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A BIVARIATE EXPONENTIAL DISTRIBUTION ARISING IN RANDOM GEOMETRY

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Summary

In this paper a new bivariate exponential distribution, arising naturally in the theory of Poisson line processes, is studied. The distribution has some interesting and useful properties which renders it suitable for use in statistical modelling work. It is presented in the spirit of adding to the repertoire of bivariate exponential forms. It joins other models, such as those of Downton (1970, *J. R. Statist. Soc.*, B, 32, 408–417), Marshall and Olkin (1967, *J. Appl. Prob.*, 4, 291–302) and Nagao and Kadoya (1971, *Bulletin of the Disaster Prevention Research Institute*, 20, 3, 183–215), which have their origins in the theory of stochastic processes.

1. Poisson line processes

Consider a Poisson line process. Recall that such a process is constructed by first defining a homogeneous Poisson point process on the cylinder $\{(p, \theta): 0 \leqq \theta < 2\pi, 0 \leqq p\}$ and then associating each point (p, θ) on the cylinder with $\mathcal{L}(p, \theta)$, a line of infinite length on the plane; p is the length of a line segment drawn from the origin O perpendicular to $\mathcal{L}(p, \theta)$ whilst θ is the angle that this segment makes with the x-axis. See Miles [4], Solomon [7] or Santaló [6] for a description.

Poisson line processes have the property that the intersection of the random lines with any reference line generates a stationary Poisson point process (of intensity λ, say). Also the number, $N(D)$, of lines intersecting with any reference domain D is Poisson distributed. In particular, if D is convex then $EN(D) = \lambda/2\ L(D)$ where $L(\cdot)$ denotes perimeter (Miles [4]).

Key words and phrases: Bivariate exponential, random geometry, Poisson line processes.

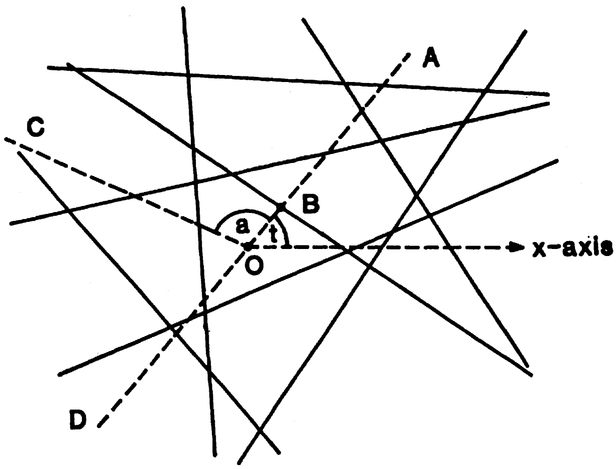

Fig. 1. A realization of a Poisson line process to illustrate a point in the text.

2. Joint distribution and correlation coefficients

Define the random variable $R(t)$ by the distance (OB in Figure 1) from the origin to the nearest line of the process in the direction of angle t. Thus we have a family of such variates for $t \in [0, 2\pi)$. For each t, $R(t)$ is distributed exponentially with density $\lambda e^{-\lambda x}$. Also for $0 \leqq a \leqq \pi$, $P\{R(t)>x,\ R(t+a)>y\}=P\{N(\Delta)=0\}$, where Δ is a triangle with O as one vertex, COA as one angle and side lengths x and y along OA and OC respectively. Obviously (using F to denote the joint distribution function of $R(t)$ and $R(t+a)$)

$$P\{R(t)>x,\ R(t+a)>y\}=\exp\left\{-\frac{\lambda}{2}(x+y+\sqrt{x^2+y^2-2xy\cos a})\right\}$$

(1)

$$F(x, y)=1-e^{-\lambda x}-e^{-\lambda y}+\exp\left\{-\frac{1}{2}(x+y+\sqrt{x^2+y^2-2xy\cos a})\right\}.$$

One can see that $R(t)$ and $R(t+\pi)$ are independent. This is consistent with the fact that they are forward and backward *recurrence times* for the Poisson point process on the line passing through AD. Such recurrence times are known to be independent (Lamperti [2]).

