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THE HETEROSCEDASTIC METHOD: MULTIVARIATE IMPLEMENTATION

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Summary

Over the past decade, procedures have been developed which allow one (in the univariate case) to make inferences about means even in the presence of unknown and unequal variances. A general method (called The Heteroscedastic Method) allowing this in all statistical problems simultaneously was formulated in 1979 and allowed specifically for the multivariate case (e.g., MANOVA and other multivariate inferences). While in the univariate case The Heteroscedastic Method is readily implemented, in the multivariate case practical implementation was not heretofore possible since a certain problem in construction of matrices required by the method had not been solved. In this paper we solve that problem and give a computer algorithm allowing for use of the solution in The Heteroscedastic Method.

1. Introduction

Suppose one has available several (k, say) sources of observations, and that source i (denoted π_i) yields observations which are normally distributed with unknown mean μ_i and unknown variance σ_i^2 ($i=1,\cdots,k$). Often one would like to make statistical inferences about $\mu_1,\cdots,\mu_k$ which have performance characteristics (e.g., power for a test, confidence coefficient for a confidence interval, etc.) which do not depend on $\sigma_1^2,\cdots,\sigma_k^2$. Procedures for achieving this when $k=1$ were first given by Stein [16] (in the case of testing and confidence intervals on μ_1), and generalized to the case $k=2$ for the Behrens-Fisher problem by Chapman [1] and Ruben [14]. While Stein [16] and others considered the general $k\geqq 2$ case, they did so only under the restriction that $\sigma_1^2=\cdots=\sigma_k^2$.

Solutions for the case $k\geqq 2$ in most instances require that one use not sample means, but a generalized sample mean ... a fact first re-

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alized and utilized by Dudewicz and Dalal [5] in their solution of ranking and selection under heteroscedasticity. Since that time solutions have appeared for cases of multiple comparisons (with and without a control), analysis of variance, various interval estimation problems, and other goals; for a review of this work see Dudewicz [3].

These solutions for $k \geqq 2$ were formalized into a general method (called The Heteroscedastic Method) for solving any statistical problem involving $\mu_1, \cdots, \mu_k$, via use of a decision-theoretic framework, by Dudewicz and Bishop [4]. The resulting method allows one to specify any problem and automatically have the required solution produced (whereas previous work started anew from first principles to produce the solution for each separate statistical problem). In addition, using a sampling scheme proposed by Chatterjee [2], it was possible to give The Heteroscedastic Method for a fully multivariate case where each of the k populations produces p-variate observations with a different (unknown) covariance matrix, thus furnishing a breakthrough in multivariate analysis. This was utilized by Dudewicz and Taneja [8] (also see Dudewicz and Taneja [7]) to extend their solution of the multivariate ranking and selection problem with known equal variance-covariance matrices to a solution of the general case of unequal and unknown variance-covariance matrices. However, heretofore it was not possible to use the multivariate procedures in practice because although certain matrices were known to exist, an algorithm for their calculation was not known. In this paper we: (1) give the multivariate sampling procedure showing the matrices required (below in this section); (2) provide an equation system whose solution will yield the required matrices (Section 2); give an explicit solution of the equation system for the bivariate ($p=2$) case (Section 3); and give a numerical example of construction of the required matrices (Section 4).

The Heteroscedastic Method is a procedure $\mathcal{P}_{HM}$ consisting of a sampling rule (telling how many observations are needed from each population) and a terminal decision rule (telling which decision to take once all observations have been made). The sampling rule involves a constant $z>0$ which (see Dudewicz and Bishop [4]) can always be chosen so as to have the procedure guarantee the desired operating characteristic requirements when a certain terminal decision rule (specified precisely by The Heteroscedastic Method, but not of explicit concern in this paper; for further details see Dudewicz and Bishop [4], Theorem (2.20)) is used. Details of the sampling rule will now be specified.

