Ann. Inst. Statist. Math.
39 (1987), Part A, 219-225

BAYESIAN ANALYSIS OF HYBRID LIFE TESTS WITH EXPONENTIAL FAILURE TIMES

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(Received Sept. 5, 1985; revised Jan. 20, 1986)

Summary

A hybrid life test procedure is discussed from the Bayesian viewpoint. A total of n items is placed on test, failed items are either not replaced or are replaced, and the test is terminated either when a pre-chosen number, K, of items have failed, or when a pre-determined time on test has been reached. Posterior and predictive distributions are obtained under the assumption of an exponential failure distribution, and point and interval estimates are given for the mean life and the life of an untested item. The results are applied to a numerical example.

1. Introduction

A series of papers by Epstein [3], [4], [5], considered life testing situations where the life X follows the exponential distribution

$$f(x|\theta)=\theta^{-1}\exp(-x/\theta)\,, \qquad \text{for } x>0,\ \theta>0\,. \tag{1.1}$$

It is required to estimate the parameter θ from test data. (If the data are not thought initially to be exponential, transformation is possible; see Draper and Guttman [2].) A variety of possibilities is available for the testing procedure. For example, one can discontinue testing

1. after a pre-selected time has elapsed. (This is often called Type I censoring.)
2. after (a pre-selected number) K items have failed. (This is often called Type II censoring.)
3. after a fixed total life has been attained during which k items have failed.

In all three cases it is possible to replace items that fail or not replace them, although case (1) is usually done without replacement.

Key words and phrases: Hybrid life test plans, posterior distribution of mean life-times, predictive distribution of future lifetimes, single exponential distribution.

A Bayesian study of both Type I and Type II censoring was made by Bhattacharya [1], who was concerned with reliability estimation.

A hybrid test procedure (due to Epstein [3]) is also possible. Specifically, we can combine the ideas of (1) and (2) and terminate the test as soon as {(1) or (3)} *or* (2) occurs. Epstein [5] provided a lower $100\cdot(1-\alpha)\%$ confidence interval for θ and conjectured what two sided intervals might be, a conjecture later investigated via Monte Carlo generations by Harter [7]. Fairbanks et al. [6] provided a rule for a two-sided interval "nearly identical with that proposed by Epstein". This is as follows.

Without replacement case

Assume that failed items are not replaced during the test and that k ($\leqq K$ necessarily) have failed at $\min(t^*, \tau_K)$, where t^* is a pre-selected termination time, and τ_K is the failure time associated with the K-th failure. Then the two sided interval is

$$(1.2)\qquad \begin{array}{ll} [(2nt^*/\chi^2_{2,\alpha/2}), \infty]\,, & \text{if } k=0\,, \\ [(2nt^*/\chi^2_{2k+2,\alpha/2}), (2nt^*/\chi^2_{2k,1-\alpha/2})]\,, & \text{if } 1\leqq k\leqq K-1\,, \\ [(2n\tau_K/\chi^2_{2K,\alpha/2}), (2n\tau_K/\chi^2_{2K,1-\alpha/2})]\,, & \text{if } k=K\,, \end{array}$$

where $\chi^2_{m,\gamma}$ is the point exceeded by a probability γ for the chi-squared distribution with m degrees of freedom.

With replacement case

When conducted with replacement, the hybrid test is terminated at minimum (T^*, T_K) where T^* is a pre-selected *total test truncation time* and T_K is the *total test time* at the K-th failure. Fairbanks et al. [6] offer parallel confidence intervals for this case by replacing nt^* and $n\tau_K$ by T^* and T_K respectively in their result above.

We now discuss a Bayesian analysis of the hybrid test; this permits a simple derivation of the confidence interval and also allows use of the predictive density to obtain predictive intervals for future observations.

2. Prior information

We assume that the experimenter's feelings about θ before the data are taken can be summarized by use of the inverted gamma prior

$$(2.1)\qquad p(\theta)\propto\theta^{-(\gamma+1)}\exp(-t_0/\theta)\,,$$

with normalizing constant $t_0^\gamma/\Gamma(\gamma)$ when $\gamma>0$, $t_0>0$. Note that this implies a "non-informative" prior of θ^{-1} if we take $\gamma=t_0=0$ in (2.1), or can represent the results of previously observing γ failures under a single exponential law (1.1) with total life time t_0 combined with a previously non-informative prior of θ^{-1}.

