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ON A CHARACTERIZATION OF MONOTONE LIKELIHOOD RATIO EXPERIMENTS

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Summary

Pfanzagl (1962, *Zeit. Wahrscheinlichkeitsth.*, **1**, 109–115) showed that a dominated family of probability measures has monotone likelihood ratios with respect to some real valued statistic if there exists a set of tests which has certain nice properties. A similar characterization was given by Dettweiler (1978, *Metrika*, **25**, 247–254), who did not assume domination. However, Pfanzagl's result is not a special case of the one proved by Dettweiler. We present a theorem which comprises the results of both authors. Our proof shows that not all conditions introduced by them are needed. Furthermore, we investigate the question concerning the generality we get if we do not assume domination.

1. Introduction

Let $(\Omega, \mathcal{A})$ be a measurable space and let $ca_1(\mathcal{A})$ be the set of all probability measures on $\mathcal{A}$. We use E_P to denote the expectation with respect to $P \in ca_1(\mathcal{A})$. We write $[P]$ instead of "P-almost everywhere" and $[\mathcal{P}]$ instead of "$[P]$ for all $P \in \mathcal{P}$" whenever $\mathcal{P} \subset ca_1(\mathcal{A})$. A test φ is a real valued measurable function defined on Ω such that $0 \leqq \varphi \leqq 1$. For $P, Q \in ca_1(\mathcal{A})$ and a test φ we write $P_\varphi Q$ if φ is most powerful for testing P against Q at level $\mathrm{E}_P\varphi$ and $1-\varphi$ is most powerful for testing Q against P at level $\mathrm{E}_Q(1-\varphi)$. If φ is fixed, this definition provides a reflexive and transitive binary relation (preorder) on $ca_1(\mathcal{A})$. In Pfanzagl's ([10]) notation $P_\varphi Q$ means "φ trennscharf $P:Q$". If φ, ψ are tests and if φ is most powerful for testing P against Q at level $\mathrm{E}_P\varphi$, then the following assertions hold.

(1.1) $\mathrm{E}_P\varphi \leqq \mathrm{E}_Q\varphi$.
(1.2) If $\mathrm{E}_P\psi = 0$, then $\varphi 1_{\{\psi>0\}} = 1_{\{\psi>0\}}[Q]$.
(1.3) If $P\{0<\varphi<1\}=0$, then $Q\{0<\varphi<1\}=0$.

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(1.4) If $\psi \leqq \varphi$ $[P]$, then $\psi \leqq \varphi$ $[Q]$.

We say that $P, Q \in ca_1(\mathcal{A})$ have monotone likelihood ratio with respect to an extended real valued statistic T and write $P \leqq_T Q$ if there is a non-decreasing function h from $\bar{R}$ to $\bar{R}$ such that

$$\frac{dQ}{d\mu}\Big/\frac{dP}{d\mu}=h \circ T[\mu] \ ,$$

where $\mu=P+Q$ and $a/0=\infty$ for $a>0$. Each pair $P, Q \in ca_1(\mathcal{A})$ has monotone likelihood ratio with respect to a statistic T which is equal to $(dQ/d\mu)/(dP/d\mu)$ on $\{dP/d\mu>0\} \cup \{dQ/d\mu>0\}$. If T is fixed, $\leqq_T$ defines a partial order on $ca_1(\mathcal{A})$. In the literature, T is usually assumed to be real valued. The set Λ of all tests φ of the type $\varphi=1_{\{T>c\}}+\gamma 1_{\{T=c\}}$, $c \in \bar{R}$, $0 \leqq \gamma \leqq 1$, has some nice and well-known properties of optimality for each pair P, Q with $P \leqq_T Q$ (see for example, Karlin and Rubin [6], Lehmann ([7], p. 68), Pfanzagl ([10], p. 112), Heyer ([5], p. 84) and the next section of this paper). Furthermore, Λ is obviously totally ordered.

The aim of the papers of Pfanzagl [8], [10] and Dettweiler [2] is to show that under suitable additional conditions families of probability measures which have monotone likelihood ratios with respect to a statistic T can be characterized by the existence of a set of tests which has some of the properties of the set Λ. We will prove a more general result. It will be shown that some of the conditions introduced by Pfanzagl [10] or Dettweiler [2] are not needed. Furthermore, we will see that families of probability measures which are totally ordered with respect to $\leqq_T$ for some statistic T and which are not necessarily dominated are majorized in the sense of Siebert [12].

2. Some properties of statistical experiments with monotone likelihood ratios

Let T be an extended real valued statistic on Ω and let Λ have the same meaning as in Section 1. For each $P \in ca_1(\mathcal{A})$ we define $H_P=\inf\{t \in \bar{R}: P\{T>t\}=0\}$ and $h_P=\sup\{t \in \bar{R}: P\{T \geqq t\}=1\}$ (we make the conventions $\inf \emptyset=\infty$ and $\sup \emptyset=-\infty$). Put $I_P=[-\infty, H_P]$ if $P\{T=H_P\}>0$ and otherwise $I_P=[-\infty, H_P[$. Furthermore, set $J_P=[h_P, \infty]$ if $P\{T=h_P\}>0$ and otherwise $J_P=]h_P, \infty]$. Define $Z_P=I_P \cap J_P$. The following lemmas are essentially known (see for example, Karlin and Rubin [6], Lehmann ([7], p. 68), Pfanzagl ([10], p. 112), or Heyer ([5], p. 84)).