Sampling rule for $\mathcal{P}_{HM}$. Select $z>0$, an integer $n_0>p$, and a $p \times p$ positive-definite matrix (α_{rs}). Take observations from each and every population π_c ($c=1, \cdots, k$) as follows. Take n_0 initial observations $X_{c1}, \cdots, X_{cn_0}$ where $X_{ci}=(X_{c1i}, X_{c2i}, \cdots, X_{cpi})'$ ($i=1, 2, \cdots, n_0$) and compute

$$\bar{X}_{ci}=\frac{1}{n_0}\sum_{l=1}^{n_0} X_{cil}\ , \tag{1.1}$$

$$S_{cij}=\sum_{l=1}^{n_0} (X_{cil}-\bar{X}_{ci})(X_{cjl}-\bar{X}_{cj})\ , \tag{1.2}$$

$$s_{cij}=\frac{1}{(n_0-1)} S_{cij}\ , \qquad i, j=1, 2, \cdots, p\ . \tag{1.3}$$

Define the positive integer N_c by

$$N_c=\max\left\{n_0+p^2, \left[z^{-1}\sum_{i,j=1}^{p} a_{ij}s_{cij}\right]+1\right\}\ , \tag{1.4}$$

where $[q]$ denotes the largest integer less than q, and select p $(p\times N_c)$ matrices for $r=1,\cdots,p$:

$$A_{cr}=(a_{cr_{ij}}) \qquad (i=1, 2,\cdots, p;\ j=1, 2,\cdots, N_c) \tag{1.5}$$

in such a way that:

(1.6a) (i) $a_{cr_{i1}}=\cdots=a_{cr_{in_0}}$;

(1.6b) (ii) $A_{cr}\eta_c=\varepsilon_r$ where η_c is the $N_c\times 1$ vector $(1, 1,\cdots, 1)'$ and ε_r is the $p\times 1$ vector whose r-th element is 1 and all other elements are zero;

and

(1.6c) (iii) $A_cA_c'=z(a^{rs})\otimes(s_c^{ij})$, where $A_c'=(A_{c1}', A_{c2}',\cdots, A_{cp}')$, $\otimes$ denotes the direct product, and (b^{ij}) denotes the inverse of the matrix (b_{ij}), $r, i=1, 2,\cdots, p$.

Next take N_c-n_0 additional observations $X_{c,n_0+1},\cdots, X_{cN_c}$ and compute

$$\tilde{\tilde{X}}_{cr}=\sum_{i=1}^{p}\sum_{l=1}^{N_c} a_{cr_{il}}X_{cil} \qquad (r=1, 2,\cdots, p)\ . \tag{1.7}$$

For π_c construct the p-dimensional vector $\tilde{\tilde{X}}_c=(\tilde{\tilde{X}}_{c1},\cdots, \tilde{\tilde{X}}_{cp})$, $c=1, 2,\cdots, k$. All decisions are based on the vector

$$(\tilde{\tilde{X}}_1, \tilde{\tilde{X}}_2,\cdots, \tilde{\tilde{X}}_k)\ . \tag{1.8}$$

The problem (which must be solved to allow practical implementation) is construction (for each population) of the matrices in (1.5) so as to satisfy (1.6a), (1.6b) and (1.6c). (The distributions of the statistics constructed in The Heteroscedastic Method are generally complicated, however recently Hyakutake and Siotani [12] have given an exact treatment of these distributions for $p=2$, and asymptotic approximations for $p>2$.)

Since the research in the present paper was finished and submitted for publication in 1984, Hyakutake [11] presented another derivation of matrices which satisfy the required properties. It is easiest to see the relationship of his work to ours by reference to the case $p=1$ (the one-dimensional case which both his work and ours extend to $p \geqq 1$). In the one-dimensional case, one has to find $p=1$ matrix A_{c1} for each population $c=1, \cdots, k$, where A_{c1} is $p \times N_c = 1 \times N_c$, satisfying (1.6a, b, c): $A=(a_1, \cdots, a_N)$ with $a_1 = \cdots = a_{n_0}$, $a_1 + \cdots + a_N = 1$, $a_1^2 + \cdots + a_N^2 = a/(\alpha s)$. As noted in Dudewicz and Dalal ([5], p. 37), there are an infinite number of solutions to this system, and in various $p=1$ case papers authors have settled on a variety of different solutions, especially those where one requires $a_1 = \cdots = a_{n_0} = a_{n_0+1} = \cdots = a_{N-1}$, and those where one requires $a_{n_0+1} = \cdots = a_N$ (each of which gives a unique solution-subject only to choice of a root of a quadratic equation). As Dudewicz and Dalal note, each of these infinitely-many solutions (such as the two just displayed implicitly) has the same properties under the model, so there is no reason to prefer one to any other. However, Dudewicz, Ramberg and Chen [6] note the different solutions will have different robustness properties, and they prefer the solution which sets $a_{n_0+1} = \cdots = a_N$ on these grounds; indeed, it has become nearly the "standard" solution in the many papers now published on this area (see Wilcox [17] for a listing of many of these). The relationship of our solution for p dimensions to that of Hyakutake [11] is that ours is a natural extension of the $p=1$ case in that requiring the additional restrictions (2.20) produces a unique solution (subject again to the multiple roots of quadratics), while Hyakutake's method yields other solutions. Hyakutake's solutions do not possess the equalities ours do, though his may be easier to compute with existing software. (Software for our solution for $p=2$ is given in Dudewicz and Taneja [9]. Solution for general p requires only a simultaneous equation solver, but for the aid of practitioners we are in process of providing software for general p.) In the case $p=1$, Hyakutake's solutions have

