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SECOND ORDER ASYMPTOTIC COMPARISON OF ESTIMATORS OF A COMMON PARAMETER IN THE DOUBLE EXPONENTIAL CASE

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Summary

The problem to estimate a common parameter for the pooled sample from the double exponential distributions is discussed in the presence of nuisance parameters. The maximum likelihood estimator, a weighted median, a weighted mean and others are asymptotically compared up to the second order, i.e. the order $n^{-1/2}$ with the asymptotic expansions of their distributions.

1. Introduction

In the regular situations the asymptotic deficiencies of the asymptotically efficient estimators were obtained in pooled samples from the same distribution by Akahira [1] and the distributions with nuisance parameters by Akahira and Takeuchi [3].

In non-regular cases the optimality of the estimators of a common parameter has been studied by Cohen [8], Bhattacharya [7], Akai [6], Akahira and Takeuchi [4] and others. In the double exponential case as a typical example of the non-regular case, the loss of information of the order statistics and related estimators is studied by Akahira and Takeuchi [5], [9].

In this paper we consider an estimation problem of a common parameter θ based on m samples of each size n from the double exponential distributions with nuisance parameters τ_i ($i=1,\cdots,m$). We obtain the asymptotic expansions of the distributions of some estimators, e.g. the maximum likelihood estimator (MLE), the weighted median and the weighted mean, and asymptotically compare them up to the second order, i.e. the order $n^{-1/2}$. We also get the bound of the asymptotic distributions of the all second order asymptotically median unbiased estimators and compare it with their asymptotic distributions up to the order $n^{-1/2}$.

Key words and phrases: Maximum likelihood estimator, weighted median, weighted mean, asymptotic expansion.

2. Estimators to be compared up to the second order

Let X_{ij} $(i=1,\cdots,m;\ j=1,\cdots,n)$ be m sets of independent samples each of size n. Suppose that for each i, X_{ij} $(j=1,\cdots,n)$ have the following density function

$$f(x,\theta,\tau_i)=\frac{1}{2\tau_i}\exp\left(-\frac{|x-\theta|}{\tau_i}\right)\qquad \text{for } -\infty<x<\infty\ ,$$

where θ and τ_i are real and positive valued parameters, respectively.

We denote the log-likelihood function for the all m samples by $L(\theta,\tau_1,\cdots,\tau_m)$. Then we have

$$L(\theta,\tau_1,\cdots,\tau_m)=-\sum_{i=1}^{m}\frac{1}{\tau_i}\sum_{j=1}^{n}|x_{ij}-\theta|-n\sum_{i=1}^{m}\log\tau_i-mn\log 2\ . \tag{2.1}$$

If τ_i $(i=1,\cdots,m)$ are known, the maximum likelihood estimator $\hat{\theta}^0_{ML}$ is given as the solution of the equation

$$\sum_{i=1}^{m}\frac{1}{\tau_i}\sum_{j=1}^{n}\operatorname{sgn}(x_{ij}-\theta)=0\ , \tag{2.2}$$

where

$$\operatorname{sgn}(x_{ij}-\theta)=\begin{cases}-1 & \text{for } x_{ij}<\theta\ ;\\ \gamma & \text{for } x_{ij}=\theta\ ;\\ 1 & \text{for } x_{ij}>\theta\ ;\end{cases}$$

with some constant γ satisfying $-1<\gamma<1$. Then $\hat{\theta}^0_{ML}$ is also regarded as a weighted median by the weights $1/\tau_i$ $(i=1,\cdots,m)$. If θ is given, then for each i the solution $\hat{\tau}_i(\theta)$ of the equation

$$\frac{\partial L(\theta,\tau_1,\cdots,\tau_m)}{\partial\tau_i}=0$$

is given by

$$\hat{\tau}_i(\theta)=\frac{1}{n}\sum_{j=1}^{n}|x_{ij}-\theta|\ . \tag{2.3}$$

Substituting (2.3) in (2.1) we have

$$L(\theta,\hat{\tau}_1(\theta),\cdots,\hat{\tau}_m(\theta))=mn\left(\log\frac{n}{2}-1\right)-n\sum_{i=1}^{m}\log\sum_{j=1}^{n}|x_{ij}-\theta|\ . \tag{2.4}$$