It can be shown for (2.1) that, if $\gamma>2$,

$$E(\theta)=t_0/(\gamma-1)\ , \qquad V(\theta)=t_0^2/\{(\gamma-1)^2(\gamma-2)\}\ . \tag{2.2}$$

Because $(2t_0/\theta)$ is distributed as $\chi^2_{2\gamma}$, prior intervals of any desired probability content can be constructed. An experimenter's prior feelings can be interpreted using such intervals and/or (2.2), to provide values for t_0 and γ. (In general, γ need not be an integer, except when it represents previous failures, as described above.)

The predictive distribution based on (2.1), sometimes known as the "no (current) data predictive distribution" is defined by

$$h(x)=\int_0^\infty f(x\,|\,\theta)p(\theta)d\theta\ . \tag{2.3}$$

Substituting the specific expressions in (1.1) and (2.1) into (2.3), and integrating, provides

$$h(x)=(\gamma/t_0)(1+x/t_0)^{-\gamma-1}\ , \tag{2.4}$$

which implies that X is distributed as $(t_0/\gamma)F_{2,2\gamma}$ where $F_{m,n}$ is a central F-variable with m and n degrees of freedom. Eq. (2.4) thus enables predictions on lifetimes to be made based only on prior information.

A drawback to the hybrid scheme of Section 1 is that, if t^* is chosen "too small", testing may terminate with no failures, providing only a limited inference on θ. Suppose, if $\gamma>0$, we choose t^* to define a suitably small upper tail area of (2.4), δ say. Then, if $F_{m,n,\delta}$ is the percentage point of $F_{m,n}$ that leaves δ in the upper tail, a choice of $t^*=(t_0/\gamma)F_{2,2\gamma,\delta}$ implies $P(X>t^*)=\delta$. This permits a sensible choice of t^* based on the (prior) information available, because it implies that, if the prior information is correct, a test item will fail before t^* with probability $1-\delta$, and thus *the chance of no failures in the interval* $(0, t^*)$ *will be small.* If $\gamma=0$, a choice of t^* in this manner is not possible, because the non-informative prior is improper.

3. Bayesian analysis of hybrid scheme

3.1. *Without replacement case*

We assume that n items whose failure times follow (1.1) are placed on test and that prior information is given by (2.1). We denote by x_i the observed failure time of the i-th item to fail and assume that failed items are not replaced, so that $x_i \leqq x_{i+1}$. Our hybrid test scheme will terminate at time minimum (t^*, τ_K) where t^* and K are chosen in advance; time t^* may or may not be chosen as in Section 2, and the integer K is such that $K \leqq n$, and it would often be selected close to $n/2$. Let $k \leqq K$ be the number of failures seen at the time the test is terminated so that $x_k \leqq t^*$ if $1 \leqq k \leqq K-1$, and $x_K < t^*$ if $k=K$, while if $k=0$, $t^* < x_1$. The likelihood is

$$l(\theta \mid \text{data}) \propto \theta^{-k} \exp[-A_K/\theta] , \qquad 0 \leqq k \leqq K , \tag{3.1}$$

where A_K, the total observed lifetime, is

$$A_k = \begin{cases} \xi(k) \sum_{j=1}^{k} x_j + (n-k)t^* , & 0 \leqq k \leqq K , \\ \sum_{j=1}^{K} x_j + (n-K)x_K , & k=K , \end{cases} \tag{3.2}$$

and where $\xi(k)=0$ if $k=0$, and 1 otherwise. Combining (2.1) with (3.1) provides the posterior

$$p(\theta \mid \text{data}) \propto \theta^{-(k+\gamma+1)} \exp[-(A_k+t_0)/\theta] , \qquad 0 \leqq k \leqq K . \tag{3.3}$$

Note that $(A_k+t_0)/\theta$ is distributed as $\chi^2_{2(k+\gamma)}/2$, a posteriori, so that

$$E(\theta \mid \text{data}) = (A_k+t_0)/(k+\gamma-1) , \tag{3.4}$$

if $k+\gamma>1$. (It is interesting to see that the posterior mean formula (3.4) coincides with Bhattacharya's (21), obtained by considering Types I and II censoring individually. A variance formula similar to Bhattacharya's (22) would also be obtained from the hybrid case.) A $100(1-\alpha)\%$ posterior interval for θ is given by

$$(2(A_k+t_0)/\chi^2_{2(k+\gamma),\alpha/2},\ 2(A_k+t_0)/\chi^2_{2(k+\gamma),1-\alpha/2}) , \tag{3.5}$$

if $k+\gamma>0$, where $\chi^2_{m,\alpha}$ denotes the point of the central χ^2 distribution with m degrees of freedom that leaves an area α in the upper tail.

