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LATENT SCALE LINEAR MODELS FOR MULTIVARIATE ORDINAL RESPONSES AND ANALYSIS BY THE METHOD OF WEIGHTED LEAST SQUARES

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Summary

A multivariate latent scale linear model is defined for multivariate ordered categorical responses and inference procedures based on the weighted least squares method are developed. Several applications of the model are suggested and illustrated through an analysis of real data. Asymptotic properties of the weighted least squares method are examined and some consequences of misspecification of the model are also discussed.

1. Introduction

In analysis of several multivariate samples with continuous variables, usually linear models for location vectors are adopted, e.g. Anderson [5], Rao [25], Gnanadesikan [14]. For multivariate quantal or categorical observations, are used log-linear models, e.g. Plackett [24], Bishop, Fienberg and Holland [9], or linear models for functions of probabilities proposed by Grizzle, Starmer and Koch [19], and Koch et al. [21]. For ordered categorical responses, however, models for multivariate analysis have been developed mainly for analysis of associations, Goodman [15]-[17], Wahrendorf [29], Clogg [13], Agresti [1], [2], and others. In the present paper, we shall consider a latent scale linear model for factorial analysis of multivariate samples with ordered categorical responses. The model, which was proposed by Uesaka and Asano [27], is a natural extension of the multivariate logit model for multivariate quantal responses by Grizzle [18], and of the latent scale linear model for the univariate ordered categorical responses considered by Uesaka and Asano [28]. A similar approach has appeared in factor analysis of multivariate dichotomous variables, Christoffersson [12],

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Muthen [22], Muthen and Christoffersson [23], where multivariate normality is assumed for the latent variables.

In Section 2, a multivariate latent scale linear model is defined. In Section 3, a method of parameter estimation is presented, and assessing validity of the model and testing linear hypotheses are discussed in Section 4. In Section 5, some applications are illustrated. Then in Section 6, we consider some consequences of misspecification of the model. In Section 7, an illustrative example of analysis of practical data is presented. The final section gives further discussion.

2. Multivariate latent scale linear model

Let $Y=(Y_1,\cdots,Y_K)'$ be a K-variate observation vector, where Y_k is measured as one of c_k ordered categories, $k=1,\cdots,K$. Without loss of generality, the u-th category is denoted by an integer u, $u=1,\cdots,c_k$, $k=1,\cdots,K$. We assume that each variable Y_k is a manifestation of a latent response or a latent trait. Thus we consider a K-variate continuous latent random vector $Z=(Z_1,\cdots,Z_K)'$, and assume that

$$(1)\qquad Y_k=u\,,\qquad \text{if } \tau_{ku-1}\leqq Z_k<\tau_{ku},\ u=1,\cdots,c_k\,,$$

where $\tau_{k0}=-\infty<\tau_{k1}<\cdots<\tau_{kc_k-1}<\tau_{kc_k}=+\infty$ and the τ_{ku}'s are unknown constants.

Consider I random samples $\{Y_{i\alpha},\ \alpha=1,\cdots,n_i\}$ and the corresponding latent vectors $\{Z_{i\alpha},\ \alpha=1,\cdots,n_i\}$, for $i=1,\cdots,I$. Assume that $Z_{i\alpha}$, $\alpha=1,\cdots,n_i$ are independently and identically distributed with c.d.f. $F_i(z)$, $i=1,\cdots,I$. The present model is that

$$(2)\qquad F_i(z)=F(z-\mu_i)\,,\qquad i=1,\cdots,I\,,$$

where $\mu_i=(\mu_{i1},\cdots,\mu_{iK})'$, $i=1,\cdots,I$ are unknown location vectors and $F(z)$ is a K-variate continuous distribution function. Further we define a linear model for location parameters as

$$(3)\qquad \mu_i=X_i\beta\,,\qquad i=1,\cdots,I\,,$$

where $X_i=[x'_{i1},\cdots,x'_{iK}]'$, $i=1,\cdots,I$ are given $K\times p$ full rank matrices and β is an unknown p-dimensional parameter vector. The basic model defined by (2) and (3) is a multivariate linear model for several multivariate samples on continuous random variables. This suggests that factorial effects can be evaluated by linear hypotheses for the location vectors, as is the case of MANOVA. In what follows, we shall develop a methods of analysis of factorial effects for an $I\times(c_1\times\cdots\times c_K)$ contingency table defined by $\{Y_{i\alpha}\}$. The observed frequency table has a product multinomial distribution whose multinomial probabilities are determined by the $F_i(z)$'s. Since the latent variables may be correlated in an unknown form, the latent distribution $F(z)$ and consequently the multinomial probabilities cannot be specified in a parametric form, and usual method such as maximum likelihood cannot be applied. Though, since location parameters can be estimated from each variable separately, it is sufficient for statistical inference that the only K univariate marginal distributions of $F(z)$ are specified.

In what follows, we assume that $F(z)$ is a continuous distribution function having the whole K-dimensional Euclidean space as the support. Further $F(z)$ is assumed to have bounded second order partial derivatives. Let the univariate marginal cumulative distributions (c.d.f.) and probability density functions (p.d.f.) of the $F(z)$ be $F_{[k]}(z)$ and $f_{[k]}(z)$, respectively, $k=1,\cdots,K$, and let q_{iku} be the cumulative probability up to the u-th category of the k-th variable of the i-th population, $u=1,\cdots,c_k$, $k=1,\cdots,K$, $i=1,\cdots,I$. Then from (1) and (2) we have

$$(M_D) \qquad F_{[k]}^{-1}(q_{iku})=\tau_{ku}-\mu_{ik} \,, \quad u=1,\cdots,c_k-1,\ k=1,\cdots,K \text{ and } i=1,\cdots,I\,.$$

This model implies (i) homogeneity of marginal distributions over the I populations except for location parameters, and (ii) homegeneity of category boundaries over the I populations. The model (3) is, under the assumption (M_D), again written as

$$(M_L) \qquad \mu_{ik}=\boldsymbol{x}_{ik}\boldsymbol{\beta}\,, \qquad k=1,\cdots,K \text{ and } i=1,\cdots,I\,.$$

Thus our model is re-expressed as a linear model for functions of univariate marginal cumulative probabilities.

