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ERROR BOUNDS FOR ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF THE MLE IN A GMANOVA MODEL

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Summary

In this paper we obtain asymptotic expansions for the distribution function and the density function of a linear combination of the MLE in a GMANOVA model, and for the density function of the MLE itself. We also obtain certain error bounds for the asymptotic expansions.

1. Introduction

Let Y be an $N\times p$ matrix whose rows have independent p-variate normal distribution with unknown covariance matrix Σ and

$$E(Y)=A\Xi B \tag{1.1}$$

where A is a known $N\times k$ matrix of rank $k\leqq N$, B is a known $q\times p$ matrix of rank $q\leqq p$ and Ξ is a $p\times q$ matrix of unknown parameters. This is known as a generalized MANOVA (GMANOVA) model (see Potthoff and Roy [9], Gleser and Olkin [5]). The MLE of Ξ is given by

$$\hat{\Xi}=(A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1} \tag{1.2}$$

where $S=Y'\{I_n-A(A'A)^{-1}A'\}Y$.

We consider the distribution of

$$\hat{\Theta}=(A'A)^{1/2}(\hat{\Xi}-\Xi)(B\Sigma^{-1}B')^{1/2} \tag{1.3}$$

and

$$\hat{\theta}=\boldsymbol{a}'(\hat{\Xi}-\Xi)\boldsymbol{b}/\sigma \tag{1.4}$$

where $\boldsymbol{a}=(a_1,\cdots,a_k)'$ and $\boldsymbol{b}=(b_1,\cdots,b_q)'$ are fixed vectors and $\sigma^2=\boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\cdot\boldsymbol{b}'(B\Sigma^{-1}B')^{-1}\boldsymbol{b}$. Gleser and Olkin [5] gave an expression for the exact density of $\hat{\Theta}$ which is an integral form. It seems that the

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exact distribution of $\hat{\theta}$ would be complicated if it could be obtained, since the distribution of $\hat{\Theta}$ itself is very complicated. In this paper we study asymptotic distributions of $\hat{\Theta}$ and $\hat{\theta}$ when $n=N-k$ is large and p, q and k are fixed. When $k=1$, the author [4] has obtained an asymptotic expansion of the distribution function of $\hat{\theta}$ and an error bound for its asymptotic expansion. It is shown that such asymptotic expansion and error bound can be obtained for the case of $k>1$. A new bound is given for the asymptotic expansion. The bound is not sharp as the previous one, but is explicitely given for any higher order asymptotic expansions. We also obtain an asymptotic expansion of the density function of $\hat{\theta}$ and an error bound for its asymptotic expansion. The results on the density function of $\hat{\theta}$ are generalized for the density function of the matrix variates $\hat{\Theta}$.

2. The distribution of $\hat{\theta}$

We can write $\hat{\theta}$ as

$$\hat{\theta}=\{\boldsymbol{b}'(B\Sigma^{-1}B')^{-1}\boldsymbol{b}\}^{-1/2}\boldsymbol{b}'(BS^{-1}B')^{-1}BS^{-1}\boldsymbol{\delta} \tag{2.1}$$

where $\boldsymbol{\delta}=\{\boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\}^{-1/2}(Y-A\Xi B)'A(A'A)^{-1}\boldsymbol{a}$. It is easily seen that $\boldsymbol{\delta}$ and S are independently distributed as $N_p(0, \Sigma)$ and $W_r(\Sigma, n)$, respectively. Therefore the distribution of $\hat{\theta}$ is essentially same as one of the statistic ξ in (1.1) of Fujikoshi [4], and we have the following Theorem 1.