The maximum likelihood estimator for θ obtained by maximizing (3.1) is $\hat{\theta}=A_k/k$ which may be compared with (3.4).

3.2 *With replacement case*

We again assume that n items whose failure times follow (1.1) are placed on test, and that prior information is given by (2.1). The hybrid test scheme will now terminate at time minimum (T^*, T_K) where T^* is a pre-selected total test truncation time, and T_K is the total life of all items subjected to test up to and including the time of the K-th failure, where K is pre-chosen. T^* may or may not be chosen as nt^* where t^* is chosen as in Section 2, and $K \leqq n$ and would often be selected close to $n/2$. Let $k \leqq K$ be the number of failures seen at the time the test is terminated. Equations (3.1), (3.3), (3.4) and (3.5) now apply to this case but with redefinition of A_k as

$$A_k = \begin{cases} T^*, & 0 \leqq k < K, \\ T_K, & k = K. \end{cases} \tag{3.6}$$

4. Predictive distribution of X (both cases)

Suppose we wish to predict the future life X of an item based on data from a hybrid test. The predictive density of X, given the data, is

$$h(x \mid \text{data}) = \int_0^\infty f(x \mid \theta) p(\theta \mid \text{data}) d\theta \tag{4.1}$$

$$\propto (1 + (A_k + t_0)^{-1} x)^{-(k+\gamma+1)} \tag{4.2}$$

using (1.1) and (3.3); compare with (2.4). This implies that X is distributed as $[(A_k + t_0)/(k+\gamma)] F_{2,2(k+\gamma)}$. Thus, a predictor for a future life time is

$$E(X \mid \text{data}) = (A_k + t_0)/(k + \gamma - 1) \tag{4.3}$$

if $k+\gamma>1$, and a $100(1-\alpha)\%$ predictive interval for X is given by

$$\{[(A_k + t_0)/(k+\gamma)] F_{2,2(k+\gamma),1-\alpha/2},\ [(A_k + t_0)/(k+\gamma)] F_{2,2(k+\gamma),\alpha/2}\} \tag{4.4}$$

if $k+\gamma>0$. Note that the right hand sides of (3.4) and (4.3) are the same.

5. Examples (without replacement case)

We illustrate use of the formulas in Section 3 with data from a without replacement situation.

Suppose ten test items ($n=10$) are placed on life test in a situation where t^* is chosen arbitrarily as 1300, $\gamma=2$, and $t_0=1000$, so that the prior mean is 1000. Four failures at times $x=836$, 974, 1108, and 1236 were observed. We shall use the same data to illustrate two separate cases: (1) $K=4$, (2) $K=5$. If we had chosen $K=4$, we would have stopped testing at time $x_4=1236<t^*=1300$. Had we chosen $K=5$, however, the test would have terminated at $t^*=1300$ because the fifth failure has not yet been observed. The relevant estimates are shown in Table 1. We notice that the posterior and predictive means are smaller when ($k=4$, $K=4$) compared with the ($k=4$, $K=5$) case. This, of course, happens because, in the latter case, more time on test *with no additional failures* is observed compared with the former case. This

Table 1. Estimates obtained from the two cases

Characteristics examined	Estimates under case	
	$K=4$	$K=5$
Posterior mean, (3.4), for θ	2514	2591
95% posterior interval, (3.5), for θ	1077, 5709	1110, 5883
Predictive mean, (4.3), for X	2514	2591
95% predictive interval, (4.4), for X	53, 10,676	55, 11,002

behavior is most reasonable.

Note that, if we were to choose t^* by applying the formula $t^* = (t_0/\gamma)F_{2,2\gamma,\delta}$ discussed in Section 2, we would obtain the value 1300 used above for $\delta=0.189$. For values of $\delta=0.25$, 0.10, 0.05, 0.01 we obtain $t^*=1000$, 2163, 3472, 9000 respectively.

6. Other distributions

For the exponential distribution,

$$1-F(x\,|\,\theta)=\theta f(x\,|\,\theta)\,, \tag{6.1}$$

where f is defined in (1.1) and F is the corresponding cdf. This facilitated the development in Section 3. For distributions for which this property does not hold, such as the gamma, Weibull, and normal, parallel steps to those in Section 3 can still be carried out in principle. However, the various integrals that arise must then be evaluated by numerical integration for particular data sets. (The difficulties are especially pronounced for the with replacement case, where the various possible patterns of failures would need careful enumeration.)

Acknowledgements

This work was partially sponsored by the United States Army under Contract No. DAAG29-80-C-0041, and by NSERC (Canada) under Grant No. A8743. We are grateful to a referee for a number of constructive comments.

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