Since in order to get θ maximizing (2.1), it is enough to obtain θ minimizing $\sum_{i=1}^{m}\log\sum_{j=1}^{n}|x_{ij}-\theta|$ by (2.4), and such a θ is given as a solution

$\hat{\theta}_{WM}$ of the equation

$$\sum_{i=1}^{m} \frac{\sum_{j=1}^{n} \operatorname{sgn}(x_{ij}-\theta)}{\sum_{j=1}^{n} |x_{ij}-\theta|} = 0 . \tag{2.5}$$

It is seen by (2.3) and (2.5) that the estimator $\hat{\theta}_{WM}$ is a weighted median by the weights $1/\hat{\tau}_i(\hat{\theta}_{WM})$ $(i=1,\cdots, m)$.

We also consider other estimators. If τ_i $(i=1,\cdots, m)$ are known and unknown, we have the weighted means $\hat{\theta}^0$ and $\hat{\theta}^*$ by the weights $1/\tau_i^2$ $(i=1,\cdots, m)$ and $1/\hat{\tau}_i^2(\hat{\theta}_i)$ $(i=1,\cdots, m)$, respectively, i.e.

$$\hat{\theta}^0 = \sum_{i=1}^{m} \frac{1}{\tau_i^2}\hat{\theta}_i \Big/ \sum_{i=1}^{m} \frac{1}{\tau_i^2} ; \qquad \hat{\theta}^* = \sum_{i=1}^{m} \frac{1}{\hat{\tau}_i^2}\hat{\theta}_i \Big/ \sum_{i=1}^{m} \frac{1}{\hat{\tau}_i^2} ,$$

where $\hat{\tau}_i = \hat{\tau}_i(\hat{\theta}_i)$ with $\hat{\theta}_i = \operatorname{med}_{1\le j\le n} X_{ij}$ $(i=1,\cdots, m)$.

In the next section we shall compare the above asymptotically efficient estimators up to the second order i.e. the order $n^{-1/2}$ and also obtain the bound of the asymptotic distributions of the second order asymptotically median unbiased estimators up to the order $n^{-1/2}$.

3. Second order asymptotic comparison of the estimators

First we shall obtain the asymptotic expansion of the distribution of the MLE $\hat{\theta}^0_{ML}$ up to the order $n^{-1/2}$ in the case when τ_i $(i=1,\cdots, m)$ are known. Since $\hat{\theta}^0_{ML}$ is the solution of the equation (2.2), the function of θ given by the left-hand side of (2.2) is locally monotone decreasing in a neighborhood of $\hat{\theta}^0_{ML}$. Hence it follows that for any positive (negative) c, $\hat{\theta}^0_{ML} \,(\lessgtr)\, \theta + (c/\sqrt{n})$ if and only if

$$\sum_{i=1}^{m} \frac{1}{\tau_i} \sum_{j=1}^{n} \operatorname{sgn}\left(x_{ij} - \theta - \frac{c}{\sqrt{n}}\right) (\lessgtr) 0 .$$

In order to obtain the asymptotic expansion of the distribution of the MLE $\hat{\theta}^0_{ML}$, without loss of generality we assume that $\theta=0$. We put $W_{ij} = \operatorname{sgn}(X_{ij} - cn^{-1/2})$ $(i=1,\cdots, m;\ j=1,\cdots, n)$. Then we have for any fixed real number c

$$E_0[W_{ij}] = -\frac{c}{\tau_i\sqrt{n}} + \frac{c^2 \operatorname{sgn} c}{2\tau_i^2 n} + o\left(\frac{1}{n}\right) . \tag{3.1}$$