As is easily seen from (M_D) some constraints must be imposed on the $\{\tau_{ku}\}$ and $\boldsymbol{\beta}$ to identify parameters. Usually one of the following restrictions is adopted;

$$\text{C1.} \qquad \tau_{k1}+\cdots+\tau_{kc_k-1}=0\,, \qquad k=1,\cdots,K\,,$$

$$\text{C2.} \qquad \boldsymbol{\mu}_1+\cdots+\boldsymbol{\mu}_I=\boldsymbol{0} \qquad \text{or} \qquad n_1\boldsymbol{\mu}_1+\cdots+n_I\boldsymbol{\mu}_I=\boldsymbol{0}\,.$$

Let c be $c_1+\cdots+c_K$ and denote by $\boldsymbol{\tau}$ the vector of free τ's, and let q be the number of free τ's. Then for C1, $q=c-2K$, and for C2, $q=c-K$.

3. Parameter estimation and testing linear hypotheses

The expressions (M_D) and (M_L) for the multivariate latent scale linear model show that a method of weighted least squares with empirical weights, Grizzle, Starmer and Koch [19] and Bhapkar [9], can be effectively applied to parameter estimation.

Let N_{iku} be the number of observations among $\{Y_{i\alpha k},\ \alpha=1,\cdots,n_i\}$ which take the values being equal to or less than u, and let Q_{iku} be a positive consistent estimate of q_{iku} such that

$$(4)\qquad Q_{iku}-N_{iku}/n_i=o_p(n_i^{-1/2}),\quad u=1,\cdots,c_k,\ k=1,\cdots,K,\ i=1,\cdots,I.$$

Let

$$(5)\qquad Z_{iku}=F_{[k]}^{-1}(Q_{iku}),\quad \text{for } u=1,\cdots,c_k-1,\ k=1,\cdots,K,\ i=1,\cdots,I.$$

Then we have

$$(6)\qquad Z_{iku}=\tau_{ku}-\boldsymbol{x}_{ik}\boldsymbol{\beta}+e_{iku}+o_p(n_i^{-1/2}),$$

where

$$(7)\qquad e_{iku}=\frac{Q_{iku}-q_{iku}}{f_{[k]}(F_{[k]}^{-1}(q_{iku}))}.$$

Hereafter, we use the symbol Z to denote a sample equivalent deviate instead of a latent variable. Let $n=n_1+\cdots+n_I$ and assume that

$$(8)\qquad \lim_{n\to\infty} n_i/n=\lambda_i>0,\qquad i=1,\cdots,I.$$

Let

$$(9)\qquad \eta_{iku}=F_{[k]}^{-1}(q_{iku}),\quad u=1,\cdots,c_k-1,\ k=1,\cdots,K,\ i=1,\cdots,I,$$

$$\boldsymbol{\eta}=(\eta_{111},\cdots,\eta_{1Kc_K-1},\cdots,\eta_{IKc_K-1})',$$

$$(10)\qquad \boldsymbol{Z}=(Z_{111},\cdots,Z_{1Kc_K-1},\cdots,Z_{IKc_K-1})',$$

$$(11)\qquad \boldsymbol{e}=(e_{111},\cdots,e_{1Kc_K-1},\cdots,e_{IKc_K-1})'.$$

Then as n tends to infinity, $\sqrt{n}\,\boldsymbol{e}$ has an asymptotic multivariate normal distribution with mean vector $\boldsymbol{0}$ and variance-covariance matrix

$$(12)\qquad \boldsymbol{\Sigma}=\mathrm{Diag}\,[\boldsymbol{\Sigma}_1/\lambda_1,\cdots,\boldsymbol{\Sigma}_I/\lambda_I],$$

where $\boldsymbol{\Sigma}_i$ is a symmetric matrix of order $c-K$ with elements

$$(13)\qquad \sigma_{i,km,uv}=\begin{cases}\dfrac{q_{i,km,uv}-q_{iku}q_{imv}}{f_{[k]}(F_{[k]}^{-1}(q_{iku}))f_{[m]}(F_{[m]}^{-1}(q_{imv}))}\,, \\ \qquad \text{for } 1\leqq u\leqq c_k-1,\ 1\leqq v\leqq c_m-1,\ 1\leqq k\neq m\leqq K\,, \\ \dfrac{q_{iku}-q_{iku}q_{ikv}}{f_{[k]}(F_{[k]}^{-1}(q_{iku}))f_{[k]}(F_{[k]}^{-1}(q_{ikv}))}\,, \\ \qquad \text{for } u\leqq v,\ k=m=1,\cdots,K\,,\end{cases}$$

where $q_{i,km,uv}$ is a bivariate cumulative probability of the joint distribution of Y_k and Y_m at the i-th population. Let V and V_i, $i=1,\cdots,I$ be consistent estimates of Σ and Σ_i, $i=1,\cdots,I$, respectively, obtained by replacing the q_{iku}'s and $q_{i,km,uv}$'s in the equation (13) by their sample values Q_{iku}'s and $Q_{i,km,uv}$'s, where $Q_{i,km,uv}$ is a positive estimate of $q_{i,km,uv}$ such that

$$(14)\qquad Q_{i,km,uv}-q_{i,km,uv}=o_p(n^{-1/2})\,.$$

The weighted least squares (WLS) estimator of $(\tau', \beta')'$ is the solution to minimization problem of the weighted sum of squares

$$(15)\qquad S(\tau,\beta)=\sum_{i=1}^{I} n_i\left(Z_i-W_i\begin{pmatrix}\tau\\ \beta\end{pmatrix}\right)'V_i^{-1}\left(Z_i-W_i\begin{pmatrix}\tau\\ \beta\end{pmatrix}\right)\,,$$

where $Z_i=(Z_{i11},\cdots,Z_{iKc_K-1})'$, $i=1,\cdots,I$, and

$$(16)\qquad W_i=\begin{bmatrix} A & \begin{matrix}-1_{c_1-1}\otimes x_{i1}\\ \vdots \\ -1_{c_K-1}\otimes x_{iK}\end{matrix}\end{bmatrix},\qquad i=1,\cdots,I\,,$$

where A is the coefficient matrix of τ and 1_{c_k-1} is the c_k-1 dimensional column vector of ones. Let $W=[W_1',\cdots,W_I']'$, $G=[W'V^{-1}W]^{-1}$ and $\Gamma=[W'\Sigma^{-1}W]^{-1}$. Then the WLS estimate of $(\tau',\beta')'$, denoted by $(\hat{\tau}',\hat{\beta}')'$, is given by

$$(17)\qquad \begin{bmatrix}\hat{\tau}\\ \hat{\beta}\end{bmatrix}=GW'V^{-1}Z\,,$$

and the estimated variance-covariance matrix is given by G. When the conditions (M_D) and (M_L) simultaneously hold, $\sqrt{n}[(\hat{\tau}',\hat{\beta}')'-(\tau',\beta')']$ has the asymptotic multivariate normal distribution with mean vector 0 and the variance-covariance matrix Γ.