LEMMA 2.1. a) *If* $P \in ca_1(\mathcal{A})$, *then* $P\{T \in Z_P\}=1$.
b) *If* $P, Q \in ca_1(\mathcal{A})$ *and* $P \leqq_T Q$, *then* $Q(\cdot \cap \{T \in I_P\}) \ll P$ *and* $P(\cdot \cap \{T \in$

$J_Q\}) \ll Q$.
c) *Suppose* $P, Q \in ca_1(\mathcal{A})$ *and* $P \leqq_T Q$. *Then* $Z_P = Z_Q$ *iff* P *and* Q *are equivalent. If* $Z_P = Z_Q$ *and* Z_P *is degenerated, then* $P = Q$.

LEMMA 2.2. a) *If* $P \in ca_1(\mathcal{A})$ *and* $\alpha \in [0, 1]$, *then there is a* $\varphi \in \Lambda$ *such that* $E_P\varphi = \alpha$.
b) *If* $P, Q \in ca_1(\mathcal{A})$, $P \leqq_T Q$, $\varphi \in \Lambda$, $0 < \mathrm{E}_P\varphi$ *and* $\mathrm{E}_Q\varphi < 1$, *then* $P_\varphi Q$.
c) *If* $P, Q, R \in ca_1(\mathcal{A})$ *such that* R *is equivalent to* P *and* $Q \geqq_T R$ *and if* $\varphi_P = 1_{\{T \notin I_P\}}$, *then* $\varphi_P \in \Lambda$, $\mathrm{E}_P\varphi_P = 0$ *and* $R_{\varphi_P} Q$.
d) *Suppose* $\emptyset \neq \mathcal{M} \subset ca_1(\mathcal{A})$ *and* $k = \sup\{H_P : P \in \mathcal{M}\}$. *Put* $D = [-\infty, k]$ *if there is a* $P \in \mathcal{M}$ *with* $P\{T = k\} > 0$, *and otherwise put* $D = [-\infty, k[$. *If* $\varphi \in \Lambda$ *such that* $\varphi = 1_{\{T \notin I_P\}}$ *or* $\mathrm{E}_P\varphi > 0$ *for some* $P \in \mathcal{M}$, *then* $\varphi 1_{\{T \notin D\}} = 1_{\{T \notin D\}}$.
e) *If* $P \in ca_1(\mathcal{A})$, $\varphi \in \Lambda$, $P\{0 < \varphi < 1\} = 0$ *and* $0 < \mathrm{E}_P\varphi < 1$, *then* $Q\{0 < \varphi < 1\} = 0$ *for all* $Q \in ca_1(\mathcal{A})$ *with the property* $Q \geqq_T P$ *or* $Q \leqq_T P$.
f) *Suppose* $\varphi = 1_{\{T > s\}} + \gamma 1_{\{T = s\}}$. *Then* $0 < \mathrm{E}_P\varphi$ *implies* $s \in I_P$ *and* $\mathrm{E}_Q\varphi < 1$ *implies* $s \in J_Q$.

A subset $\mathcal{M} \subset ca_1(\mathcal{A})$ is called majorized if there is a measure μ on $\mathcal{A}$ such that every $P \in \mathcal{M}$ has a density with respect to μ (see Siebert [12]).

PROPOSITION 2.3. *A subset* $\mathcal{M}$ *of* $ca_1(\mathcal{A})$ *which is totally ordered with respect to* $\leqq_T$ *is majorized.*

PROOF. We define an equivalence relation on $\bigcup_{P \in \mathcal{M}} Z_P$. Two points x, y of this set are called equivalent if there is a non-empty finite subset $\mathcal{F} \subset \mathcal{M}$ such that $\bigcup_{P \in \mathcal{F}} Z_P$ is an interval and $x, y \in \bigcup_{P \in \mathcal{F}} Z_P$. Let $\{A_i, i \in I\}$ be the family of all equivalence classes where $i \in A_i \subset \bar{R}$ for all $i \in I$.

Suppose A_i is not degenerated. It easily follows from the above definition that there is a countable subset $\mathcal{W}_i \subset \mathcal{M}$ with $A_i = \bigcup_{Q \in \mathcal{W}_i} Z_Q$. If $P \in \mathcal{M}$ and $Z_P \subset A_i$, then $P(A) = P(\bigcup_{Q \in \mathcal{W}_i} (A \cap \{T \in Z_Q\})) \leqq \sum_{Q \in \mathcal{W}_i} P(A \cap \{T \in Z_Q\})$. Hence Lemma 2.1.b) implies that P is absolutely continuous with respect to $\mathcal{W}_i$ since $\mathcal{M}$ is totally ordered. We conclude that for each $i \in I$ there is a $\mu_i \in ca_1(\mathcal{A})$ such that $P \ll \mu_i$ for all $P \in \mathcal{M}$ with $Z_P \subset A_i$ (if A_i is degenerated, use Lemma 2.1.c)). $\mathcal{M}$ is majorized by the measure $\mu = \sum_{i \in I} \mu_i$.

Remark 2.4. a) If $\Omega = R$, $\mathcal{A} = \mathcal{B}_1$ and $T = id_R$, then Z_P is identified with Pfanzagl's ([11], p. 1219) convex support of P. The set of all $i \in I$ such that A_i is not degenerated is countable. $\mathcal{M}$ is dominated if it contains at most countably many Dirac measures (then the set of all $i \in I$ such that A_i is degenerated is at most countable). This result was proved by Pfanzagl ([11], Theorem 3) by means of a topological argument.

b) Let $\hat{\mathcal{A}}$ consist of all sets $A \subset \Omega$ such that $A \cap \{T \in Z_P\} \in \mathcal{A}$ for every $P \in \mathcal{M}$. From the definition of μ (proof of Proposition 2.3) it is clear that it can be extended to a measure $\hat{\mu}$ on $\hat{\mathcal{A}}$ and that $(\Omega, \hat{\mathcal{A}}, \hat{\mu})$ is strictly localizable (Fremlin [3], p. 172, a direct sum of finite measure spaces). Experiments majorized by a localizable measure retain several properties of experiments dominated by a finite measure (see for example, Ghosh et al. [4] and the references given there).