$$A=(a_1, \cdots, a_N) = \frac{1}{N} e j'_N + K G^{-1} \left[\frac{-1}{N-m} j_m j'_{N-m} ; I_m \right]$$

where $G_{m \times m}$ and $K_{1 \times m}$ are not unique, and need not have $a_{n_0+1} = \cdots = a_N$.

Now that solutions to the problem of the construction of the matrices needed by The Heteroscedastic Method of Dudewicz and Bishop are available, work on these multivariate methods is proceeding rapidly and internationally. In particular, Dudewicz and Taneja [9] have given software already described above, and have given an application to selecting the best of $k=4$ bivariate populations in an accounting example in the accounting literature; this work uses the solutions of this present paper. Also, Hyakutake, Siotani, Li and Mustafid [13] have given percentage points and power functions for various statistics in the Heteroscedastic Method, while, using the solutions of Hyakutake [11], Siotani, Hyakutake, Li and Mustafid [15] have considered simultaneous confidence intervals of given length for mean vectors. We hope that these results will lead to full exploitation of the benefits of The Heteroscedastic Method in the years ahead. In addition, more work is needed on the question of which solutions of the matrix problem are most robust under reasonable deviations from the assumed model, as well as on choice of the matrix (α).

2. An equation system yielding $A_{c1},\cdots,A_{cp}$

Our goal in this section is to provide a method of constructing matrices $A_{c1},\cdots,A_{cp}$ in (1.5) which satisfy (1.6a), (1.6b) and (1.6c). In carrying out The Heteroscedastic Method's procedure $\mathcal{P}_{HM}$, this will be done independently for each population (i.e., for $c=1,\cdots,k$). Since results from one population do not affect those for other populations, we can (without loss) drop the subscript c in our work in this section. It will also be useful to denote N_c of (1.4) as $n=n_0+m-1$, and we will make use of the following

LEMMA 2.1. *Let E be an $r+p+q$ by $r+p+q$ matrix partitioned as*

$$E=\begin{bmatrix} A_{r\times r} & 0_{r\times p} & B_{r\times q} \\ 0_{p\times r} & F_{p\times p} & 0_{p\times q} \\ C_{q\times r} & O_{q\times p} & D_{q\times q} \end{bmatrix} \tag{2.2}$$

where subscripts (which shall subsequently be dropped for simplicity) refer to sizes of submatrices, 0 is a matrix with all entries zero, and $F=1_p+I_p$, where 1_p is the $p\times p$ matrix all of whose entries are 1, and I_p is the $p\times p$ identity matrix. Then the determinant

$$|E|=(p+1)\begin{vmatrix} A & B \\ C & D \end{vmatrix}\,. \tag{2.3}$$

PROOF. Elementary operations on the middle p rows of E result in changing F to

$$F'=\begin{bmatrix} I_{p-1} & 0_{p-1\times 1} \\ 0_{1\times p-1} & p+1 \end{bmatrix}\,. \tag{2.4}$$

Therefore (expanding successively by the $r+1$-st to $r+p$-th columns of the modified E) we find the desired result as claimed in (2.3). The following result (from Lemma 3 on pp. 139–140 of Chatterjee [2]) will also be useful below.

LEMMA 2.5 (Chatterjee [2]). *Let* $(a_{i1}, a_{i2}, \cdots, a_{im})'$, $i=1, 2, \cdots, l$, *be* $l<m$ *linearly independent** *m-vectors, let* $v_{ij}=\sum_{k=1}^{m} a_{ik}a_{jk}$ $(i, j=1, 2, \cdots, l)$, $V=(v_{ij})$, *and let* $c_{1\times l}=(c_1, c_2, \cdots, c_l)$ *where* $c_1, c_2, \cdots, c_l$ *are given real numbers. Then, subject to the conditions*

$$a_{i1}x_1+a_{i2}x_2+\cdots+a_{im}x_m=c_i \qquad (i=1, 2, \cdots, l)\,, \tag{2.6}$$