4. Assessing validity of the model and testing linear hypotheses

4.1. *Under the null hypothesis*

The validity of the model can be checked by residual sum of squares

$S(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\beta}})$. When the model (M_L) is true, this statistic has an asymptotic chi-square distribution with $I\times(c-K)-(p+q)$ degrees of freedom. If this statistic is large enough to doubt the validity of the model (M_L), $S(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\beta}})$ may be decomposed into two components like $S(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\beta}})=S(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}})+S(\hat{\boldsymbol{\beta}}\mid\tilde{\boldsymbol{\mu}})$, where $S(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\beta}})$ is the weighted sum of squares of residuals under the model (M_D), and $S(\hat{\boldsymbol{\beta}}\mid\tilde{\boldsymbol{\mu}})$ is the increment of residual sum of squares due to additional constraints imposed by (M_L). This sum of squares is equal to the weighted sum of squares for the linear model

$$\begin{bmatrix}\tilde{\boldsymbol{\tau}}\\ \tilde{\boldsymbol{\mu}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}\end{bmatrix}\begin{bmatrix}\boldsymbol{\tau}\\ \boldsymbol{\beta}\end{bmatrix}+\begin{bmatrix}\boldsymbol{e}_\tau\\ \boldsymbol{e}_\mu\end{bmatrix}, \qquad \text{with weights } n\boldsymbol{G}^{-1}, \tag{18}$$

where $\boldsymbol{e}_\tau$ and $\boldsymbol{e}_\mu$ are error vectors associated with $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\mu}}$, respectively, and $\boldsymbol{X}=[\boldsymbol{X}_1',\cdots,\boldsymbol{X}_I']'$. When (M_D) is true $S(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}})$ has an asymptotic chi-square distribution with $(I-1)(c-2K)$ degrees of freedom. Further when the model (M_L) is true $S(\hat{\boldsymbol{\beta}}\mid\tilde{\boldsymbol{\mu}})$ has an asymptotic chi-square distribution with $IK-p$ or $IK-K-p$ degrees of freedom according to the identifiability condition C1 or C2. Further examination of validity of the model can be done by a close inspection of standardized residuals given by

$$R_{iku}=\frac{\sqrt{n_i}\,(Z_{iku}-\hat{\tau}_{ku}+\hat{\mu}_{ik})}{\sqrt{\hat{\sigma}_{i,kk,uu}-\boldsymbol{w}_{iku}\boldsymbol{G}\boldsymbol{w}_{iku}'}}\,, \tag{19}$$
$$u=1,\cdots,c_k-1,\ k=1,\cdots,K,\ i=1,\cdots,I,$$

where $\boldsymbol{w}_{iku}$ is the iku-th row vector of $\boldsymbol{W}$.

Consider a linear hypothesis H_0 that $\boldsymbol{H\beta}=\boldsymbol{0}$, where $\boldsymbol{H}$ is an $r\times p$ full rank matrix. The test of H_0 against an alternative H_1 that $\boldsymbol{H\beta}\neq\boldsymbol{0}$ can be made on the basis of a chi-squared statistic

$$X^2=(\boldsymbol{H}\hat{\boldsymbol{\beta}})'(\boldsymbol{H}\boldsymbol{V}(\hat{\boldsymbol{\beta}})\boldsymbol{H}')^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}})\,, \tag{20}$$

where $\boldsymbol{V}(\hat{\boldsymbol{\beta}})$ is an estimated variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and is given by the last $p\times p$ submatrix of $\boldsymbol{G}$. When the null hypothesis H_0 is true, X^2 has asymptotically the chi-square distribution with r degrees of freedom. It should be pointed out that X^2 of (20) is equal to $S(\hat{\boldsymbol{\beta}}\mid\boldsymbol{\beta}, \boldsymbol{H})$, the difference of residual sum of squares under the hypothesis H_0 and under the model (M_L).

4.2. *Asymptotic properties under Pitman type alternatives*

Assume that the conditions (M_D) and (M_L) simultaneously hold and consider the asymptotic power for a Pitman type sequence of alternatives. Let the location vector of the i-th population for the sample of size n_i be

$$\boldsymbol{\mu}_{i(n)}=\boldsymbol{X}_i\boldsymbol{\beta}_{(n)}\,, \qquad i=1,\cdots,I, \tag{21}$$

and let

$$\boldsymbol{\beta}_{(n)}=\boldsymbol{\beta}_{(0)}+\boldsymbol{\xi}/\sqrt{n}\ , \tag{22}$$

where $\boldsymbol{\beta}_{(0)}$ and $\boldsymbol{\xi}$ are constant p-vectors such that

$$\boldsymbol{H\beta}_{(0)}=\mathbf{0} \quad \text{and} \quad \boldsymbol{H\xi}\neq\mathbf{0}\ . \tag{23}$$

Denote the parent univariate and bivariate marginal cumulative probabilities of the i-th sample, for $i=1,\cdots,I$, be $\{q_{iku}^{(n)}\}$ and $\{q_{i,km,uv}^{(n)}\}$, respectively, and let their limiting values for infinitely large n be $\{q_{iku}^{(0)}\}$ and $\{q_{i,km,uv}^{(0)}\}$, respectively. Then since $F(\boldsymbol{z})$ has bounded and continuous second order partial derivatives, we have

$$q_{iku}^{(n)}=q_{iku}^{(0)}+O(n^{-1/2})\ ,\qquad q_{i,km,uv}^{(n)}=q_{i,km,uv}^{(0)}+O(n^{-1/2})\ . \tag{24}$$

Applying a convergence theorem, e.g. Rao [26], p. 385, we have the following theorem.