Let Λ^* be the set of all $\varphi \in \Lambda$ such that for some $P \in \mathcal{M}$, we have $\varphi = 1_{\{T \notin I_P\}}$ or $0 < \mathrm{E}_P\varphi < 1$ and $P\{0 < \varphi < 1\} = 0$. The following properties Propositions 2.5. a) and b) are quite obvious for a dominated set $\mathcal{M}$ which is totally ordered with respect to $\leqq_T$ if the map S is defined by $S(\varphi) = \mathrm{E}_\mu(1-\varphi)$ for some equivalent finite dominating measure μ.

PROPOSITION 2.5. *Let $\mathcal{M}$ be a non-empty subset of $ca_1(\mathcal{A})$ which is totally ordered with respect to $\leqq_T$. Then there is a map S from Λ^* to $\bar{R}$ with the following properties.*

a) *If φ, $\psi \in \Lambda^*$ and $\varphi \leqq \psi$ $[\mathcal{M}]$, then $S(\varphi) \geqq S(\psi)$; moreover, if $\varphi \leqq \psi$ $[\mathcal{M}]$ and $P\{\varphi < \psi\} > 0$ for some $P \in \mathcal{M}$, then $S(\varphi) > S(\psi)$.*

b) *If $\varphi \in \Lambda^*$ and if Γ is a non-empty, countable subset of Λ^* such that $\psi \leqq \varphi$ $[\mathcal{M}]$ for all $\psi \in \Gamma$ and* $\inf\{S(\psi) : \psi \in \Gamma\} = S(\varphi)$, *then* $\sup\{\mathrm{E}_Q\psi : \psi \in \Gamma\} = \mathrm{E}_Q\varphi$ *for all $Q \in \mathcal{M}$.*

PROOF. Let $\{A_i, i \in I\}$ be defined as in the proof of Proposition 2.3. We assume that i is a real interior point of A_i if A_i is not degenerated. We define a partition of Λ^*. Put $\Lambda_i = \{1_{\{T > i\}}\}$ if $i \in I$ and A_i is degenerated. If A_i is not degenerated, then let Λ_i denote the set of all $\varphi \in \Lambda^*$ of the type $\varphi = 1_{\{T > s\}} + \gamma 1_{\{T = s\}}$ for some $s \in A_i$. Since $\mathcal{M}$ is totally ordered with respect to $\leqq_T$, we get from Lemma 2.2.e) that $P\{0 < \varphi < 1\} = 0$ for all $P \in \mathcal{M}$ and $\varphi \in \Lambda^*$. We have $\Lambda^* = \bigcup_{i \in I} \Lambda_i$: If $\varphi = 1_{\{T > s\}} + \gamma 1_{\{T = s\}} \in \Lambda^*$ and $0 < \mathrm{E}_P\varphi < 1$, then $s \in Z_P$ and $Z_P \subset A_i$ for some $i \in I$ (see Lemma 2.2.f)). This A_i is not degenerated. Thus $\varphi \in \Lambda_i$. If $\varphi \in \Lambda^*$ and $\mathrm{E}_Q\varphi \in \{0, 1\}$ for all $Q \in \mathcal{M}$, then $\varphi = 1_{\{T \notin I_P\}}$ for some $P \in \mathcal{M}$. Hence $\varphi = 1_{\{T > i\}}$ or $\varphi = 1_{\{T \geqq i\}}$ for some $i \in I$. If the corresponding A_i is degenerated, we have $\varphi = 1_{\{T > i\}}$. Indeed, $\varphi = 1_{\{T \geqq i\}}$ would imply that $Z_P \cup \{i\}$ is a non-degenerated interval which is contained in A_i.

Obviously, for each $i \in I$ such that A_i is not degenerated, there is a $c_i > 0$ with the property that $i + c_i$ is an interior point of A_i. We define S by $S(\varphi)=c_i\mathrm{E}_{\mu_i}(1-\varphi)+i$ if $\varphi\in\Lambda_i$ and Λ_i is not degenerated and $S(\varphi)=S(1_{\{T>i\}})=i$ if $\varphi\in\Lambda_i$ and Λ_i is degenerated (μ_i as in the proof of Proposition 2.3). Straightforward calculations show that S has the above properties.

3. Conditions for the existence of monotone likelihood ratios

In order to characterize (among other things) subsets $\mathcal{M}\subset ca_1(\mathcal{A})$ which are totally ordered with respect to $\leqq_T$ for some suitable statistic T, we introduce the following set-up. Let $\mathcal{M}$ be a non-empty subset of $ca_1(\mathcal{A})$. Suppose $\mathcal{M}\subset\mathcal{P}\subset ca_1(\mathcal{A})$ in such a way that for each $P\in\mathcal{P}$ there is a $Q\in\mathcal{M}$ which is equivalent to P. For each $P\in\mathcal{P}$ let $\mathcal{W}_P$ be a subset of $ca_1(\mathcal{A})$ with $P\in\mathcal{W}_P$. Assume that for $P, Q\in\mathcal{M}$ we have $P\in\mathcal{W}_Q$ or $Q\in\mathcal{W}_P$. Put $\mathcal{Q}=\bigcup_{P\in\mathcal{P}}\mathcal{W}_P$.