THEOREM 1. *Let $F(x)$ be the distribution function of $\hat{\theta}$ and let s be a positive integer. Assume that $n-r-2s+1>0$, where $n=N-k$ and $r=p-q$. Then it holds that*

$$|F(x)-F_{s-1}(x)|\leqq c_s \tag{2.2}$$

where

$$F_{s-1}(x)=\Phi(x)+\sum_{j=1}^{s-1}(1/2^j j!)h_j\Phi^{(2j)}(x)\,, \tag{2.3}$$

$$c_s=l_{2s}h_s/\{2^s s!\}\,, \tag{2.4}$$

$$h_j=\frac{r(r+2)\cdots(r+2(j-1))}{(n-r-1)(n-r-3)\cdots(n-r-2j+1)}\,, \tag{2.5}$$

$l_j=\sup|\Phi^{(j)}(x)|$, and $\Phi^{(j)}(x)$ is the j-th derivative of the standard normal distribution function $\Phi(x)$.

It may be noted that $F_{s-1}(x)$ is an asymptotic expansion of $F(x)$

up to the order $O(n^{-(s-1)})$ and the order of c_s is $O(n^{-s})$. The values of l_{2s} for $s=1, 2, 3$ are

$$(2.6) \qquad l_2=\frac{1}{\sqrt{2\pi}}, \quad l_4=\Phi^{(4)}(\sqrt{3-\sqrt{6}})=\frac{1.38\cdots}{\sqrt{2\pi}}, \quad l_6=\frac{5.78\cdots}{\sqrt{2\pi}},$$

but those for $s\geqq 4$ have been not given explicitely. So, the bounds are not feasible for $s\geqq 4$. In the following we shall derive a new bound, based on the characteristic function of $\hat{\theta}$.

LEMMA 1. *Under the same assumption* "$n-r-2s+1>0$" *as in Theorem* 1 *it holds that*

$$(2.7) \qquad |F(x)-F_{s-1}(x)|\leqq\tilde{c}_s,$$

$$(2.8) \qquad \tilde{c}_s=h_s/(\pi s).$$

PROOF. Using (2.9) in Fujikoshi [4] we can write the characteristic function of $\hat{\Theta}$ as follows:

$$(2.9) \qquad \psi(t)=\psi_{s-1}(t)+R_s(t)$$

where

$$(2.10) \qquad \psi_{s-1}(t)=\exp\left(-\frac{1}{2}t^2\right)\sum_{j=0}^{s-1}\left(-\frac{1}{2}t^2\right)^j h_j\Big/j!,$$

$$(2.11) \qquad R_s(t)=\left\{\left(-\frac{1}{2}t^2\right)^s\Big/s!\right\}\exp\left(-\frac{1}{2}t^2\right)E_v\left[v^s\exp\left(-\frac{1}{2}\eta t^2 v\right)\right]$$

and η $(0<\eta<1)$ is the constant that appeared in the remainder term of Taylor's expansion of e^x. Here v is a nonnegative random variable, satisfying

$$E(v^j)=h_j, \qquad j=1, 2, \cdots, s.$$

Nothing that $F_{s-1}(x)$ is obtained by inverting $\psi_{s-1}(t)$, and using the fundamental inequality for error estimates (see, e.g., Feller [3], p. 538), we obtain

$$(2.12) \qquad |F(x)-F_{s-1}(x)|\leqq\frac{1}{\pi}\int_{-\infty}^{\infty}|\psi(t)-\psi_{s-1}(t)|/|t|dt$$

$$\leqq h_s(\pi 2^s s!)^{-1}\int_{-\infty}^{\infty}|t|^{2s-1}\exp\left(-\frac{1}{2}t^2\right)dt=\tilde{c}_s.$$

It may be noted that Kariya and Maekawa [7] have obtained certain bounds for approximations to the distribution function of a generalizes LSE, by the method as in the proof of Lemma 1. We shall show that the bound $\tilde{c}_s$ can be replaced by a better bound $\bar{c}_s=\tilde{c}_s/2$.

LEMMA 2. *It holds that*

$$c_s \leqq \bar{c}_s = \frac{1}{2}\tilde{c}_s . \tag{2.13}$$

PROOF. Using an integral representation to Hermite Polynomials (see, e.g., Erdélyi and et al. [2], p. 194) we can write

$$\Phi^{(2s)}(x) = (-1)^s \frac{1}{2\pi} 2^{2s+1} \int_0^\infty t^{2s-1} \exp(-2t^2) \sin(2xt) dt .$$

Therefore we obtain

$$\begin{aligned} |\Phi^{(2s)}(x)| &\leqq \frac{1}{2\pi} 2^{2s+1} \int_0^\infty t^{2s-1} \exp(-2t^2) dt \\ &= \frac{1}{\pi} 2^{s-1}(s-1)! . \end{aligned}$$

This implies the disired result.