THEOREM 1. *When the sample sizes tend to infinity, the WLS estimate* $(\hat{\boldsymbol{\tau}}_n', \hat{\boldsymbol{\beta}}_n')'$ *is asymptotically multivariate normal, i.e.,*

$$\sqrt{n}\begin{pmatrix}\hat{\boldsymbol{\tau}}_n-\boldsymbol{\tau}\\ \hat{\boldsymbol{\beta}}_n-\boldsymbol{\beta}_{(n)}\end{pmatrix}\xrightarrow{\mathcal{L}} N(0,\,[\boldsymbol{W}'\boldsymbol{\Sigma}^{(0)-1}\boldsymbol{W}]^{-1})\ ,$$

where $\boldsymbol{\Sigma}^{(0)}$ *is the matrix defined by* (13) *with replacing* $\{q_{iku}\}$ *and* $\{q_{i,km,uv}\}$ *by* $\{q_{iku}^{(0)}\}$ *and* $\{q_{i,km,uv}^{(0)}\}$, *respectively, and* $\mathcal{L}$ *stands for a convergence in law.*

COROLLARY 1. *The chi-square test statistic defined by* (20) *has asymptotically a noncentral chi-square distribution with r degrees of freedom and noncentrality parameter*

$$\gamma^2=(\boldsymbol{H\xi})'[\boldsymbol{H}(\boldsymbol{W}'\boldsymbol{\Sigma}^{(0)-1}\boldsymbol{W})\boldsymbol{H}']^{-1}(\boldsymbol{H\xi})\ . \tag{25}$$

5. Illustrations for applications

Later we shall give an analysis of real data. At present we illustrate a few applications.

1. Analysis of one-way layout and testing homogeneity of several samples. Suppose we wish to test that the I populations are homogeneous. In our formulation, this means that we test the null hypothesis

$$H_0:\ \boldsymbol{\mu}_1=\cdots=\boldsymbol{\mu}_I$$

against the alternative H_1 that at least one of the equalities does not hold. It should be noted that under the null hypothesis, model (M_D) holds for any univariate continuous c.d.f. $G(\boldsymbol{z})$, but the power of the test may decrease. This will be discussed in Section 6.

2. Factorial analysis in a two-way layout. For a two-way layout experiment with factors A and B, we wish to test whether there exists no interaction between A and B, whether there exists no main effect of A or B. Let the linear hypotheses corresponding to these statements be $\boldsymbol{H}_{A\times B}\boldsymbol{\beta}=\mathbf{0}$, $\boldsymbol{H}_A\boldsymbol{\beta}=\mathbf{0}$ and $\boldsymbol{H}_B\boldsymbol{\beta}=\mathbf{0}$, respectively. The test statistics are computed from the equation (20) by replacing the matrix $\boldsymbol{H}$ with $\boldsymbol{H}_{A\times B}$, $\boldsymbol{H}_A$ and $\boldsymbol{H}_B$, respectively. Then we can construct a MANOVA table. Note that these chi-squared statistics are not mutually independent, though the $\boldsymbol{H}_{A\times B}$, $\boldsymbol{H}_A$ and $\boldsymbol{H}_B$ are mutually orthogonal. An example of analysis of a real data of this form will be given in Section 7.

3. Analysis of repeated measures experiments. Suppose the K response variables are obtained on the same variable with c ordered categories under K different conditions. It may be reasonable to assume that the relationship between the observed category and the latent variable may be identical over the K conditions and that the K marginal distributions $F_{[k]}(z)$, $k=1,\cdots,K$ are identical. Thus the constraints

$$\tau_{1u}=\tau_{2u}=\cdots=\tau_{Ku}\ , \qquad u=1,\cdots,c-1\ , \tag{26}$$

and

$$F_{[1]}(z)=\cdots=F_{[K]}(z) \tag{27}$$

are introduced. Whole procedure of analysis is the same as described in the previous sections except for the restrictions defined by the equations (26) and (27).

4. Multivariate logit analysis or approximate analysis of multivariate probit. When all variables are quantal and the K univariate variables are assumed to have the logistic distributions, our method agrees with the multivariate logit analysis proposed by Grizzle [18]. If we take $F_{[k]}(z)$ to be the standard normal distribution, our model gives an appoximate solution for multivariate probit model.

5. Analysis of a randomized block experiment. Suppose that we have n subjects and K treatments are randomly allocated to K homogeneous experimental units on every subject. Then a response variable with c ordered categories is observed for every treatment. The observations are arranged into a $c\times c\times\cdots\times c$ table. It is assumed that the latent response vectors $\boldsymbol{Z}_\alpha$, $\alpha=1,\cdots,n$ are independently and identically distributed in a K variate continuous c.d.f. $F(\boldsymbol{z})$ which is assumed to be symmetric with respect to the K arguments. The interest here is to evaluate whether the K treatments have the same effect or not. Setting I to be 1, the same model as that for the repeated measures experi-

ments described above can be applied. The linear hypothesis is that

$$H_0:\ \mu_1=\cdots=\mu_K\ .$$

6. Validity and efficiency of a specific distributional model

Suppose that the multivariate latent scale linear model (M_L) holds for a latent distribution $F(z)$. The true latent distribution is, however, not known, and one should assume the latent marginal distributions to be, say, $G_{[k]}(z)$, $k=1,\cdots,K$. Thus it is important to know some consequences of misspecification of the distributional model. Here we shall investigate this problem.

6.1. *General*

Let

(28)
$$\eta^*_{iku}=G^{-1}_{[k]}(q_{iku})\ ,\qquad Z^*_{iku}=G^{-1}_{[k]}(Q_{iku})\ ,$$
$$e^*_{iku}=\frac{Q_{iku}-q_{iku}}{g_{[k]}(\eta^*_{iku})}\ ,\qquad u=1,\cdots,c_k-1,\ k=1,\cdots,K,\ i=1,\cdots,I,$$

where $g_{[k]}(z)$ is the p.d.f. of $G_{[k]}(z)$, $k=1,\cdots,K$. Further let

(29)
$$\boldsymbol{Z}^*=(Z^*_{111},\cdots,Z^*_{1Kc_K-1},\cdots,Z^*_{IKc_K-1})'\ ,$$

and

(30)
$$\boldsymbol{e}^*=(e^*_{111},\cdots,e^*_{1Kc_K-1},\cdots,e^*_{IKc_K-1})'\ .$$

Then we have

(31)
$$\boldsymbol{Z}^*=\boldsymbol{\eta}^*+\boldsymbol{e}^*+o_p(n^{-1/2})\ .$$

Let $\sigma^*_{i,km,uv}$ be the covariance between $\sqrt{n_i}\,e^*_{iku}$ and $\sqrt{n_i}\,e^*_{imv}$, and denote

(32)
$$\boldsymbol{\Sigma}^*_i=(\sigma^*_{i,km,uv})\ ,\qquad i=1,\cdots,I,$$
$$\boldsymbol{\Sigma}^*=\text{Diag.}\,[\boldsymbol{\Sigma}^*_1/\lambda_1,\cdots,\boldsymbol{\Sigma}^*_I/\lambda_I]\ .$$