Remark 3.1. It is not generally true that an order $\leqq$ is defined by "$V\leqq W$ iff $W\in\mathcal{W}_V$". Indeed, put $\mathcal{M}=\mathcal{P}=\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ and $\mathcal{W}_{\varepsilon_i}=\{\varepsilon_{(i+1)[\mathrm{mod}\ 3]}\}$, where ε_i denotes the Dirac measure at i. But if $\leqq$ is an arbitrary partial order on a subset $\mathcal{P}'\subset ca_1(\mathcal{A})$ such that $\mathcal{M}\subset\mathcal{P}'$ and $\mathcal{M}$ is totally ordered with respect to $\leqq$, then the above conditions on $\mathcal{W}_P$ are fulfilled with $\mathcal{M}\subset\mathcal{P}\subset\mathcal{P}'$ and $\mathcal{W}_P=\{Q\in\mathcal{P}' : Q\geqq P\}$, $P\in\mathcal{P}$.

Let $\Phi'\subset\Phi$ be sets of tests on $(\Omega, \mathcal{A})$. Our aim it to give conditions on Φ, Φ', $\mathcal{M}$, $\mathcal{P}$ and $\{\mathcal{W}_P, P\in\mathcal{P}\}$ which imply that there is a statistic T such that each pair P, Q with $P\in\mathcal{P}$ and $Q\in\mathcal{W}_P$ has monotone likelihood ratio with respect to T.

CONDITION 3.2. (On Φ, Φ', $\mathcal{M}$, $\mathcal{P}$ and $\{\mathcal{W}_P, P\in\mathcal{P}\}$) a) For every $\alpha\in]0,1[$ and $P\in\mathcal{P}$ there is a $\varphi\in\Phi$ such that $\mathrm{E}_P\varphi=\alpha$.
b) If $P\in\mathcal{P}$, $Q\in\mathcal{W}_P$, $\varphi\in\Phi$, $0<\mathrm{E}_P\varphi$ and $\mathrm{E}_Q\varphi<1$, then $P_\varphi Q$.
c) For every $P\in\mathcal{P}$ there is an $M_P\in\mathcal{M}$ which is equivalent to P and a $\xi_P\in\Phi$ such that $\mathrm{E}_P\xi_P=0$ and $P_{\xi_P}Q$ for every $Q\in\mathcal{W}_P$. Moreover, $(M_P)_{\xi_P}V$ for every $V\in\mathcal{W}_{M_P}\cap\mathcal{M}$.
d) If $\varphi, \psi\in\Phi$ and $\varphi\leqq\psi\,[\mathcal{M}]$, then $\varphi\leqq\psi\,[\mathcal{Q}]$. Φ' is the set of all $\varphi\in\Phi$ such that $\varphi=\xi_P$ for some $P\in\mathcal{P}$ or $\varphi=1_{\{dQ/d(P+Q)>c\,dP/d(P+Q)\}}\,[P]$ and $0<\mathrm{E}_P\varphi<1$ for some $P\in\mathcal{P}$, $Q\in\mathcal{W}_P$ and $0\leqq c<\infty$. There is a map $S:\Phi'\to\bar{R}$ with the following properties.
e) If $\varphi, \psi\in\Phi'$ and $\varphi\leqq\psi\,[\mathcal{P}]$, then $S(\varphi)\geqq S(\psi)$; moreover, if $\varphi\leqq\psi\,[\mathcal{P}]$ and $P\{\varphi<\psi\}>0$ for some $P\in\mathcal{P}$, then $S(\varphi)>S(\psi)$.
f) If $\varphi\in\Phi'$ and if Γ is a non-empty, countable subset of Φ' such that $\psi\leqq\varphi\,[\mathcal{P}]$ for all $\psi\in\Gamma$ and $\inf\{S(\psi):\psi\in\Gamma\}=S(\varphi)$, then $\sup\{\mathrm{E}_Q\psi:\psi\in\Gamma\}=\mathrm{E}_Q\varphi$ for every $Q\in\mathcal{Q}$.

Example 3.3. a) Suppose $\mathcal{M}$ is totally ordered by $\leqq_T$ for some statistic T, $\mathcal{P}=\mathcal{M}$, $\mathcal{W}_P=\{Q\in ca_1(\mathcal{A}): Q\geqq_T P, Q\ll\mathcal{M}\}$, $\Phi=\Lambda$ and $\xi_P=1_{\{T\notin I_P\}}$. If $0<\alpha=P\{dQ/d\mu>cdP/d\mu\}<1$ where $0\leqq c<\infty$, $P, Q\in ca_1(\mathcal{A})$, $\mu=P+Q$ and $P\leqq_T Q$, then there is a $\varphi\in\Lambda$ (see Section 1) with $\varphi=1_{\{dQ/d\mu>cdP/d\mu\}}$ $[P]$. Indeed, by Lemma 2.2. a) there is a $\varphi\in\Lambda$ with $\mathrm{E}_P\varphi=\alpha$. The rest follows from the Neyman-Pearson-Lemma. Thus we have $\Phi'\subset\Lambda^*$. Using Lemma 2.2 and Proposition 2.5, we see that Condition 3.2 is fulfilled.

b) If $\mathcal{M}$ is given as in a), $\mathcal{P}\supset\mathcal{M}$, and for each $P\in\mathcal{P}$ there is a $Q\in\mathcal{M}$ which is equivalent to P, then, by Lemma 2.2 and Condition 3.2 is fulfilled if $\mathcal{W}_P=\{Q\in ca_1(\mathcal{A}):\ Q\geqq_T P,\ Q(\cdot\cap\{T\in D\})\ll\mathcal{M}\}$ $(P\in\mathcal{P})$, $\Phi=\{\varphi\in\Lambda: \varphi=1_{\{T\notin I_P\}}$ or $\mathrm{E}_P\varphi>0$ for some $P\in\mathcal{M}\}$ and $\xi_P=1_{\{T\notin I_P\}}$.