The correlation coefficient of $R(t)$ and $R(t+a)$ can be readily found using the following formula for non-negative random variables. Here we use X and Y interchangeably with $R(t)$ and $R(t+a)$.

$$\rho(a)=\frac{1}{\sigma_X\sigma_Y}\int_0^\infty\int_0^\infty [F(x, y)-F_X(x)F_Y(y)]dxdy$$

$$=\lambda^2\int_0^\infty\int_0^\infty\left[\exp\left\{-\frac{\lambda}{2}(x+y+\sqrt{x^2+y^2-2xy\cos a})\right\}\right.$$
$$\left.-\exp\{-\lambda(x+y)\}\right]dxdy$$
$$=\begin{cases}1 & a=0\,;\\ -1+\dfrac{4}{1+\cos a}\left[1-\dfrac{1-\cos a}{1+\cos a}\log\dfrac{2}{1-\cos a}\right] & 0<a<\pi\,;\\ 0 & a=\pi\,.\end{cases}$$

Alternatively, one may define a new coefficient of association, $\eta=(1+\cos a)/2$, and write ρ as

$$\rho=-1+\frac{2}{\eta}\left[1+\frac{1-\eta}{\eta}\log(1-\eta)\right]\qquad \eta\neq 0,\ \eta\neq 1\,.$$

Thus, ρ runs from 0 to 1 as η runs from 0 to 1, or as the angle a runs from π to 0. Since $\rho'(\eta)=2(\eta-2)\log(1-\eta)/\eta^3>0$, ρ is monotone in η and hence monotone in a. Thus $\rho\in[0,1]$, with $\rho=0$ implying that X and Y are independent. Table 1 shows $\rho(a)$ for a range of a.

Table 1. Correlation coefficient $\rho(a)$ for a range of a

a	0	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$\rho(a)$	1	.891	.728	.571	.434	.320	.227	.153	.096	.053	.023	.006	0

The joint distribution function is absolutely continuous with respect to (x, y). Thus F has a density. Furthermore this density is also smooth with derivatives everywhere. The density does however have a singularity at (0, 0). Figure 2 shows a typical contour plot of the probability density function, with λ standardized at 1 and $a=\pi/6$. Thus $\rho=.728$. Figure 3 shows the comparable contour plot for F.

Writing S^2 for $x^2+y^2-2xy\cos a=(x+y)^2-4xy\eta$, the joint p.d.f. is

$$f(x,y)=\frac{\lambda(1-\eta)}{2S^3}[4\eta xy+S\{S(x+y)+x^2+y^2+2xy\eta\}]\exp\left[-\frac{1}{2}\lambda(x+y+S)\right]$$

for $x\geqq 0$, $y\geqq 0$.

In the standard form (1), where X and Y have identical marginal distributions their means variances are $1/\lambda$ and $1/\lambda^2$ respectively. It is clear, however, that either or both variables can be scaled to have different means.

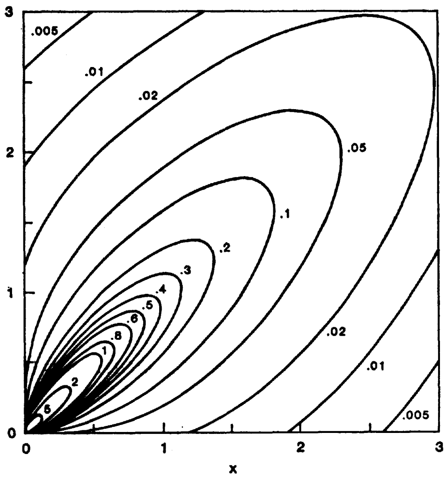

Fig. 2. The contours of the joint probability density function of X and Y. Here λ has been set at 1 and the angle a set at $\pi/6$. Note that all contour lines intersect both x and y axes, the points of intersection being away from the origin (though often very close).