$x_1^2+\cdots+x_m^2$ *has as its minimum value*

$$-\frac{\begin{vmatrix} V & c'_{1\times l} \\ c_{1\times l} & 0 \end{vmatrix}}{|V|}\,. \tag{2.7}$$

Our problem now is to find p matrices (each $p\times n$, where $n=n_0+m-1$)

$$A_1, A_2, \cdots, A_p \tag{2.8}$$

(these are the matrices in (1.5) for some fixed c, $1\leqq c\leqq k$) which satisfy (1.6a, b, c). Note that, since use of Lemma 2.5 will require $l=p^2$ independent vectors to construct $A_1, \cdots, A_p$, we will need $m\geqq 1+p^2$ (or, $N_c\geqq n_0+p^2$ at (1.4)). (It is known from Chatterjee [2] that these exist for n as specified in (1.4).) Now finding such $A_1, A_2, \cdots, A_p$ is equivalent to finding $B_1, B_2, \cdots, B_p$, denoted (for $r=1, \cdots, p$) by

$$B_r=(b_{r\cdot ij})=(\beta_{r1}\,\beta_{r2}\cdots\beta_{rp})'\,, \tag{2.9}$$

such that the p^2 m-vectors β_{ri} $(r, i=1, 2, \cdots, p)$ satisfy conditions (C*):

$$\text{(C*)}\begin{cases} \beta'_{ri}\delta=\eta_{ri} \text{ (with } \delta \text{ an } m\text{-vector given by } \delta'=(1, 1, \cdots, 1, \sqrt{n_0}),\ \eta_{ri} \text{ being 1 if } i=r \text{ and 0 otherwise), and} \\ \left.\begin{array}{l} \beta'_{ri}\beta_{jt}=\alpha^{rj}s^{it}z \quad (j=1, 2, \cdots, r-1) \\ \beta'_{ri}\beta_{ru}=\alpha^{rr}s^{iu}z \end{array}\right\} \text{ for } t=1, 2, \cdots, p,\ u=1, 2, \cdots, i-1\,. \end{cases} \tag{2.10}$$

After finding such $B_1, \cdots, B_p$ which satisfy (C*) one takes $A_1, \cdots, A_p$ as given by (for $r, i=1, \cdots, p$)

$$\begin{aligned} &a_{r\cdot i, n_0+u}=b_{r\cdot iu} \qquad (u=1, 2, \cdots, m-1) \\ &a_{r\cdot i1}=\cdots=a_{r\cdot in_0}=b_{r\cdot im}/\sqrt{n_0}\,. \end{aligned} \tag{2.11}$$

In general there will be infinitely many $B_1, \cdots, B_p$ which will satisfy (C*). To obtain a unique solution within this class, we will impose the

* In applications, it is sometimes convenient to replace independence of the l m-vectors by the equivalent condition of non-singularity of the $l\times l$ matrix (v_{ij}).

additional conditions

$$\text{(2.12)}\quad (\mathrm{C}^{**})\quad (\beta_{ri})_1=(\beta_{ri})_2=\cdots=(\beta_{ri})_{m-p(r-1)-i}\,,\qquad (r,\, i=1,\cdots,p)\,.$$

It can be shown that $B_1,\cdots,B_p$ satisfying (C*) *and* (C**) exist iff

$$\text{(2.13)}\qquad \min\{\beta'_{ri}\beta_{ri}\colon (\mathrm{C}^*) \text{ satisfied})\} \\ =\min\{\beta'_{ri}\beta_{ri}\colon (\mathrm{C}^*) \text{ and } (\mathrm{C}^{**}) \text{ satisfied}\}\,,$$

as we shall now show in several Propositions.

PROPOSITION 2.14. *Assume that vectors* $\beta_{11}, \beta_{12},\cdots,\beta_{1,i-1}$ *satisfying* (C*) *and* (C**) *have been found. Then a* β_{1i} *satisfying the corresponding conditions exists.*

The proof of Proposition 2.14 follows the lines of the more general

PROPOSITION 2.15. *Assume that vectors* $\beta_{11}, \beta_{12},\cdots,\beta_{1p}; \beta_{21}, \beta_{22},\cdots, \beta_{2p};\cdots; \beta_{j-1,1}, \beta_{j-1,2},\cdots,\beta_{j-1,p}; \beta_{j1}, \beta_{j2},\cdots,\beta_{j,i-1}$ *satisfying* (C*) *and* (C**) *have been found. Then a* β_{ji} *satisfying the corresponding conditions exists.*