It is easily seen that $\boldsymbol{Z}^*$ converges in probability to $\boldsymbol{\eta}^*$ and $\sqrt{n}\,(\boldsymbol{Z}^*-\boldsymbol{\eta}^*)$ is asymptotically normal with mean $\boldsymbol{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}^*$. The vector $\boldsymbol{\eta}^*$ may not lie in the subspace spanned by the column vectors of $\boldsymbol{W}$. Let

(33)
$$Z^*_{iku}=\tau^*_{ku}-\boldsymbol{x}_{ik}\boldsymbol{\beta}^*+\xi^*_{iku}+e^*_{iku}+o_p(n^{-1/2})\ ,$$

where $\boldsymbol{\tau}^*=\{\tau^*_{ku}\}$ and $\boldsymbol{\beta}^*$ are any constant vectors and the ξ^*_{iku}'s are deviations determined depending on $\{\tau^*_{ku}\}$ and $\boldsymbol{\beta}^*$. Now the WLS estimate of $(\boldsymbol{\tau}^{*\prime},\boldsymbol{\beta}^{*\prime})'$ is

$$\begin{pmatrix} \hat{\boldsymbol{\tau}}^* \\ \hat{\boldsymbol{\beta}}^* \end{pmatrix} = \boldsymbol{G}^*\boldsymbol{W}'\boldsymbol{V}^{*-1}\boldsymbol{Z}^* , \tag{34}$$

where $\boldsymbol{V}^*$ is an estimate of $\boldsymbol{\Sigma}^*$ and $\boldsymbol{G}^* = [\boldsymbol{W}'\boldsymbol{V}^{*-1}\boldsymbol{W}]^{-1}$. By Taylor expansion we can write

$$\boldsymbol{G}^*\boldsymbol{W}'\boldsymbol{V}^{*-1} = \boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1} + \boldsymbol{E} + o_p(n^{-1/2}) , \tag{35}$$

where $\boldsymbol{\Gamma}^*$ is the p-limit of $\boldsymbol{G}^*$ and is equal to $[\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{W}]^{-1}$, and

$$\boldsymbol{E} = \sum_{i=1}^{I}\sum_{k=1}^{K}\sum_{u=1}^{c_k}\left[\frac{\partial}{\partial q_{iku}}\boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\right](Q_{iku} - q_{iku}) . \tag{36}$$

Since $(\hat{\boldsymbol{\tau}}^{*\prime}, \hat{\boldsymbol{\beta}}^{*\prime})'$ converges to $\boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{\eta}^*$, we have

$$\boldsymbol{\xi}^* = \boldsymbol{\eta}^* - \boldsymbol{W}\boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{\eta}^* . \tag{37}$$

Denote

$$\begin{pmatrix} \boldsymbol{\tau}^{**} \\ \boldsymbol{\beta}^{**} \end{pmatrix} = \boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{\eta}^* , \tag{38}$$

then

$$\begin{aligned} \begin{pmatrix} \hat{\boldsymbol{\tau}}^* \\ \hat{\boldsymbol{\beta}}^* \end{pmatrix} &= \boldsymbol{G}^*\boldsymbol{W}'\boldsymbol{V}^{*-1}\left(\boldsymbol{W}\begin{pmatrix} \boldsymbol{\tau}^{**} \\ \boldsymbol{\beta}^{**} \end{pmatrix} + \boldsymbol{\xi}^* + \boldsymbol{e}^*\right) + o_p(n^{-1/2}) \\ &= \begin{pmatrix} \boldsymbol{\tau}^{**} \\ \boldsymbol{\beta}^{**} \end{pmatrix} + \boldsymbol{G}^*\boldsymbol{W}'\boldsymbol{V}^{*-1}(\boldsymbol{\xi}^* + \boldsymbol{e}^*) + o_p(n^{-1/2}) . \end{aligned} \tag{39}$$

Using (35), (37), (38) and that $\boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{\xi}^* = \boldsymbol{0}$, we have

$$\begin{pmatrix} \hat{\boldsymbol{\tau}}^* \\ \hat{\boldsymbol{\beta}}^* \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau}^{**} \\ \boldsymbol{\beta}^{**} \end{pmatrix} + \boldsymbol{E}\boldsymbol{\xi}^* + \boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{e}^* + o_p(n^{-1/2}) . \tag{40}$$

This leads to the following theorem.

THEOREM 2. *Suppose that for a K-variate continuous c.d.f.* $F(\boldsymbol{z})$ *the linear model defined by* (M_D) *and* (M_L) *is satisfied and that* $G_{[k]}(z)$, $k=1, \cdots, K$ *are the hypothesized univariate marginal c.d.f.'s of the* $\boldsymbol{Z}$. *Then, as n tends to infinity under the condition* (8), *the WLS estimator of* $(\boldsymbol{\tau}', \boldsymbol{\beta}')'$ *converges in probability to* (38). *Further* $\sqrt{n}([\hat{\boldsymbol{\tau}} - \boldsymbol{\tau}^{**}]', [\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{**}]')'$ *has an asymptotic multivariate normal distribution with mean vector* $\boldsymbol{0}$ *and variance-covariance matrix* $\boldsymbol{\Gamma}^{**}$, *where* $\boldsymbol{\Gamma}^{**}$ *is the asymptotic variance-covariance matrix of* $\sqrt{n}(\boldsymbol{E}\boldsymbol{\xi}^* + \boldsymbol{\Gamma}^*\boldsymbol{W}'\boldsymbol{\Sigma}^{*-1}\boldsymbol{e}^*)$.

COROLLARY 2. *If as n tends to infinity* $\boldsymbol{\xi}^*$ *tends to* $\boldsymbol{0}$ *with order* $o(1)$, $\sqrt{n}(\hat{\boldsymbol{\tau}}', \hat{\boldsymbol{\beta}}')'$ *has the asymptotic variance-covariance matrix* $\boldsymbol{\Gamma}^*$.

6.2. *Asymptotic efficiency of the test for homogeneity of I populations*

In order to see asymptotic efficiency of our procedure, let us consider testing homogeneity of I multivariate samples which was stated in the preceding section. Assume that n tends to infinity with satisfying the condition (8). Suppose the null hypothesis H_0 given in the Section 5 is tested against an alternative defined by

$$(41)\qquad H_n:\ \boldsymbol{\mu}_i=\boldsymbol{\beta}_i/\sqrt{n}\ ,\qquad i=1,\cdots,I,$$

where $\boldsymbol{\beta}_i=(\beta_{i1},\cdots,\beta_{iK})'$, $i=1,\cdots,I$ are unknown K-dimensional vectors such that $\lambda_1\boldsymbol{\beta}_1+\cdots+\lambda_I\boldsymbol{\beta}_I=\mathbf{0}$.