From Theorem 1 and Lemma 2 we obtain the following Theorem 2.

THEOREM 2. *Under the same assumption "$n-r-2s+1>0$" as in Theorem 1 it holds that*

$$|F(x) - F_{s-1}(x)| \leqq \bar{c}_s = h_s/(2\pi s) . \tag{2.14}$$

The bound $\bar{c}_s$ is not sharp as the bound c_s, but is given explicitely for any s. The sharpness of $\bar{c}_s$ to c_s may be measured by

$$e_s = c_s/\bar{c}_s = \pi l_{2s}/\{2^{s-1}(s-1)!\} . \tag{2.15}$$

For $s=1, 2, 3$,

$$e_1 \doteqdot 0.76 , \quad e_2 \doteqdot 0.86 , \quad e_3 \doteqdot 0.91 .$$

THEOREM 3. *Let s be a positive integer. Let $f(x)$ be the density function of $\hat{\theta}$, and let*

$$f_{s-1}(x) = \phi(x) + \sum_{j=1}^{s-1} (1/2^j j!) h_j \phi^{(2j)}(x) \tag{2.16}$$

where $\phi^{(j)}(x)$ is the j-th derivative of the density function $\phi(x)$ of $N(0, 1)$. Assume that $n-r-2s+1>0$. Then

$$|f(x) - f_{s-1}(x)| \leqq d_s = h_s (2s)! / \{\sqrt{2\pi}(2^s s!)^2\} .$$

PROOF. Since the Fourier transform of $f_{s-1}(x)$ is $\psi_{s-1}(t)$, from (2.9) we have

$$
\begin{aligned}
|f(x)-f_{s-1}(x)| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \exp(itx)R_s(t)dt\right| \\
&\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} |R_s(t)|dt \\
&\leqq [h_s/\{2\pi 2^s s!\}]\int_{-\infty}^{\infty} t^{2s}\exp\left(-\frac{1}{2}t^2\right)dt \\
&= d_s .
\end{aligned}
$$

3. The distribution of $\hat{\Theta}$

We shall derive an asymptotic expansion of the density function $f(X)$ of $\hat{\Theta}$ and an error bound for the asymptotic expansion. First we give a reduction for the distribution of $\hat{\Theta}$.

LEMMA 3. *The random matrix $\hat{\Theta}$ defined by* (1.3) *can be expressed as*

$$\hat{\Theta}=Z-U \tag{3.1}$$

where the random matrices Z: $k\times q$ and U: $k\times q$ are the following properties:

(i) *the elements of Z are independently distributed as $N(0, 1)$,*
(ii) *Z and U are independent,*
(iii) *$U=MW^{-1/2}L$, where the elements of M: $k\times r$ and L: $r\times q$ are independently distributed as $N(0, 1)$, W: $r\times r$ is distributed as $W_r(I_r, n)$, and L, M and W are independent.*

PROOF. Gleser and Olkin [5] have essentially showed this result in a canonical form of the GMANOVA model. Here we shall see how the matrices L, M and W are defined in the terms of the original model. We can write $\hat{\Theta}$ as

$$\hat{\Theta}=(A'A)^{-1/2}A'\bar{Y}\bar{S}^{-1}\bar{B}'(\bar{B}\bar{S}^{-1}\bar{B}')^{-1}(\bar{B}\bar{B}')^{1/2}$$

where $\bar{Y}=(Y-A\Xi B)\Sigma^{-1/2}$, $\bar{S}=\Sigma^{-1/2}S\Sigma^{-1/2}$ and $\bar{B}=B\Sigma^{-1/2}$. Let $H=[H_1, H_2]$ be an orthogonal matrix such that $H_1=\bar{B}'(\bar{B}\bar{B}')^{-1/2}$. We define