PROOF. It is known (see the discussion above and following (2.9)) that vectors satisfying (C*) exist, and that $\min\{\beta'_{ri}\beta_{ri}\colon(\mathrm{C}^*)\}\leqq\max\{\beta'_{ri}\beta_{ri}\colon (\mathrm{C}^*)\}=+\infty$. Hence, if (2.13) holds, at least one set of such vectors can be found which satisfies (C**) as well as (C*), without further restricting the feasible values of $\beta'_{ri}\beta_{ri}$. It follows that it suffices to show (2.13) in this case, which we denote as (2.16) for reference. Let (for $a=1, 2,\cdots, j-1;\ b=1, 2,\cdots, p$)

$$\text{(2.17)}\qquad \beta_{a,b}=(x_{1,a,b},\, x_{2,a,b},\cdots,\, x_{m,a,b})'$$

and (for $k=1, 2,\cdots, i-1$)

$$\text{(2.18)}\qquad \beta_{jk}=(w_{1,k},\, w_{2,k},\cdots,\, w_{m,k})'$$

denote the given vectors which satisfy (C*) and (C**). The vector to be found will be denoted by

$$\text{(2.19)}\qquad \beta_{ji}=(z_1,\, z_2,\cdots,\, z_m)'\,.$$

Then conditions (C*), with the additional (C**) restrictions, say (2.20) are equivalent to the system

$$\text{(2.21)}\qquad \begin{aligned} & z_1+z_2+z_3+\cdots+z_{m-1}+\sqrt{n_0}\,z_m=\eta_{ji} \\ & z_1-z_e=0 \qquad (e=2, 3,\cdots, m-p(j-1)-i) \\ & x_{1,a,b}z_1+x_{2,a,b}z_2+\cdots+x_{m,a,b}z_m=\alpha^{ja}s^{ib}z \\ & \qquad\qquad (b=1, 2,\cdots, p;\ a=1, 2,\cdots, j-1) \end{aligned}$$

$$W'_{m\times i-1}Z_{m\times 1}=\alpha^{jj}s^{id}z\ ,$$

where $W=(w_{cd})$ $(c=1,2,\cdots,m;\ d=1,2,\cdots,i-1)$ and $Z=(z_1,\cdots,z_m)'$. These are $l=m-1$ equations in the m unknowns $z_1,\cdots,z_m$, and must be independent. For, let V denote the $p\times((j-1)p+i)$ matrix

$$V\equiv(\delta,\beta_{11},\beta_{12},\cdots,\beta_{ip},\beta_{21},\beta_{22},\cdots,\beta_{2p},\cdots,\beta_{j1},\cdots,\beta_{j,i-1})\ .$$

$$V'V=\left(\begin{array}{c|c} n_0+m-1 & \varepsilon' \quad 0_{1\times i-1} \\ \hline \begin{matrix}\varepsilon \\ 0_{i-1\times 1}\end{matrix} & (\alpha^{rt})\otimes s^{-1}z \end{array}\right)$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{j-1})'$ and ε_q is as defined in (1.6b). By the hypotheses of Proposition 2.15 the vectors $\beta_{11},\beta_{12},\cdots,\beta_{1p};\ \beta_{21},\beta_{22},\cdots,\beta_{2p};\cdots;\ \beta_{j-1,1},\beta_{j-1,2},\cdots,\beta_{j-1,p};\ \beta_{j1},\beta_{j2},\cdots,\beta_{j,i-1}$ satisfy (C*) and (C**), and of course exist. Note that (see p. 147 of Chatterjee [2]) $|V'V|>0$, hence $V'V$ is non-singular. We will now apply Lemma 2.6 for which we need

$$\left\{\begin{array}{l} v_{11}=n_0+m-1\ ,\qquad v_{1,m-a'p-i+(j-a')}=1\quad (a'=j-1,j-2,\cdots)\ , \\ v_{1,l}=0\quad \text{(all other } l) \end{array}\right.$$

$$(2.22)\quad \begin{array}{l} \left\{\begin{array}{l} \text{For } a=2,3,\cdots,m-p(j-1)-i,\ v_{aa}=2 \text{ and} \\ v_{aq}=\left\{\begin{array}{ll} 1 & \text{for } q=3,4,\cdots,m-p(j-1)-i \\ 0 & \text{for all } q>m-p(j-1)-i \end{array}\right. \end{array}\right. \\ \left\{\begin{array}{l} \text{For } b=1,2,\cdots,p, \\ v_{m-p(j-1)-i+1,m-p(j-l)-i+b}=\alpha^{l1}s^{b1}z\quad (l=1,2,\cdots,j-1) \\ \vdots \\ v_{m-p(j-1)-i+1,m-1}=\alpha^{j1}s^{i-1,1}z \end{array}\right. \\ \vdots \\ v_{m-i+1,m-i+1}=\alpha^{j-1,j-1}s^{pp}z \\ \vdots \\ v_{m-1,m-1}=\alpha^{jj}s^{i-1,i-1}z\ . \end{array}$$