By Taylor expansion, we have

$$(42)\qquad \eta^*_{iku}=\tau^*_{ku}-\frac{\beta_{ik}}{\sqrt{n}}\frac{f_{[k]}(\tau_{ku})}{g_{[k]}(\tau^*_{ku})}+o(n^{-1/2})\ ,$$

where $\tau^*_{ku}=G^{-1}_{[k]}(F_{[k]}[\tau_{ku}])$, $u=1,\cdots,c_k-1$, $k=1,\cdots,K$. Thus

$$(43)\qquad \boldsymbol{\eta}^*=[\mathbf{1}_I\otimes\boldsymbol{A}]\boldsymbol{\tau}^*-\boldsymbol{\xi}^*/\sqrt{n}+o(n^{-1/2})\ ,$$

where

$$(44)\qquad \xi^*_{iku}=\beta_{ik}f_{[k]}(\tau_{ku})/g_{[k]}(\tau^*_{ku})\ ,\quad u=1,\cdots,c_k-1,\ k=1,\cdots,K,\ i=1,\cdots,I.$$

Further as n tends to infinity under condition (8), the asymptotic variance-covariance matrices $\boldsymbol{\Sigma}^*_i$, $i=1,\cdots,I$ tend to the same matrix, say, $\boldsymbol{\Sigma}^*_0=(\sigma^*_{0,km,uv})$, where the elements are defined in the same manner as $\sigma_{i,km,uv}$'s with replacing $q_{i,km,uv}$ and q_{iku} by $q_{0,km,uv}$ and q_{0ku}, respectively.

For testing H_0 against H_n, let $\boldsymbol{\tau}=(\tau_{11},\cdots,\tau_{1c_1-1},\cdots,\tau_{Kc_K-1})'$, $\boldsymbol{\beta}$ be any $(I-1)K$ dimensional column vector and

$$(45)\qquad \boldsymbol{X}=\begin{bmatrix} \boldsymbol{I}_K & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{I}_K \\ -\dfrac{\lambda_1}{\lambda_I}\boldsymbol{I}_K & \cdots & -\dfrac{\lambda_{I-1}}{\lambda_I}\boldsymbol{I}_K \end{bmatrix}.$$

Thus H_0 is defined as $\boldsymbol{\beta}=\mathbf{0}$. Further let

$$(46)\qquad \boldsymbol{X}_0=\begin{bmatrix} \mathbf{1}_{c_1-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{1}_{c_K-1} \end{bmatrix},$$

then $\boldsymbol{\Gamma}^*$ defined in Section 6.1 becomes

$$\Gamma^* = \begin{bmatrix} A'\Sigma_0^{*-1}A & 0 \\ 0 & \Lambda \otimes X_0'\Sigma_0^{*-1}X_0 \end{bmatrix}^{-1}, \tag{47}$$

where $\Lambda = (\lambda_i \delta_{ii'} - \lambda_i \lambda_{i'}/\lambda_I)$ and $\delta_{ii'}$ is the Kronecker's delta. Thus it follows that $\tau^{**} = \tau^*$ and

$$\beta^{**} = [\Lambda \otimes X_0'\Sigma_0^{*-1}X_0]^{-1}[I_{I-1} \otimes X_0'\Sigma_0^{*-1}]D(\xi^*), \tag{48}$$

where

$$D(\xi^*) = [\lambda_1(\xi_1^* - \xi_I^*)', \cdots, \lambda_I(\xi_{I-1}^* - \xi_I^*)']'.$$

Consequently the noncentrality parameter of X^2 statistic for testing H_0 against H_n is written as

$$D(\xi^*)'\Lambda^{-1} \otimes (X_0'\Sigma_0^{*-1})'[X_0'\Sigma_0^{*-1}X_0]^{-1}(X_0'\Sigma_0^{*-1})D(\xi^*). \tag{49}$$

For obtaining a simple expression of γ^2, let

$$\begin{aligned} f_k &= [f_{[k]}(\tau_{k1}), \cdots, f_{[k]}(\tau_{kc_k-1})]', \qquad k = 1, \cdots, K, \\ g_k &= [g_{[k]}(\tau_{k1}^*), \cdots, g_{[k]}(\tau_{kc_k-1}^*)]', \qquad k = 1, \cdots, K, \end{aligned} \tag{50}$$

and

$$\tilde{F} = \text{Diag.}\ [f_1', \cdots, f_K'], \qquad \tilde{G} = \text{Diag.}\ [g_1', \cdots, g_K'], \tag{51}$$

and further let

$$\begin{aligned} Q_{km} &= [q_{0,km,uv} - q_{0ku}q_{0mv}], \qquad 1 \leqq k \neq m \leqq K, \\ Q_{kk} &= [q_{0ku} - q_{0ku}q_{0kv}], \qquad 1 \leqq u \leqq v \leqq c_{k-1}, \text{ for } k = 1, \cdots, K, \end{aligned} \tag{52}$$

and

$$Q = (Q_{km}).$$

Then we have

$$\xi_i^* = \tilde{G}^{-1}\tilde{F}X_0\beta_i, \tag{53}$$

and

$$\Sigma_0^* = \tilde{G}^{-1}Q\tilde{G}^{-1}. \tag{54}$$

Further

$$X_0'\Sigma_0^{*-1}X_0 = [g_k Q^{km} g_m], \tag{55}$$

and

$$\xi_i^{*\prime}\Sigma_0^{*-1}X_0 = \beta_i'[f_k' Q^{km} g_m], \tag{56}$$

where Q^{km} is the (k, m) block matrix of Q^{-1}. After some manipulations, we have

$$(57) \qquad \gamma^2=\sum_{i=1}^{I} \lambda_i \beta_i'[f_k'Q^{km}g_m][g_k'Q^{km}g_m]^{-1}[g_k'Q^{km}f_m]\beta_i \ .$$

This gives

THEOREM 3. *The test statistic X^2 for testing H_0 against H_n has an asymptotic noncentral chi-square distribution with $(I-1)K$ degrees of freedom and noncentrality parameter given by* (57).

COROLLARY 3. *When $c_1=\cdots=c_K=2$, we have*

$$(58) \qquad \gamma^2=\sum_{i=1}^{I} \lambda_i \beta_i' D(F) Q^{-1} D(F) \beta_i \ ,$$

where $Q=[p_{km}-p_k p_m]$, $p_{km}=\Pr.(Y_k=1$ and $Y_m=1)$, $p_k=\Pr.(Y_k=1)$ and $D(F)=\text{Diag.}[f(\tau_{11}),\cdots, f(\tau_{K1})]$.