Putting

$$\psi_i(a) = \frac{1}{\tau_i} \sum_{j=1}^{n} \operatorname{sgn}(X_{ij} - a) \qquad (i=1,\cdots, m) ,$$

we obtain by (3.1)

$$E_0\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{m}\psi_i\left(\frac{c}{\sqrt{n}}\right)\right]=\frac{1}{\sqrt{n}}\sum_{i=1}^{m}\frac{1}{\tau_i}\sum_{j=1}^{n}E_0(W_{ij}) = -cp_2+\frac{p_3c^2\operatorname{sgn} c}{2\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right), \tag{3.2}$$

where $p_k=\sum_{i=1}^{m}\frac{1}{\tau_i^k}$ $(k=1, 2, \cdots)$. Since

$$E_0[W_{ij}^2]=E_0\left[\left\{\operatorname{sgn}\left(X_{ij}-\frac{c}{\sqrt{n}}\right)\right\}^2\right]=1,$$

it follows from (3.1) that the asymptotic variance of W_{ij} at $\theta=0$ is given by

$$V_0(W_{ij})=1+\frac{c^2}{\tau_i^2 n}+o\left(\frac{1}{n}\right). \tag{3.3}$$

Then we obtain from (3.3)

$$V_0\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{m}\psi_i\left(\frac{c}{\sqrt{n}}\right)\right)=p_2+\frac{p_4c^2}{n}+o\left(\frac{1}{n}\right). \tag{3.4}$$

We also have

$$E_0\left[\left\{\psi_i\left(\frac{c}{\sqrt{n}}\right)\right\}^2\right]=\frac{n}{\tau_i^2}\{1+(n-1)\mu_i^2\}; \tag{3.5}$$

$$E_0\left[\left\{\psi_i\left(\frac{c}{\sqrt{n}}\right)\right\}^3\right]=\frac{n}{\tau_i^3}\{\mu_i+3(n-1)\mu_i+(n-1)(n-2)\mu_i^3\}, \tag{3.6}$$

where

$$\mu_i=E_0(W_{i\alpha})=-\frac{c}{\tau_i\sqrt{n}}+\frac{c^2\operatorname{sgn} c}{2\tau_i^2 n}+o\left(\frac{1}{n}\right).$$

By (3.5) and (3.6) we obtain the third order cumulant

$$\mathcal{K}_3\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{m}\psi_i\left(\frac{c}{\sqrt{n}}\right)\right)=\frac{2c}{n}p_4+o\left(\frac{1}{n}\right). \tag{3.7}$$

Remark 3.1. In order to find the asymptotic distribution of the MLE $\hat{\theta}_{ML}^0$ up to the second order, i.e. the order $n^{-1/2}$, it is enough to show that the third order cumulant is of order n^{-1}, but the value given by (3.7) may be necessary in the case of higher order than second one.

It follows from (3.2), (3.4) and (3.7) that the asymptotic expansion of the distribution of the MLE $\hat{\theta}_{ML}^0$ up to the order $n^{-1/2}$ is given by

$$(3.8)\qquad P\{\sqrt{n}\hat{\theta}^0_{ML}<c\}=P\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{m}\frac{1}{\tau_i}\sum_{j=1}^{n}\operatorname{sgn}\left(X_{ij}-\frac{c}{\sqrt{n}}\right)<0\right\}$$
$$=\Phi(c\sqrt{p_2})-\frac{p_3c^2}{2\sqrt{p_2n}}\phi(c\sqrt{p_2})\operatorname{sgn}c+o\left(\frac{1}{\sqrt{n}}\right),$$

where $\Phi(x)=\int_{-\infty}^{x}\phi(u)du$ with $\phi(u)=(1/\sqrt{2\pi})e^{-u^2/2}$. We also have by (3.8)

$$(3.9)\qquad P\{\sqrt{p_2n}(\hat{\theta}^0_{ML}-\theta)<t\}=\Phi(t)-\frac{p_3t^2}{2p_2^{3/2}\sqrt{n}}\phi(t)\operatorname{sgn}t+o\left(\frac{1}{\sqrt{n}}\right).$$