c) Suppose $\Omega=R$, $\mathcal{A}$ is the power set of R, $\mathcal{M}=\mathcal{P}$ is the set of all Dirac measures on $\mathcal{A}$, Φ is the set of all tests of the type $\varphi=\alpha 1_{\{x\}}+1_{]x,\infty[}$, $\alpha, x\in R$. Then Condition 3.2 is fulfilled with $M_P=P$ for all $P\in\mathcal{P}$, $\mathcal{W}_{\varepsilon_x}=\{\varepsilon_y: y\geqq x\}$, $\xi_{\varepsilon_x}=1_{]x,\infty[}$, $\Phi'=\{1_{]x,\infty[}: x\in R\}$ and $S(1_{]x,\infty[})=x$, $x\in R$.

PROPOSITION 3.4. *Suppose that Condition* 3.2 *holds. If* $\varphi, \psi\in\Phi'$, *then* $\varphi\leqq\psi$ $[\mathcal{M}]$ *or* $\psi\leqq\varphi$ $[\mathcal{M}]$.

PROOF. First we prove that $\xi_Q\leqq\xi_P$ $[\mathcal{M}]$ if $P, Q\in\mathcal{P}$ and $M_Q\in\mathcal{W}_{M_P}$. By Conditions 3.2. b), c) and (1.1), $\mathrm{E}_{M_P}\xi_Q=0$. Applying Condition 3.2. b) and (1.1) gives $\mathrm{E}_V\xi_Q=0$ for every $V\in\mathcal{M}$ such that $M_P\in\mathcal{W}_V$. By (1.4) and Condition 3.2. c), $\xi_Q\leqq\xi_P\,[V]$ for all $V\in\mathcal{W}_{M_P}\cap\mathcal{M}$.

Next we suppose $P\in\mathcal{M}$ and $0<\mathrm{E}_P\varphi<1$. Then $\psi\leqq\varphi\,[P]$ implies $\psi\leqq\varphi\,[\mathcal{M}]$, and $\varphi\leqq\psi\,[P]$ implies $\varphi\leqq\psi\,[\mathcal{M}]$. Indeed, suppose $\psi\leqq\varphi\,[P]$. Then we have $\mathrm{E}_P\psi\leqq\mathrm{E}_P\varphi<1$. If $V\in\mathcal{W}_P$, then $\mathrm{E}_V\varphi=1$ or, by Condition 3.2. b), $P_\varphi V$, and (1.4) implies $\psi\leqq\varphi\,[V]$. If $V\in\mathcal{M}$ and $P\in\mathcal{W}_V$, then $\mathrm{E}_V\psi=0$ or, by Condition 3.2. b), $V_\psi P$. Then (1.4) implies $\psi\leqq\varphi\,[V]$. The proof in the case $\varphi\leqq\psi\,[P]$ is analogous.

It remains to prove that $\psi\leqq\varphi\,[P]$ or $\varphi\leqq\psi\,[P]$ if $\varphi=1_{\{dQ/d\mu>cdP/d\mu\}}\,[P]$ and $0<\mathrm{E}_P\varphi<1$, where $P\in\mathcal{P}$, $Q\in\mathcal{W}_P$, $0\leqq c<\infty$ and $\mu=P+Q$. The proof of this assertion is more or less the same as the one of Pfanzagl's ([10], Hilfssatz 2). If $\mathrm{E}_P\psi=0$ or $\mathrm{E}_P\psi=1$, there is nothing to be proved. Suppose $0<\mathrm{E}_P\psi<1$. It follows from Condition 3.2. b) that ψ is most powerful for testing P against Q at level $\mathrm{E}_P\psi$. The Neyman-Pearson-Lemma shows that there is a $k\in[0,\infty[$ such that $1_{\{dQ/d\mu\geqq kdP/d\mu\}}\geqq\psi\geqq 1_{\{dQ/d\mu>kdP/d\mu\}}\,[\mu]$. The rest of the proof is easy.

In Proposition 3.4 we have $P\{0<\varphi<1\}=0$ for every $P\in\mathcal{M}$ and $\varphi\in\Phi'$. Indeed, by (1.2) and Condition 3.2. c), $V\{0<\xi_P<1\}=0$ for all $V\in\mathcal{W}_{M_P}\cap\mathcal{M}$. On the other hand, by Condition 3.2. b) and (1.1), $\mathrm{E}_V\xi_P=0$ for all $V\in\mathcal{M}$ such that $M_P\in\mathcal{W}_V$. Now suppose $0<\mathrm{E}_Q\varphi<1$ and $Q\{0<\varphi<1\}=0$ for some $Q\in\mathcal{M}$. Then Condition 3.2. b) and (1.3) imply that $V\{0<\varphi<1\}=0$ for every $V\in\mathcal{W}_Q$ and $V\{0<\varphi<1\}=0$ for every $V\in\mathcal{M}$

such that $Q \in \mathcal{W}_V$. We conclude that instead of tests φ from Φ' we could have used the sets $\{\varphi=1\}$ or $\{\varphi=0\}$. In analogous situations, Pfanzagl ([10], p. 114, the set C) and Dettweiler ([2], p. 250) considered sets of the type $\{\varphi=0\}$. Proposition 3.4 replaces Pfanzagl's ([10], Hilfssatz 2) and Dettweiler's ([2], assertion (F)). If we replace R in Example 3.3. c) by the set of all rational numbers, then in this example Pfanzagl's set C consists of all intervals $]-\infty, x[$ and $]-\infty, x]$, x rational.

