=\Phi(u_\alpha-a)+\frac{\phi(u_\alpha-a)}{N^{1/2}}\kappa_3(0)(u_\alpha^2-1)+o(N^{-1/2}) ,$$

and $\phi(w_\alpha-a)=\phi(u_\alpha-a)+o(1)$. Further from Conditions 1 and 2, we have

$$\xi_1^{2\prime}(0)=\frac{4}{9}\int f^2(x)dx-\frac{8}{9}\int (F(x)f(x))^2dx .$$

Through simple and direct computation we have

$$\kappa_3(0)=\frac{6\sqrt{3}}{5} \qquad \text{and} \qquad \xi_1^2(0)=\frac{1}{27} .$$

Combining the above discussions we have the desired result.