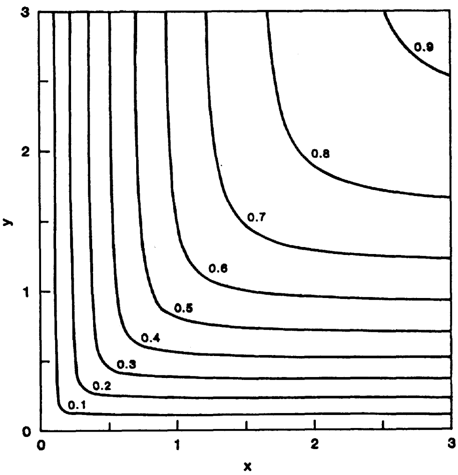

Fig. 3. Contours of the joint distribution function F, for $\lambda=1$ and $a=\pi/6$.

3. Conditional moments

The regression, let us say of Y upon X, does not take a simple form. Using S and η as before, we can write the conditional distribution function of Y given x, and numerically calculate $E(Y|x)$ and $E(Y^2|x)$ using the formulae $E(Y|x)=\int_0^\infty [1-F(y|x)]dy$ and $E(Y^2|x)=2\int_0^\infty y[1-F(y|x)]dy$.

$$F(y|x)=1-\frac{1}{2S}[S+x+y(1-2\eta)]\exp\left[-\frac{1}{2}\lambda(y-x+S)\right].$$

There are numerical instabilities for very small x, but analytic considerations show that $E(Y|x)\to(1-\cos a)/(2\lambda)$ and $E(Y^2|x)\to(1-\cos a)/\lambda^2$ as $x\to 0$. Figure 4 shows the nature of $E(Y|x)$ for $\lambda=1$ and a range of angle a. Figure 5 shows the conditional standard deviation of Y given x for a similar range.

4. Joint characteristic function

It is possible to find the joint characteristic function of X and Y. Denoting $E[\exp(isX+itY)]$ by $\phi(s,t)$, we have for $a\neq\pi$,

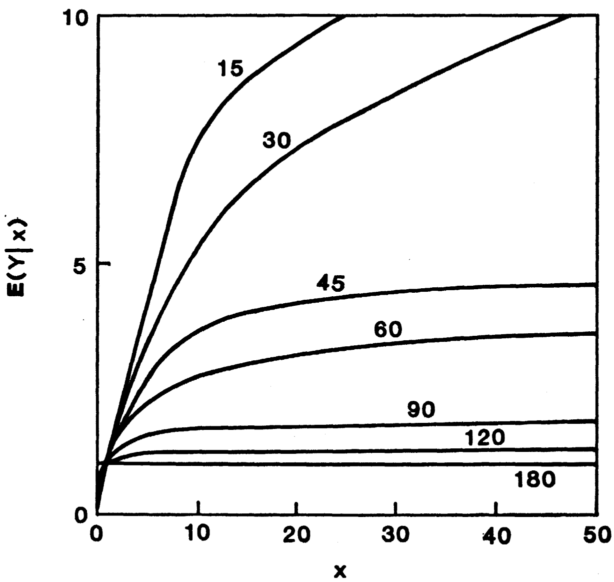

Fig. 4. The conditional expectation of Y given X for this bivariate exponential distribution. λ is held fixed at 1 whilst values of the angle a, in degrees, are shown for each curve.