By Lemma 2.5, the minimum of the right hand side of (2.16) can therefore be expressed as in (2.7). Let ε_r be as in (1.6b), the p-dimensional vector with r-th element 1 and all other elements zero, and Λ the $(j-1)p+i-1$ by $(j-1)p+i-1$ matrix $(\alpha^{rt})\otimes s^{-1}$. Then by Lemma 2.1 the minimum value of the right hand side of (2.16) is, letting $\varepsilon=(\varepsilon_1'\ \varepsilon_2'\cdots\varepsilon_{j-1}')'$,

$$
\left(\begin{array}{c|cc|c}
n_0-m+1 & \varepsilon' & 0_{1\times i-1} & \eta_{ji} \\
\hline
\varepsilon & & & \alpha^{j1}s^{i1}z \\
 & \Lambda z & & \vdots \\
0_{i-1\times 1} & & & \alpha^{jj}s^{i,i-1}z \\
\hline
\eta_{ji} & \alpha^{j1}s^{i1}z\cdots\alpha^{jj}s^{i,i-1}z & & 0
\end{array}\right) \tag{2.23}
$$

divided by the determinant of the same matrix with the right column and bottom row deleted. However, this is the same minimum as shown by Chatterjee [2], p. 147, for the left hand side of (2.16), which completes the proof of Proposition 2.15.

If we substitute $j=1$ in Proposition 2.15, we obtain Proposition 2.14 as a special case. If we choose $i=1$ in Proposition 2.15 we see that, assuming vectors $\beta_{11}, \beta_{12}, \cdots, \beta_{1p}; \beta_{21}, \cdots, \beta_{2p}; \cdots; \beta_{j-1,1}, \cdots, \beta_{j-1,p}$ have been found satisfying (C*) and (C**), then β_{j1} (the first row vector of the j-th matrix) can be constructed.

PROPOSITION 2.24. *Vector β_{11} satisfying* (C*) *and* (C**) *can be found, completing the proof that $B_1, \cdots, B_p$ satisfying* (C*) *and* (C**) *exist.*

PROOF. Denote $\beta_{11}=(x_1, x_2, \cdots, x_m)'$. Then conditions (C*) and (C**) are

$$
\begin{aligned}
&\beta'_{11}\beta_{11}=\alpha^{11}s^{11}z\equiv c \quad \text{(say)}\,, \\
&\beta'_{11}\delta=\eta_{11}=1\,, \\
&x_1=x_2=\cdots=x_{m-1}\equiv a \quad \text{(say)}\,,
\end{aligned} \tag{2.25}
$$

or

$$
\begin{aligned}
&(m-1)a^2+x_m^2=c\,, \\
&(m-1)a+\sqrt{n_0}\,x_m=1\,, \\
&x_1=x_2=\cdots=x_{m-1}=a\,,
\end{aligned} \tag{2.26}
$$

or

$$
\begin{aligned}
x_1=x_2=\cdots=x_{m-1}=a& \\
=\left\{1\pm\sqrt{1-(1-n_0\alpha^{11}s^{11}z)\left(1+\frac{n_0}{m-1}\right)}\right\}&\Big/(n_0+m-1) \\
x_m=(1-(m-1)a)/\sqrt{n_0}\,,&
\end{aligned} \tag{2.27}
$$

which is unique if we choose the $+\sqrt{}$ for a. (Recall that $m\geqq 1+p^2 \geqq 2$, so $m-1\geqq 1$, and $n_0+m-1\geqq 1$.) The discriminant in (2.27) will be $\geqq 0$ iff (subject to the linear constraints in (2.26)) c is a possible value of $x_1^2+\cdots+x_{m-1}^2+x_m^2$. However, by Lemma 2.5 subject to the linear constraints in (2.26) the possible values of $x_1^2+\cdots+x_m^2$ are all values in the closed interval $[1/(n_0+m-1), \infty)$. Hence the discriminant in (2.27) will be $\geqq 0$ iff