4. Conditions of Pfanzagl and Dettweiler

Pfanzagl ([10], p. 110) considered a set of tests Φ and a non-empty dominated set $\mathcal{M}\subset ca_1(\mathcal{A})$ which bears a total order $\leqq$. He introduced Conditions 3.2. a) to c) where $\mathcal{M}=\mathcal{P}=\mathcal{Q}$, $\mathcal{W}_P=\{Q\in\mathcal{M}: Q\geqq P\}$, $M_P=P$ for all $P\in\mathcal{P}$. Obviously, this implies that all of Condition 3.2 are fulfilled (we may put $S(\varphi)=\mathrm{E}_\mu(1-\varphi)$ for all $\varphi\in\Phi'$, where μ is a dominating finite measure). Under the additional assumption that for every $P\in\mathcal{Q}$ there is a $\psi\in\Phi$ such that $\mathrm{E}_P\psi=1$ and $Q_\psi P$ for every $Q\leqq P$, Pfanzagl ([10], p. 110) proved that there is a real valued statistic T such that $P\leqq_T Q$ whenever $P, Q\in\mathcal{Q}$ and $P\leqq Q$. We shall see that only Condition 3.2 is needed.

Dettweiler ([2], Theorem 1) considered a set of tests Φ and non-empty subsets $\mathcal{P}\subset\mathcal{Q}\subset ca_1(\mathcal{A})$ where $\mathcal{Q}$ bears a partial order $\leqq$ with the property that for all $P\in\mathcal{P}$ and $Q\in\mathcal{Q}$ there is a $K\in\mathcal{P}$ such that $K\leqq Q$ and $K\leqq P$.

PROPOSITION 4.1. *Suppose that for every $\alpha\in[0,1]$ and $P\in\mathcal{P}$, there is a $\varphi\in\Phi$ such that $\mathrm{E}_P\varphi=\alpha$ and that $P_\varphi Q$ if $P\in\mathcal{P}$, $Q\in\mathcal{Q}$, $P\leqq Q$ and $\varphi\in\Phi$. Then all probability measures from $\mathcal{P}$ are equivalent and there is a $G\in\mathcal{A}$ such that $P(G)=1$, $Q(\cdot\cap G)\ll P$ and $\chi(1-1_G)=1-1_G\,[Q]$ for all $P\in\mathcal{P}$, $Q\in\mathcal{Q}$ and $\chi\in\Phi$. Furthermore, Condition 3.2 is fulfilled with $\mathcal{W}_P=\{Q\in\mathcal{Q}: Q\geqq P\}$, $\mathcal{M}=\{K\}$, $M_P=K$ and $S(\varphi)=\mathrm{E}_K(1-\varphi)$ for all $P\in\mathcal{P}$ and $\varphi\in\Phi'$, where $K\in\mathcal{P}$ is arbitrary.*

PROOF. Suppose $K, P\in\mathcal{P}$. We will show that K and P are equivalent. There is a $Q\in\mathcal{P}$ such that $Q\leqq P$ and $Q\leqq K$. We can find a $\varphi\in\Phi$ such that $\mathrm{E}_Q\varphi=1$, $Q_\varphi K$ and $Q_\varphi P$. Now a simple application of (1.2) gives $Q\ll P$ and $Q\ll K$. On the other hand, there exist ψ, $\chi\in\Phi$ such that $\mathrm{E}_K\psi=0$, $Q_\psi K$, $\mathrm{E}_P\chi=0$ and $Q_\chi P$. Then (1.2) shows that $K\ll Q$ and $P\ll Q$. Thus K and P are equivalent.

Let $K\in\mathcal{P}$ be fixed. There are tests $\varphi, \psi\in\Phi$ such that $\mathrm{E}_K\varphi=0$, $\mathrm{E}_K\psi=1$, $P_\varphi Q$ and $P_\psi K$ whenever $P\in\mathcal{P}$, $Q\in\mathcal{Q}$, $P\leqq Q$ and $P\leqq K$. Define $G=\{\varphi=0\}\cap\{\psi=1\}$.

For a fixed $Q\in\mathcal{Q}$ there is a $P\in\mathcal{P}$ such that $P\leqq Q$ and $P\leqq K$. If $K(B)=0$, then from (1.2) we get $(1-\psi)1_B=1_B\,[P]$. Using again (1.2), this implies $\varphi 1_{\{\psi=1\}\cap B}=1_{\{\psi=1\}\cap B}$ [Q]. Hence $Q(G\cap B)=0$. From $P\ll K$, we get $P(\{\varphi>0\}\cup\{\psi<1\})=0$. If $\chi\in\Phi$, then $P_\chi Q$. Applying (1.2) gives $\chi 1_{\{\varphi>0\}\cup\{\psi<1\}}=1_{\{\varphi>0\}\cup\{\psi<1\}}$ [Q]. The rest of the proof is obvious.

Dettweiler ([2], Theorem 1) showed that under the premise of Proposition 4.1 and two additional assumptions there is a real valued statistic T such that $P\leqq_T Q$ whenever $P\in\mathcal{P}$, $Q\in\mathcal{Q}$ and $P\leqq Q$. We shall see that this result follows from Condition 3.2 alone. Dettweiler assumed that there is a sequence (P_n) in $\mathcal{P}$ which has the property that for each $P\in\mathcal{P}$ there is a positive integer m such that $P_m\leqq P$, and that $Q_\varphi P$ if $P\in\mathcal{P}$, $Q\in\mathcal{Q}$, $Q\leqq P$ and $\varphi\in\Phi$.