$$[Z, M]=(A'A)^{-1/2}A'\bar{Y}[H_1, H_2]$$

$$W=\tilde{S}_{22}\,, \qquad L=\tilde{S}_{22}^{-1/2}\tilde{S}_{21}$$

where

$$\tilde{S}=H'\bar{S}H=\begin{bmatrix}\tilde{S}_{11} & \tilde{S}_{12}\\ \tilde{S}_{21} & \tilde{S}_{22}\end{bmatrix}, \qquad \tilde{S}_{11}: q\times q .$$

Then, we can see that $\hat{\Theta}=Z-U$, by using that $H_2'\bar{S}^{-1}\bar{B}'=-\tilde{S}_{22}^{-1}\tilde{S}_{21}\tilde{S}_{11\cdot 2}^{-1}$

$\times(\bar{B}\bar{B}')^{1/2}$, $(\bar{B}\bar{S}^{-1}\bar{B}')^{-1}=(\bar{B}\bar{B}')^{-1/2}\tilde{S}_{11\cdot2}^{-1}(\bar{B}\bar{B}')^{-1/2}$, where $\tilde{S}_{11\cdot2}=\tilde{S}_{11}-\tilde{S}_{12}\tilde{S}_{22}^{-1}\tilde{S}_{21}$. Further, it is easily seen that Z, M, L and W satisfy the properties (i), (ii) and (iii).

From Lemma 3 we can write the characteristic function of $\hat{\Theta}$ as

$$\Psi(T)=E[\text{etr}\,(iT'\hat{\Theta})]$$
$$=\text{etr}\left(-\frac{1}{2}T'T\right)E_{L,W}\left[\text{etr}\left(-\frac{1}{2}LT'TL'W^{-1}\right)\right] \tag{3.2}$$

where T is a $k\times q$ matrix and etr denotes the exponential of trace. A Taylor's expansion of $\text{etr}\left(-\frac{1}{2}LT'TL'W^{-1}\right)$ yields

$$\Psi(T)=\Psi_{s-1}(T)+R_s(T) \tag{3.3}$$

where

$$\Psi_{s-1}(T)=\text{etr}\left(-\frac{1}{2}T'T\right)\sum_{j=0}^{s-1}\frac{1}{j!}\left(-\frac{1}{2}\right)^j E_{L,W}[(\text{tr}\,LT'TL'W^{-1})^j] \tag{3.4}$$

$$R_s(T)=\text{etr}\left(-\frac{1}{2}T'T\right)\times\frac{1}{s!}\left(-\frac{1}{2}\right)^s$$
$$\times E_{L,W}\left[(\text{tr}\,LT'TL'W^{-1})^s\,\text{etr}\left(-\frac{1}{2}\eta LT'TL'W^{-1}\right)\right], \tag{3.5}$$

and η satisfies $0<\eta<1$. This reduction holds under the assumption that $E_{L,W}[(\text{tr}\,LT'TL'W^{-1})^s]$ exists. From the following Lemma 3 we can see that the assumption is satisfied if $n-r-2s+1>0$.

LEMMA 3. (Constantine [1], Khatri [8]). *Let $C_\kappa(\Omega)$ be a zonal polynomial of the $p\times p$ symmetric matrix Ω corresponding to a partition $\kappa=(k_1,\cdots,k_p)$, $k=k_1+\cdots+k_p$, $k_1\geqq\cdots\geqq k_p\geqq0$ of the integer k. Suppose that S is distributed as $W_p(\Sigma, n)$, and k is a positive integer satisfing $n-p-2k+1>0$. Then*

$$E_s[C_\kappa(\Omega S^{-1})]=e_\kappa C_\kappa(\Omega\Sigma^{-1}) \tag{3.6}$$

where $e_\kappa=1\Big/\left[(-2)^k\left(-\frac{n}{2}+\frac{p+1}{2}\right)_\kappa\right]$, $(a)_\kappa=\prod_{i=1}^{p}\left(a-\frac{1}{2}(i-1)\right)_{k_i}$ *and* $(a)_m=a(a+1)\cdots(a+m-1)$.