$$\frac{1}{n_0+m-1} \leqq \alpha^{11}s^{11}z \,. \tag{2.28}$$

Recall that, for any two positive-definite $p\times p$ matrices (α_{ij}) and (s_{rt}) with respective inverses (α^{ij}) and (s^{rt}),

$$((\alpha^{ij})\otimes(s^{rt}))^{-1}=(\alpha_{ij})\otimes(s_{rt}) \,, \tag{2.29}$$

hence (by positive-definiteness) $\alpha^{11}s^{11}>0$ and (2.28) holds iff

$$n_0+m-1 \geqq \frac{(\alpha^{11}s^{11})^{-1}}{z} \,. \tag{2.30}$$

However, it follows from (1.4) (since $N_c=n=n_0+m-1$) that

$$n_0+m-1 \geqq \frac{\sum_{i,j=1}^{p} \alpha_{ij}s_{ij}}{z} \,, \tag{2.31}$$

from which (2.30) clearly follows in the case $p=1$. (When $p>1$, one finds as in Chatterjee [2] that the needed discriminants are all $\geqq 0$ iff (2.31) holds, as it does by (1.4).)

3. Explicit solution of the bivariate ($p=2$) case

In this section we provide an explicit solution of the equation system yielding the $p=2$ matrices A_{c1}, A_{c2}, each $2\times m$, which satisfy (1.6a), (1.6b) and (1.6c) in the $p=2$ case. Equivalently, we construct explicitly the $p=2$ matrices

$$B_1=\begin{pmatrix}\beta'_{11}\\ \beta'_{12}\end{pmatrix}\equiv\begin{pmatrix}x_1 & x_2 & \cdots & x_{m-1} & x_m\\ y_1 & y_2 & \cdots & y_{m-1} & y_m\end{pmatrix} \tag{3.1}$$

$$B_2=\begin{pmatrix}\beta'_{21}\\ \beta'_{22}\end{pmatrix}\equiv\begin{pmatrix}u_1 & u_2 & \cdots & u_{m-1} & u_m\\ w_1 & w_2 & \cdots & w_{m-1} & w_m\end{pmatrix} \tag{3.2}$$

satisfying (2.10) and (2.12) (after which A_1, A_2 are constructed via (2.11)). Here, by (2.12), we take

$$\begin{aligned} &x_1=x_2=\cdots=x_{m-4}=x_{m-3}=x_{m-2}=x_{m-1}\\ &y_1=y_2=\cdots=y_{m-4}=y_{m-3}=y_{m-2}\\ &u_1=u_2=\cdots=u_{m-4}=u_{m-3}\\ &w_1=w_2=\cdots=w_{m-4} \,. \end{aligned} \tag{3.3}$$

By (2.10),

$$x_m = \frac{1-(m-1)x_{m-1}}{\sqrt{n_0}} , \tag{3.4}$$

and

$$\beta'_{11}\beta_{11} = (m-1)x_{m-1}^2 + x_m^2 = z\alpha^{11}s^{11} . \tag{3.5}$$

Using x_m from (3.4) in (3.5) we find

$$(m-1)(n_0+m-1)x_{m-1}^2 - 2(m-1)x_{m-1} - (n_0 z\alpha^{11}s^{11}-1) = 0 . \tag{3.6}$$

Hence β_{11} can be constructed from (3.4) *and* (3.6).

Now let

$$\alpha = x_{m-1} - \frac{x_m}{\sqrt{n_0}} , \qquad \gamma = z\alpha^{11}s^{21} . \tag{3.7}$$

Then, again by (2.10), after simplification,

$$\begin{aligned} & y_m = -\frac{1}{\sqrt{n_0}}((m-2)y_{m-2} + y_{m-1}) \\ & y_{m-1} = \frac{\gamma}{\alpha} - (m-2)y_{m-2} \\ & (m-2)(m-1)y_{m-2}^2 - 2\frac{\gamma}{\alpha}(m-2)y_{m-2} + \frac{n_0+1}{n_0}\frac{\gamma^2}{\alpha^2} - z\alpha^{11}s^{22} = 0 , \end{aligned} \tag{3.8}$$

from which β_{12} can be constructed.