Next we shall obtain the asymptotic expansion of the distribution of the estimator $\hat{\theta}_{WM}$, i.e. the weighted median by the weights $1/\hat{\tau}_i(\hat{\theta}_{WM})$ $(i=1,\cdots,m)$ up to the order $n^{-1/2}$ in the case when τ_i $(i=1,\cdots,m)$ are unknown. Since $\hat{\theta}_{WM}$ is the solution of the equation (2.5), the function of θ given by the left-hand side of (2.5) is locally monotone decreasing in a neighbourhood of $\hat{\theta}_{WM}$. Hence it follows that for any fixed positive (negative) c, $\hat{\theta}_{WM}\,{}_{(}\lesssim_{)}\,\theta+(c/\sqrt{n})$ if and only if

$$\sum_{i=1}^{m}\frac{\dfrac{1}{\sqrt{n}}\sum_{j=1}^{n}\operatorname{sgn}\left(x_{ij}-\theta-\dfrac{c}{\sqrt{n}}\right)}{\dfrac{1}{n}\sum_{j=1}^{n}\left|x_{ij}-\theta-\dfrac{c}{\sqrt{n}}\right|}\;{}_{(}\lesssim_{)}\,0\,.$$

In order to obtain the asymptotic expansion of the distribution of the estimator $\hat{\theta}_{WM}$, without loss of generality we assume that $\theta=0$. For each $i=1,\cdots,m$ we put

$$Y_i=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\operatorname{sgn}\left(X_{ij}-\frac{c}{\sqrt{n}}\right);\qquad Z_i=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left|X_{ij}-\frac{c}{\sqrt{n}}\right|-\tau_i\sqrt{n}\,.$$

Then we have

$$(3.10)\qquad \sum_{i=1}^{m}\frac{\dfrac{1}{\sqrt{n}}\sum_{j=1}^{n}\operatorname{sgn}\left(X_{ij}-\dfrac{c}{\sqrt{n}}\right)}{\dfrac{1}{n}\sum_{j=1}^{n}\left|X_{ij}-\dfrac{c}{\sqrt{n}}\right|}$$
$$=\sum_{i=1}^{m}Y_i\frac{1}{\tau_i}\left\{1-\frac{Z_i}{\tau_i\sqrt{n}}+\frac{Z_i^2}{\tau_i^2n}+o_p\left(\frac{1}{n}\right)\right\}=\sum_{i=1}^{m}Y_iU_i\qquad\text{(say)}.$$

Since

$$E_0\left[\operatorname{sgn}\left(X_{ij}-\frac{c}{\sqrt{n}}\right)\left|X_{ij}-\frac{c}{\sqrt{n}}\right|\right]=-\frac{c}{\sqrt{n}}$$
$$(i=1,\cdots,m;\ j=1,\cdots,n)$$

and

$$E_0\left[\left|X_{ij}-\frac{c}{\sqrt{n}}\right|\right]=\tau_i+\frac{c^2}{2\tau_i n}+o\left(\frac{1}{n}\right) \tag{3.11}$$
$$(i=1,\cdots,m;\ j=1,\cdots,n),$$

it follows that

$$E_0(Y_i Z_i)=o(1). \tag{3.12}$$

Since for each $i=1,\cdots,m$, Y_i and Z_i are asymptotically normally distributed, it follows from (3.11) that Y_i and Z_i are asymptotically independent. By (3.1) and (3.11) we have for each $i=1,\cdots,m$

$$E_0[Y_i]=-\frac{c}{\tau_i}+\frac{c^2\,\mathrm{sgn}\,c}{2\tau_i^2\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3.13}$$

$$E_0[Z_i]=\frac{c^2}{2\tau_i\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.14}$$

By (3.13) and (3.14) we have

$$E_0[Y_i U_i]=-\frac{c}{\tau_i^2}+\frac{c^2\,\mathrm{sgn}\,c}{2\tau_i^3\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.15}$$

Hence we obtain

$$E_0\left[\sum_{i=1}^{m} Y_i U_i\right]=-p_2 c+\frac{p_3 c^2\,\mathrm{sgn}\,c}{2\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.16}$$