The noncentrality parameter (58) does not depend on the assumed distribution G.

COROLLARY 4. *When $K=1$, the noncentrality parameter γ^2 reduces to*

$$(59) \qquad \gamma^2=\left(\sum_{i=1}^{I} \lambda_i \beta_i\right)^2 \sigma^2(F)\rho(F, G)^2 \ ,$$

where

$$(60) \qquad \rho(F, G)=\sum_{j=1}^{c} p_j \Delta_j(F)\Delta_j(G)/\sigma(F)\sigma(G) \ ,$$

$$\sigma^2(F)=\sum_{j=1}^{c} p_j \Delta_j(F)^2 \ , \qquad \sigma^2(G)=\sum_{j=1}^{c} p_j \Delta_j(G)^2 \ ,$$

$$\Delta_j(F)=\{f(\tau_j)-f(\tau_{j-1})\}/p_j \ ,$$

$$\Delta_j(G)=\{g(\tau_j^*)-g(\tau_{j-1}^*)\}/p_j \ , \qquad j=1,\cdots, c \ .$$

The next theorem states that misspecification of the latent distribution usually reduces the power. For this purpose, let us denote by $\gamma^2(G|F)$ the noncentrality parameter of the chi-square statistic obtained on the basis of the distribution $G(z)$ when the true one is $F(z)$. We have

THEOREM 4. *For any continuous K variate c.d.f. $G(z)$ the inequality*

$$(61) \qquad \gamma^2(F|F) \geqq \gamma^2(G|F)$$

holds. The equality holds when every β_i, $i=1,\cdots, I$ lies in the null space *of the matrix appears in* (64) *below.*

PROOF. Let

$$\bar{G}=Q^{-1/2}\tilde{G}X_0 \quad \text{and} \quad \bar{F}=Q^{-1/2}\tilde{F}X_0 ,$$

where $Q^{-1/2}$ is a matrix satisfying $(Q^{-1/2})^2=Q^{-1}$. Then (55) and (56) can be written as $\bar{G}'\bar{G}$ and $\beta_i\bar{F}\bar{G}'$. Thus we have

$$\gamma^2(G|F)=\sum_{i=1}^{I}\lambda_i\beta_i'(\bar{F}\bar{G}')(\bar{G}\bar{G}')^{-1}\bar{G}\bar{F}'\beta_i , \tag{62}$$

and

$$\gamma^2(F|F)=\sum_{i=1}^{I}\lambda_i\beta_i(\bar{F}\bar{F}')\beta_i , \tag{63}$$

thus

$$\gamma^2(F|F)-\gamma^2(G|F)=\sum_{i=1}^{I}\lambda_i\beta_i'\bar{F}[I-\bar{G}'(\bar{G}\bar{G}')^{-1}\bar{G}]\bar{F}'\beta_i . \tag{64}$$

Since $I-\bar{G}'(\bar{G}\bar{G}')^{-1}\bar{G}$ is an idempotent matrix, the quadratic form is positive semi-definite. The condition under which the equality holds is obvious.

6.3. *Asymptotic efficiency for a randomized block experiment*

Let us consider a randomized block experiment described in Section 5. The latent responses on a subject have a joint distribution $F(z-\mu_n)$ and consider a sequence of alternative hypotheses

$$H_n:\ \mu_n=\mu_0+\beta/\sqrt{n} , \tag{65}$$

where $\beta=(\beta_1,\cdots,\beta_K)'$ is a nonzero constant vector such that $\beta_1+\cdots+\beta_K=0$. The null hypothesis H_0 is that $\beta=0$. For notational simplicity, let $F(z)$ and $G(z)$ be the true and assumed univariate marginal c.d.f. of the latent variables and $f(z)$ and $g(z)$ be their p.d.f.'s, respectively. Then we obtain the following results.

THEOREM 5. *The chi-squared statistic for testing H_0 against H_n has asymptotically a noncentral chi-square distribution with $K-1$ degrees of freedom and noncentrality parameter*

$$\gamma^2(G|F)=\{\sigma(G,F)^2/\sigma(G)^2\}\sum_{k=1}^{K}\beta_k^2 , \tag{66}$$

where

$$\sigma(G,F)=g'(Q_0-Q_1)^{-1}f , \qquad \sigma(G)^2=g'(Q_0-Q_1)^{-1}g , \tag{67}$$

and Q_0 is the common limit of the variance-covariance matrix of $\sqrt{n}\cdot$

$(Q_{k1}, \cdots, Q_{kc-1})$, *and* $\boldsymbol{Q}_1$ *is the common limit of the covariance matrix between* $\sqrt{n}(Q_{k1}, \cdots, Q_{kc-1})$ *and* $\sqrt{n}(Q_{m1}, \cdots, Q_{mc-1})$ *for any pair of* (k, m), $k \neq m$.

The proof is omitted.

Let

$$\sigma(F)^2 = \boldsymbol{f}'(\boldsymbol{Q}_0 - \boldsymbol{Q}_1)^{-1}\boldsymbol{f}\,. \tag{68}$$

Then by Cauchy-Shwarz inequality we have

COROLLARY 5. $\gamma^2(G|F) \leqq \gamma^2(F|F)$.

7. An illustrative example

Ashiya Public Health Center has been conducting mental health examination for infants of three years of age for about twenty years with collaboration of psychiatrists Dr. Kuromaru and his collaborators. A cross-classification table given in Table 1 was obtained from the examinations of infants who lived in Ashiya City and marked the third birthday between October 1981 and September 1982. Table 1 consists of three two-way marginal tables of the three variables Y_1, Y_2 and Y_3 in four groups. The observed variables and groups are defined as follows;

Response variables:

Y_1: Performance of form matching task ($c_1=3$; 0–2, 3, 4), which

Table 1. The two-way marginal frequencies of the three response variables on every four groups classified by sex and sibling condition of infants of three years of age.

Sex	Sibling condition		Y_1-Y_2			Y_1-Y_3		Y_2-Y_3	
			1	2	3	1	2	1	2
Male	Alone or first	1	5	16	28	25	24	18	1
		2	10	52	70	44	88	42	42
		3	4	16	24	19	25	28	94
	Otherwise	1	5	18	22	26	19	14	4
		2	11	54	82	77	70	51	34
		3	2	13	30	24	21	62	72
Female	Alone or first	1	2	22	24	11	37	6	1
		2	3	45	93	34	107	29	59
		3	2	21	32	14	41	24	125
	Otherwise	1	4	20	21	23	22	11	0
		2	6	46	71	54	69	48	34
		3	1	16	24	19	22	37	79

is a measure of intellectual development.