Letting

$$\phi_j = y_{m-j} - \frac{y_m}{\sqrt{n_0}} \qquad (j=1, 2) \tag{3.9}$$

$$g = z\alpha^{21}s^{12} , \qquad f = z\alpha^{21}s^{11} , \qquad \eta = \frac{g\alpha - \phi_1 f}{\alpha(\phi_2 - \phi_1)} , \tag{3.10}$$

by (2.10), and again after simplifying, we find

$$\begin{aligned} & u_m = -\frac{1}{\sqrt{n_0}}((m-3)u_{m-3} + u_{m-2} + u_{m-1}) \\ & u_{m-1} = \frac{f}{\alpha} - (m-3)u_{m-3} - u_{m-2} \\ & u_{m-2} = \eta - (m-3)u_{m-3} \\ & (m-3)(m-2)u_{m-3}^2 - 2(m-3)\eta u_{m-3} + 2\eta^2 \\ & \qquad -2\frac{f}{\alpha}\eta + \frac{n_0+1}{n_0}\frac{f^2}{\alpha^2} - z\alpha^{22}s^{11} = 0 , \end{aligned} \tag{3.11}$$

from which β_{21} *can be constructed.*

Finally, using (2.10) we have the β_{22} relations which, setting

$$g^* = z\alpha^{21}s^{21}\,, \qquad \nu = z\alpha^{21}s^{22}\,, \qquad t = z\alpha^{22}s^{21} \tag{3.12}$$

and simplifying, yield

$$\begin{aligned}
&w_m = \frac{1}{\sqrt{n_0}}(1-(m-4)w_{m-4}-w_{m-3}-w_{m-2}-w_{m-1})\\
&w_{m-1} = \delta_1-(m-4)w_{m-4}-w_{m-3}-w_{m-2}\\
&w_{m-2} = \delta_2-(m-4)w_{m-4}-w_{m-3}\\
&w_{m-1} = \delta_3-(m-4)w_{m-4}\\
&(m-4)(m-3)w_{m-4}^2-2\delta_3(m-4)w_{m-4}+2(\delta_2^2+\delta_3^2)\\
&\qquad -2\delta_2(\delta_1+\delta_3)+\frac{n_0+1}{n_0}\delta_1^2+\frac{1-2\delta_1}{n_0}-z\alpha^{22}s^{22}=0
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
&\delta_1 = \frac{g^*}{\alpha}-\frac{x_m}{\alpha\sqrt{n_0}}\,, \qquad \delta_2 = \frac{\nu-\phi_1\delta_1}{\phi_2-\phi_1}-\frac{y_m}{\sqrt{n_0}\,(\phi_2-\phi_1)}\,,\\
&\phi_i = u_{m-i}-\frac{u_m}{\sqrt{n_0}} \qquad (i=1,2,3)\,,\\
&\delta_3 = \frac{t-\phi_1\delta_1-(\phi_2-\phi_1)\delta_2}{\phi_3-\phi_2}-\frac{u_m}{\sqrt{n_0}\,(\phi_3-\phi_2)}\,.
\end{aligned} \tag{3.14}$$

Thus, β_{22} *can be constructed from* (3.13). This completes construction of B_1 and B_2, hence of A_1 and A_2.

4. Numerical example of construction

For illustration, take $p=2$, $z=1$, $(\alpha_{rs})=I_2$, and $n_0=10$ is part of the Iris data of Fisher [10], so that the first stage data is

$$\begin{pmatrix} 7.0 & 6.4 & 6.9 & 5.5 & 6.5 & 5.7 & 6.3 & 4.9 & 6.6 & 5.2 \\ 3.2 & 3.2 & 3.1 & 2.3 & 2.8 & 2.8 & 3.3 & 2.4 & 2.9 & 2.7 \end{pmatrix}.$$

Then

$$\bar{X} = \begin{pmatrix} 6.10 \\ 2.87 \end{pmatrix}, \qquad s = \begin{pmatrix} .5289 & .1933 \\ .1933 & .1157 \end{pmatrix}, \qquad N=14\,.$$

The additional data is

$$\begin{pmatrix} 5.0 & 5.9 & 6.0 & 6.1 \\ 2.0 & 3.0 & 2.2 & 2.9 \end{pmatrix},$$

and we find the required matrices to be

$$A_1=\begin{pmatrix} -.29848\cdots -.29848 & .996199 & .996199 & .996199 & .996199 \\ .62751\cdots \quad .62751 & -.55304 & -.55304 & -.55304 & -4.6160 \end{pmatrix},$$

$$A_2=\begin{pmatrix} .00000\cdots .00000 & .90005 & .90005 & -1.80010 & .00000 \\ .07695\cdots .07695 & .61354 & -3.5270 & 3.05651 & .08746 \end{pmatrix}.$$

Then $\tilde{\tilde{X}}=(5.34655, -1.15755)'$.

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