Now we consider the Fourier inverse transform $f_{s-1}(X)$ of $\Psi_{s-1}(T)$, which is given by

$$f_{s-1}(X)=\left(\frac{1}{2\pi}\right)^{kq}\int\cdots\int\text{etr}\,(iX'T)\Psi_{s-1}(T)dT \tag{3.7}$$

$$=\left(\frac{1}{2\pi}\right)^{kq/2}\operatorname{etr}\left(-\frac{1}{2}X'X\right)\sum_{j=0}^{s-1}\frac{1}{j!}\left(-\frac{1}{2}\right)^{j}g_j(X)$$

where

$$g_j(X)=\left(\frac{1}{2\pi}\right)^{kq/2}\operatorname{etr}\left(\frac{1}{2}X'X\right)\int\cdots\int\operatorname{etr}(iX'T) \times\operatorname{etr}\left(-\frac{1}{2}T'T\right)E_{L,W}[(\operatorname{tr}LT'TL'W^{-1})^{j}]dT\,. \tag{3.8}$$

By the same method as in Lemma 3 we can obtain an error bound when $f(X)$ is approximated by $f_{s-1}(X)$, which is given in the following Theorem 4.

THEOREM 4. *Let $f(X)$ be the density function of $\hat{\Theta}$, and let s be a positive integer satisfying $n-r-2s+1>0$. Then it holds that*

$$|f(X)-f_{s-1}(X)|\leqq D_s \tag{3.9}$$

where $f_{s-1}(X)$ is given by (3.7) *and*

$$D_s=\frac{1}{2^s s!}\left(\frac{1}{2\pi}\right)^{kq}\int\cdots\int\operatorname{etr}\left(-\frac{1}{2}T'T\right)\times E_{L,W}[(\operatorname{tr}LT'TL'W^{-1})^{s}]dT\,. \tag{3.10}$$

PROOF. Using (3.3) we have

$$|f(X)-f_{s-1}(X)|=\left|\left(\frac{1}{2\pi}\right)^{kq}\int\cdots\int\operatorname{etr}(iT'X)R_s(T)dT\right| \leqq\left(\frac{1}{2\pi}\right)^{kq}\int\cdots\int|R_s(T)|dT\,.$$

Since $\left|\operatorname{etr}\left(-\frac{1}{2}\eta LT'TL'W^{-1}\right)\right|<1$, the last expression in the above inequality is bounded by

$$\left(\frac{1}{2\pi}\right)^{kq}\frac{1}{2^s s!}\int\cdots\int\operatorname{etr}\left(-\frac{1}{2}T'T\right)\times E_{L,W}[(\operatorname{tr}LT'TL'W^{-1})^{s}]dT$$

which is equal to D_s.

We shall reduce the formula (3.9) to a practically useful form for $s=1, 2$. We use the following Lemma 4.

LEMMA 4. *Let L: $r\times q$ and W: $r\times r$ be independently distributed as a rq-variate normal distribution with mean vector* 0 *and covariance matrix I_{rq} and a Wishart distribution $W_r(I_r, n)$, respectively. Then*

(i) $E_{L,W}[\operatorname{tr}LT'TL'W^{-1}]=\dfrac{r}{n-r-1}\operatorname{tr}T'T$, *if* $n-r-1>0$,

(ii) $E_{L,W}[(\mathrm{tr}\, LT'TL'W^{-1})^2]$

$$=\frac{r}{(n-r)(n-r-1)(n-r-3)}[\{(n-r-2)r+2\}(\mathrm{tr}\, T'T)^2 +2(n-1)\,\mathrm{tr}\,(T'T)^2]\,, \qquad if\ n-r-3>0\,.$$