Y_2 : Utterance ($c_2=3$; a word, two-word, three-word utterance).

Y_3 : Articulation ($c_3=2$; poor, good).

Factors:

A : Sex (1 : male, 2 : female)

B : Sibling condition (1 : alone or the first, 2 : otherwise).

We can assume the existence of the latent traits related to the above observed variables. Let the mean vector of the latent traits be $\boldsymbol{\mu}_{ij}=(\mu_{ij1}, \mu_{ij2}, \mu_{ij3})'$, $i=1, 2$ for the factor A and $j=1, 2$ for the factor B. The free threshold parameters are $\boldsymbol{\tau}=(\tau_{11}, \tau_{21})'$, and $\tau_{31}=0$, so that the coefficient matrix of $\boldsymbol{\tau}$ is

$$A=\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\end{bmatrix}'.$$

Firstly we considered the unstructed model, that is, X was the 12×12 identity matrix. Assuming that the three distributions of the latent variables belong to the same location family, we tried four distributions; normal, logistic and double exponential of the first and second

Table 2. Goodness of fit statistics and groupwise indices of goodness of fit for four latent distributional models.

Group	Normal	Logistic	DE* First	DE Second
Male 1	0.716	0.814	0.307	1.001
Male 2	1.401	1.312	1.221	1.238
Female 1	2.367	3.095	1.341	3.349
Female 2	0.556	0.645	0.440	0.726
Total	5.040	5.867	3.308	6.315

* DE stands for the double exponential distribution.

kind. The weighted sum of residuals $S(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}})$ for each distributional model are given in Table 2, where the samplewise values are defined by

$$n_l\left(\boldsymbol{Z}_l-\boldsymbol{W}_l\begin{pmatrix}\tilde{\boldsymbol{\tau}}\\ \tilde{\boldsymbol{\mu}}\end{pmatrix}\right)'\boldsymbol{V}_l^{-1}\left(\boldsymbol{Z}_l-\boldsymbol{W}_l\begin{pmatrix}\tilde{\boldsymbol{\tau}}\\ \tilde{\boldsymbol{\mu}}\end{pmatrix}\right), \qquad l=1, 2, 3, 4.$$

Though every distribution did not show any significant departure from the data, we adopted the double exponential distribution of the first kind, since it gave the smallest $S(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}})$. Table 3 shows the estimated location vectors, threshold values and their estimated standard deviations. A table of analysis of variance is given in Table 4. This suggests that the two factors do not affect the development of the general intelligence, but do independently the development of the faculty of speech. Following the results in Table 4, we proceed to fit a simple model having only main effects of A and B on Y_3 and of A on Y_2. The design matrix is

$$X=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

Table 3. Estimated location vectors and threshold values and their estimated standard deviations.

		Latent variables		
		Y_1	Y_2	Y_3
Location vectors	Male 1	−0.551 (0.083)	0.323 (0.074)	−0.066 (0.088)
	Male 2	−0.487 (0.081)	0.380 (0.072)	−0.475 (0.097)
	Female 1	−0.468 (0.079)	0.588 (0.073)	0.344 (0.080)
	Female 2	−0.546 (0.086)	0.427 (0.077)	−0.248 (0.096)
Threshold values		−0.971 (0.037)	−0.605 (0.027)	

Table 4. MANOVA and ANOVA for sex and sibling conditions.

Variables	(Y_1, Y_2, Y_3)		Y_1		Y_2		Y_3	
Factors	X^2	df	X^2	df	X^2	df	X^2	df
Sex	13.427	3	0.022	1	4.418	1	12.344	1
Sibling condition	31.917	3	0.008	1	0.500	1	30.653	1
Interaction	3.120	3	0.812	1	2.175	1	1.306	1
Total	52.286	9	0.851	3	7.296	3	47.870	3

The goodness of fit statistic of this model was $S(\hat{\beta}\,|\,\tilde{\mu})=3.308$ with 6 degrees of freedom and $S(\hat{\tau}, \hat{\beta})=6.897$ with 12 degrees of freedom. The estimated main effects are given below with estimated standard deviations (in the parentheses).

Y_2:	main effect of sex	−0.080 (0.0369)
Y_3:	main effect of sex	−0.164 (0.0445)
	main effect of sibling condition	0.243 (0.0428)

8. Further discussion

The WLS analysis of the multivariate latent scale linear model is a natural extension of the maximum likelihood (ML) analysis of the univariate latent scale linear model. In univariate case both WLS and ML estimators of $(\boldsymbol{\tau}', \boldsymbol{\beta}')'$ are BAN, so that they are equivalent without the term of order $o_p(n^{-1/2})$. Similarly the WLS procedure for the present model is asymptotically equivalent to the ML procedure in the following sense; if the $I\times(c_1\times\cdots\times c_K-1)-(p+q)$ unspecified functions of multinomial probabilities of the contingency table were arbitrarily defined, the ML estimator of $(\boldsymbol{\tau}', \boldsymbol{\beta}')$ had the same asymptotic variance-covariance matrix as the WLS estimator. This can be easily checked in the same way as Amemiya [4]. However, since our method does not utilize second order information, as a result, the convergence to the asymptotic properties may be slow. In order to utilize second order information, all bivariate marginal distributions are to be specified. This requires additional assumptions on the latent variables. Suppose that the latent response vectors are multivariate normal and variance-covariance matrices are common. Every higher order marginal probability is defined in a parametric form, therefore a WLS procedure involving bivariate marginal probabilities could be available and might yield a more efficient estimator than that considered in the present paper. In a bivariate case, an ML procedure can be applied as has been discussed by Uesaka [26]. Amemiya [3] showed, in bivariate quantal responses, that the ML estimator is more efficient than the one given by a weighted least squares method based on the only univariate marginal probabilities.

A similar approach to analysis of multivariate discrete responses is done in the context of factor analysis of dichotomous variables, Christoffersson [13], Muthen [22] and Muthen and Christoffersson [23]. They assume multivariate normality for the variables underlying quantal responses, and utilize generalized weighted least squares method involving bivariate marginal probabilities. Their method might be applied to our model if every bivariate marginal probability were specified, however, such a formulation is not studied here.

A computer software program for analysis of multivariate latent scale linear model was written by the author and implemented to **NISAN** system developed by Asano and others, Asano et al. [6].

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