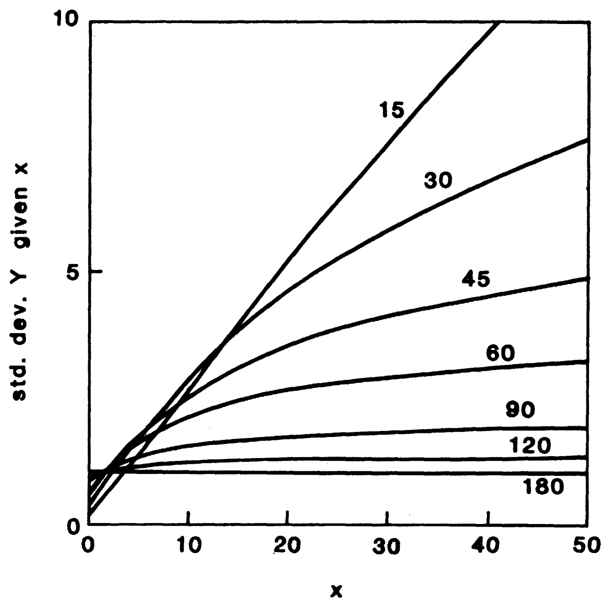

Fig. 5. The standard deviation of Y conditional upon x for this bivariate exponential distribution. λ is held fixed at 1 whilst the angle a is varied.

$$\phi(s, t)=\frac{\lambda^2+st}{(\lambda-is)(\lambda-it)}-\frac{4st}{\lambda^2\zeta^2}\left[\theta(s, t)+\theta(t, s)-\frac{4\eta(1-\eta)}{\zeta}\log\left(\frac{\psi+\zeta}{\psi-\zeta}\right)\right]$$

where $\theta(s, t)=\dfrac{\eta\lambda+i(s-t-2s\eta)}{\lambda-is}$,

$$\psi=\frac{2}{\lambda}[2\lambda-\eta\lambda-i(s+t)] ,$$

$$\zeta^2=\frac{4}{\lambda^2}[\eta^2\lambda^2-4\eta st-(s-t)^2-2i\eta\lambda(s+t)] .$$

When $a=\pi$, $\phi(s, t)=\lambda^2/[(\lambda-is)(\lambda-it)]$. The foregoing formula collapses to this simple form when $a=\pi$ and $\zeta\neq 0$. ($\zeta=0$ only if $a=\pi$ and the real numbers s and t satisfy $4(s^2+t^2)=\lambda^2$.)

5. Minimum of X and Y

The minimum of X and Y is also exponentially distributed. From (1)

$$P\{\min(X, Y)>z\}=\exp\left[-\frac{1}{2}\lambda z(2+\sqrt{2-2\cos a})\right]$$

and so it is easily seen that the min (X, Y) is exponentially distributed with mean $[\lambda(2+\sqrt{2-2\cos a})/2]^{-1}$. Indeed the multivariate version of this distribution enjoys the same property; the minimum is exponentially distributed. The multivariate version is found by considering n directions instead of 2. If all directions lie within a semi-circle then (using the $n-1$ angles $a_1, a_2, \cdots, a_{n-1}$) one can show that the minimum distance to a random line exceeds z with probability

$$\exp\left[-\frac{1}{2}\lambda z\left(2+\sum_{i=1}^{n-1}\sqrt{2-2\cos a_i}\right)\right] .$$

If the n directions do not lie within a semi-circle then the minimum distance exceeds z with probability

$$\exp\left[-\frac{1}{2}\lambda z\sum_{i=1}^{n}\sqrt{2-2\cos a_i}\right] ,$$

where a_n is the angle "2π minus the sum of the other a_i".

This pleasing property of the minimum variate extends to an uncountably infinite collection of such variates, since

$$P\{\inf_{t\in[0,2\pi)} R(t)>z\}=P\{N\text{ (ball radius } z)=0\}=\exp(-\lambda\pi z) .$$

As a slight generalization we note that

$$P\{\inf_{t\in[0,h)} R(t)>z\}=\begin{cases}\exp\left[-\frac{1}{2}\lambda z(2+h)\right], & h\in[0,\pi]\\ \exp\left[-\frac{1}{2}\lambda z(h+\sqrt{2-2\cos h})\right], & h\in[\pi,2\pi].\end{cases}$$

6. A trivariate generalization

The multivariate distribution mentioned in the previous section is a special case of a more elaborate distribution. We illustrate by sketching the trivariate case. Consider the Poisson plane process in R^3 introduced by Miles [4]. This is a statistically homogeneous collection of planes in R^3 with the following properties.