PROOF. Considering the expectation with respect to L, we have

$$E_L[\mathrm{tr}\, LT'TL'W^{-1}]=(\mathrm{tr}\, T'T)\,\mathrm{tr}\, W^{-1}\,,$$

$$E_L[(\mathrm{tr}\, LT'TL'W^{-1})^2]=(\mathrm{tr}\, T'T)^2(\mathrm{tr}\, W^{-1})^2+2\,\mathrm{tr}\,(T'T)^2\,\mathrm{tr}\, W^{-2}\,.$$

The expectations of the above expressions with respect to W are evaluated by Lemma 2 and the following identities (see, James [6]):

$$\mathrm{tr}\, W^{-1}=C_{(1)}(W^{-1})$$

$$\begin{bmatrix}(\mathrm{tr}\, W^{-1})^2\\ \mathrm{tr}\, W^{-2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}C_{(2)}(W^{-1})\\ C_{(1,1)}(W^{-1})\end{bmatrix}.$$

THEOREM 5. *Let $f(X)$ be the density function of $\hat{\Theta}$. Then it holds that*

(i) *if* $n-r-1>0$,

$$|f(X)-f_0(X)|\leqq D_1 \tag{3.11}$$

(ii) *if* $n-r-3>0$,

$$\left|f(X)-f_0(X)\left\{1+\frac{r}{2(n-r-1)}(\mathrm{tr}\, X'X-kq)\right\}\right| \leqq\frac{D_1}{4(n-r)(n-r-3)}[\{(n-r-2)r+2\}(kq+2)+2(n-1)(k+q+1)]\,, \tag{3.12}$$

where $f_0(X)=(2\pi)^{-kq/2}\,\mathrm{etr}\left(-\frac{1}{2}X'X\right)$ *and* $D_1=(rkq)/\{2(n-r-1)(2\pi)^{kq/2}\}$.

PROOF. The formulas (3.9) with $s=1$ and $s=2$ are reduced to (3.11) and (3.12), respectively. This reduction can be obtained by using Lemma 4 and the following identities:

$$\left(\frac{1}{2\pi}\right)^{kq}\int\cdots\int \mathrm{etr}\left(iT'X-\frac{1}{2}T'T\right)\mathrm{tr}\, T'TdT=f_0(X)(kq-\mathrm{tr}\, X'X)\,,$$

$$\left(\frac{1}{2\pi}\right)^{kq/2}\int\cdots\int \mathrm{etr}\left(-\frac{1}{2}T'T\right)\mathrm{tr}\, T'TdT=kq\,,$$

$$\left(\frac{1}{2\pi}\right)^{kq/2}\int\cdots\int \mathrm{etr}\left(-\frac{1}{2}T'T\right)(\mathrm{tr}\, T'T)^2dT=kq(kq+2)\,,$$

$$\left(\frac{1}{2\pi}\right)^{kq/2}\int\cdots\int \operatorname{etr}\left(-\frac{1}{2}T'T\right)\operatorname{tr}(T'T)^2 dT = kq(k+q+1)\,.$$

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REFERENCES

[1] Constantine, A. G. (1966). The distribution of Hotelling's generalized T_0^2, *Ann. Math. Statist.*, **37**, 215-225.

[2] Erdélyi, A. et al. (1953). *Higher Transcendental Functions, 2*, McGraw-Hill, New York.

[3] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, 2, Wiley, New York.

[4] Fujikoshi, Y. (1985). An error bound for an asymptotic expansion of the distribution function of an estimate in a multivariate linear model, *Ann. Statist.*, **13**, 827-831.

[5] Gleser, L. J. and Olkin, I. (1970). Linear models in multivariate analysis, in *Essays in Probability and Statistics* (ed. R. C. Bose), 267-292, University of North Carolina Press, Chapel Hill.

[6] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[7] Kariya, T. and Maekawa, K. (1982). A method for approximations to the pdf's and cdf's of GLSE's and its application to the seemingly unrelated regression model, *Ann. Inst. Statist. Math.*, **34**, 281-297.

[8] Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors, *Ann. Math. Statist.*, **37**, 468-479.

[9] Potthoff, R. F. and Roy, S. N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems, *Biometrika*, **51**, 313-326.