Example 4.2. a) In Example 3.3. c) $\mathcal{P}$ is not dominated. Therefore neither Pfanzagl's nor Dettweiler's conditions hold.
b) If in Example 3.3. c) we take the set of rational numbers instead of R, then Pfanzagl's conditions are fulfilled, but Dettweiler's ones are not, since the premise of Proposition 4.1 does not hold: If $0<\gamma<1$, $P=\varepsilon_1$, $Q=\varepsilon_2$ and $\varphi=1_{\{1\}}+\gamma 1_{]1,\infty[}$, then $P_\varphi Q$ is not true.

5. Main results

Suppose that Condition 3.2 holds. The aim of this section is to introduce a statistic T which has the property that $P\leqq_T Q$ whenever $Q\in\mathcal{W}_P$ and $P\in\mathcal{P}$. Let $F\subset\bar{R}$ be the image of Φ' under S. Put

$$F_0=\bigcup_{n\geqq 1}\{b\in F\colon\ ]b,b+1/n[\subset\bar{R}\setminus F\}\qquad\text{and}\qquad F_1=F\setminus F_0\,.$$

Then F_0 is countable. Furthermore, let F_2 denote a countable dense subset of F_1. For every $b\in F_0\cup F_2$ we choose a $\varphi\in\Phi'$ such that $S(\varphi)=b$; in this way, we get a countable subset $\Psi\subset\Phi'$. The definition of Ψ differs from the definition of certain countable sets which were used by Pfanzagl ([8], p. 171) and Dettweiler ([2], p. 251) for the same purpose. It resembles more Dettweiler's definition than Pfanzagl's one. Dettweiler's condition (H), (C2) makes no sense; this is certainly due to misprints.

The following definition is completely analogous to those given by Pfanzagl ([8], p. 171) and Dettweiler ([2], p. 252). For each $\omega\in\Omega$ we put

$$T(\omega)=\inf\{S(\psi)\colon\psi\in\Psi;\ \psi(\omega)=0\}$$

if $\omega\in\bigcup_{\chi\in\Psi}\{\chi=0\}$ and otherwise $T(\omega)=\infty$.

Remark 5.1. Without loss of generality we could have assumed S to be real valued and bounded. In this case, in the definition of T,

∞ could be replaced by a real number which is greater than every $S(\varphi)$, and T would become a bounded real valued function.

The next result replaces Pfanzagl's [8], Hilfssatz 1 (see also Pfanzagl [10], p. 114, (10)). The proof is almost the same as part (a) and (b) of Pfanzagl's proof. Part (c) of that proof is not needed because of our definition of F_0. Moreover, the argument of part (c) cannot be used under our conditions. This can be seen by means of Eexample 3.3. c), since then, roughly speaking, $\sup\{S(\psi): \psi \in \Gamma\}=S(\varphi)$ does not imply $\inf\{\mathrm{E}_P\psi: \psi \in \Gamma\}=\mathrm{E}_P\varphi$.

LEMMA 5.2. *Suppose that Condition 3.2 holds. Let S be real valued and $T(\omega)>-\infty$ for all $\omega \in \Omega$. Then T is measurable and, if $\varphi \in \Phi'$, then $\varphi=1_{\{T>S(\varphi)\}}$ $[Q]$.*

PROOF. The measurability follows from

$$\{T<a\}=\bigcup_{\psi \in \Psi, S(\psi)<a}\{\psi=0\}\,. \tag{5.1}$$

It suffices to show that $\{\varphi=0\}=\{T\leqq S(\varphi)\}$ $[Q]$. From (5.1), Conditions 3.2. e) and d) we get

$$\{T<S(\varphi)\}\subset\{\varphi=0\}\ [Q] \tag{5.2}$$

for all $\varphi \in \Phi'$. For each $\varphi \in \Psi$ we have

$$\{\varphi=0\}\subset\{T\leqq S(\varphi)\}\,. \tag{5.3}$$

Suppose $\varphi \in \Phi'$.

Case 1. $S(\varphi)\notin F_0$. Then there exists a $\chi \in \Psi$ such that $S(\chi)>S(\varphi)$, and

$$S(\varphi)=\inf\{S(\psi): \psi \in \Psi,\ S(\psi)>S(\varphi)\}\,.$$

Now Condition 3.2. f) implies

$$\mathrm{E}_Q\varphi=\sup\{\mathrm{E}_Q\psi: \psi \in \Psi,\ S(\psi)>S(\varphi)\} \qquad \text{for all} \quad Q \in \mathcal{Q}\,.$$

This is equivalent to

$$Q\{\varphi=0\}=\inf\{Q\{\psi=0\}: \psi \in \Psi,\ S(\psi)>S(\varphi)\} \qquad \text{for all} \quad Q \in \mathcal{Q}\,.$$

Using Conditions 3.2. e) and d), we have $\{\psi=0\}\supset\{\varphi=0\}$ $[Q]$ whenever $S(\psi)>S(\varphi)$. Therefore we conclude

$$\{\varphi=0\}=\bigcap_{\psi \in \Psi, S(\psi)>S(\varphi)}\{\psi=0\}\ [Q]\,.$$

On the other hand,

$$\{T \leqq S(\varphi)\} = \bigcap_{\psi \in \Psi, S(\psi) > S(\varphi)} \{T < S(\psi)\} = \bigcap_{\psi \in \Psi, S(\psi) > S(\varphi)} \{T \leqq S(\psi)\} ,$$

and by (5.2) and (5.3), we get

$$\{T \leqq S(\varphi)\} = \bigcap_{\psi \in \Psi, S(\psi) > S(\varphi)} \{\psi = 0\} \ [Q] .$$

Case 2. $S(\varphi) \in F_0$. Then there is a $\xi \in \Psi$ such that $\xi = \varphi$ $[\mathcal{P}]$. Hence $\xi = \varphi$ $[Q]$, by Condition 3.2. d). From the definition of T and Ψ it follows that $\{T = S(\varphi)\} \subset \{\xi = 0\}$. Hence $\{T = S(\varphi)\} \subset \{\varphi = 0\}$ $[Q]$. By (5.2) and (5.3), we get $\{T \leqq S(\varphi)\} = \{\varphi = 0\}$ $[Q]$.