(a) The planes cut any reference line at points which form a Poisson point process, of rate λ say, along this line.

(b) The planes cut any reference plane with lines which form a Poisson line process on this two-dimensional plane.

(c) The number of planes cutting any convex body D in R is distributed as a Poisson variate with mean $\lambda M(D)$, where $M(D)$ is the "mean caliper diameter" of D.

The mean caliper diameter is the mean length of the orthogonal projection of D onto a random isotropic line. For a convex polyhedron (Santaló [6])

$$M=\frac{1}{4\pi}\sum(\pi-\alpha_i)x_i \tag{2}$$

where x_i and α_i are the edge lengths and corresponding dihedral angles of the polyhedron. Now consider three directions emanating from the O of R^3. Define X, Y and Z as the distances along these directional lines until a plane of the random plane process is encountered. Each of these three variates is exponentially distributed with mean $1/\lambda$. Their joint distribution is given by:

$$P(X>x,\ Y>y,\ Z>z)=\exp[-\lambda M(\Delta)] \tag{3}$$

where Δ is now a tetrahedron with one vertex at O, edges emanating from O of lengths x, y and z, with dihedral angles determined by x, y and z, and the chosen directions. Thus the exponent in (3) has 6 terms each involving an edge length and a dihedral angle as in (2). There is no short expression for all dihedral angles of a tetrahedron.

If the three chosen directions are coplanar then, clearly, the trivariate distribution collapses to that mentioned in Section 5.

7. The associated bivariate uniform distribution

It is of some interest to enlarge the repertoire of bivariate uniform distributions since they provide a basis for the generation of bivariate forms having quite general marginals. The variates U and V, defined by $U=\exp(-\lambda X)$ and $V=\exp(-\lambda Y)$, are jointly uniform. Their joint distribution function, $G(u, v)$, their correlation coefficient and their regression curves are given below for completeness. For $u, v \in (0, 1)$ $\eta \in [0, 1]$,

$$G(u, v)=\sqrt{uv}\exp\left[-\frac{1}{2}\sqrt{(\log uv)^2-4\eta\log u\log v}\right]$$

$$\rho(\eta)=\frac{3}{8+\eta}\left[4-\eta-\frac{8(1-\eta)}{\xi}\log\frac{(\eta-\xi)(3\eta+\xi)}{(\eta+\xi)(3\eta-\xi)}\right],$$

where $\xi^2=\eta(8+\eta)$.

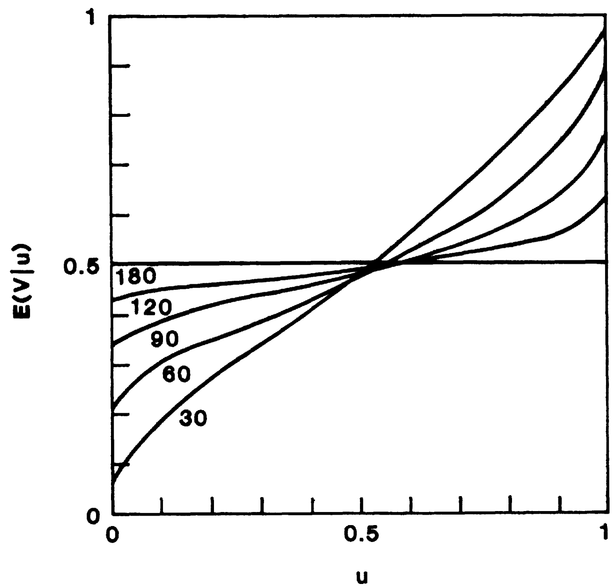

Fig. 6. The conditional expectation of V given U for the bivariate uniform distribution. The angle a is marked for each curve.

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