We need one more lemma which easily follows from Condition 3.2 and (1.2).

LEMMA 5.3. *Suppose that Condition* 3.2 *holds,* $P \in \mathcal{P}$, $Q \in \mathcal{W}_P$ *and* $\mu = P + Q$.
a) *If* $P\{dQ/d\mu > cdP/d\mu\} = 0$ *for some* $c \in [0, \infty]$, *then*

$$1_{\{dQ/d\mu > cdP/d\mu\}} = 1_{\{dQ/d\mu > 0, dP/d\mu = 0\}} = \xi_P \ [\mu] .$$

b) *If* $P\{dQ/d\mu > cdP/d\mu\} = 1$ *for some* $c \in [0, \infty]$, *then*

$$\{dQ/d\mu > cdP/d\mu\} = \Omega \ [\mu] .$$

Now we can prove our main result. One essential difference to the methods used in an analogous situation by Pfanzagl ([8], pp. 174–176 and [10], p. 115) and Dettweiler ([2], pp. 152–153) is that we make use of a well-known factorization theorem due to Doob.

PROPOSITION 5.4. *Suppose that Condition* 3.2 *holds. Then there is a statistic* T *such that* $P \leqq_T Q$ *for all* $P \in \mathcal{P}$ *and* $Q \in \mathcal{W}_P$.

PROOF. Because of Remark 5.1, we assume without loss of generality that S is real valued and $T(\omega) > -\infty$ for all $\omega \in \Omega$. Put $\mu = P + Q$. If $c \in [0, \infty]$ and $0 < P\{dQ/d\mu > cdP/d\mu\} < 1$, then there is a test $\varphi_c \in \Phi'$ such that $\varphi_c = 1_{\{dQ/d\mu > cdP/d\mu\}}$ $[\mu]$. This follows from Condition 3.2. a) and the Neyman-Pearson-Lemma.

If $c \in [0, \infty]$ and $P\{dQ/d\mu > cdP/d\mu\} = 0$, we define $\varphi_c = \xi_P$ (see Lemma 5.3. a)). If $P\{dQ/d\mu > cdP/d\mu\} = 1$, we put $\varphi_c = 1_\Omega$ (see Lemma 5.3. b)).

Obviously, $\varphi_{c'} \leqq \varphi_c$ $[P]$ for $c \leqq c'$. As in the proof of Proposition 3.4, we get $\varphi_{c'} \leqq \varphi_c$ $[\mathcal{P}]$ for $c \leqq c'$. Therefore a non-decreasing map m from $[0, \infty]$ to $R \cup \{-\infty\}$ is defined by $m(c) = S(\varphi_c)$ if $P\{dQ/d\mu > cdP/d\mu\} < 1$ and $m(c) = -\infty$ otherwise. Put

$$g(\omega) = \begin{cases} \dfrac{dQ}{d\mu}(\omega) \Big/ \dfrac{dP}{d\mu}(\omega) & \text{if } \dfrac{dP}{d\mu}(\omega) > 0 \\ \infty & \text{otherwise} . \end{cases}$$

If $P\{dQ/d\mu > cdP/d\mu\} < 1$ and $c < \infty$, by Lemmas 5.2 and 5.3. a), we get

$$\{g>c\}=\{dQ/d\mu>cdP/d\mu\}=\{\varphi_c=1\}=\{T>m(c)\}\ [\mu]\ .$$

If $P\{dQ/d\mu > cdP/d\mu\} = 1$, then $c < \infty$ and, by Lemma 5.3. b),

$$\{g>c\}=\{dQ/d\mu>cdP/d\mu\}=\Omega=\{T>-\infty\}=\{T>m(c)\}\ [\mu]\ .$$

If $c=\infty$, then, by Lemmas 5.2 and 5.3. a),

$$\{g=\infty\}=\{dP/d\mu=0,\ dQ/d\mu>0\}=\{\xi_P=1\}=\{T>m(\infty)\}\ [\mu]\ .$$

Now we use the proof of the well-known factorization theorem due to Doob (see Dellacherie and Meyer [1], 13-I-18): If we define

$$g_n=\sum_{k=1}^{\infty} k2^{-n}\ 1_{\{k2^{-n}<g\leq(k+1)2^{-n}\}}+\infty 1_{\{g=\infty\}}$$

for each positive integer n, then $g=\sup_n g_n$. We have $g_n=h_n\circ T\ [\mu]$, where

$$h_n(t)=\sum_{k=1}^{\infty} k2^{-n}\ 1_{]m(k2^{-n}),m((k+1)2^{-n})]}(t)+\infty 1_{]m(\infty),\infty[}(t)\ .$$

Since m is non-decreasing, each h_n is non-decreasing. Arguing similar as Dellacherie and Meyer [1], we see that there is a non-decreasing map $h: \bar{R}\to\bar{R}$ such that $g=h\circ T\ [\mu]$.

Remark 5.5. Proposition 5.4 obviously implies that $\mathcal{M}$ is totally ordered with respect to $\leqq_T$.

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