By (3.1) we obtain for each $i=1,\cdots,m$

$$E_0[Y_i^2]=1+\frac{c^2}{\tau_i^2}-\frac{c^3\,\mathrm{sgn}\,c}{\tau_i^3\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.17}$$

Since

$$E_0\left[\left(X_{ij}-\frac{c}{\sqrt{n}}\right)^2\right]=2\tau_i^2+\frac{c^2}{n},$$

it follows from (3.11) that

$$E_0[Z_i^2]=\tau_i^2+o(1). \tag{3.18}$$

Since by (3.17) and (3.18)

$$E_0[Y_i^2 U_i^2]=\frac{1}{\tau_i^2}+\frac{c^2}{\tau_i^4}-\frac{c^3\,\mathrm{sgn}\,c}{\tau_i^5\sqrt{n}}+o\left(\frac{1}{n}\right), \tag{3.19}$$

it follows from (3.16) that

$$V_0(Y_i U_i)=\frac{1}{\tau_i^2}+o\left(\frac{1}{\sqrt{n}}\right).$$

Hence we have

$$V_0\left(\sum_{i=1}^m Y_i U_i\right)=p_2+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.20}$$

Since by (3.6)

$$E_0[Y_i^3]=-\frac{3c}{\tau_i}-\frac{c^3}{\tau_i^3}+\frac{3c^2 \operatorname{sgn} c}{2\tau_i^2\sqrt{n}}+\frac{3c^4 \operatorname{sgn} c}{2\tau_i^4\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right),$$

it follows from (3.14) that

$$E_0[Y_i^3 U_i^3]=-\frac{3c}{\tau_i^4}-\frac{c^3}{\tau_i^6}-\frac{3c^2 \operatorname{sgn} c}{2\tau_i^5\sqrt{n}}+\frac{3c^4 \operatorname{sgn} c}{2\tau_i^7\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.21}$$

Since by (3.15), (3.19) and (3.21) the third order cumulant of $Y_i U_i$ is given by $\mathcal{K}_3(Y_i U_i)=o(1/\sqrt{n})$, it follows that that of $\sum_{i=1}^m Y_i U_i$ is done by

$$\mathcal{K}_3\left(\sum_{i=1}^m Y_i U_i\right)=o\left(\frac{1}{\sqrt{n}}\right). \tag{3.22}$$

Hence it follows from (3.16), (3.20) and (3.22) that the asymptotic expansion of the distribution of the estimator $\hat{\theta}_{WM}$ up to the order $n^{-1/2}$ is given by

$$\begin{aligned} P\{\sqrt{n}\hat{\theta}_{WM}<c\} &= P\left\{\sum_{i=1}^m Y_i U_i<0\right\} \\ &= \Phi(c\sqrt{p_2})-\frac{p_3 c^2}{2\sqrt{p_2 n}}\phi(c\sqrt{p_2}) \operatorname{sgn} c+o\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{3.23}$$

We also have by (3.23)

$$P\{\sqrt{p_2 n}(\hat{\theta}_{WM}-\theta)<t\}=\Phi(t)-\frac{p_3 t^2}{2p_2^{3/2}\sqrt{n}}\phi(t) \operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.24}$$

In a similar way to Akahira and Takeuchi ([2], page 97) it is shown that the bound of the asymptotic distributions of the all second order asymptotically median unbiased estimators based on the samples $\{X_{ij}\}$ is given by

$$\Phi(t)-\frac{p_3 t^2}{6p_2^{3/2}\sqrt{n}}\phi(t) \operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right)=F^*(t) \qquad \text{(say)},$$

that is, for any second order asymptotically median unbiased estimator θ_n

$$P\{\sqrt{p_2 n}(\hat{\theta}_n-\theta)\leqq t\}\leqq F^*(t) \qquad \text{for all } t>0;$$

$$P\{\sqrt{p_2 n}(\hat{\theta}_n-\theta)\leqq t\}\geqq F^*(t) \qquad \text{for all } t<0.$$

Here an estimator $\hat{\theta}_n$ of θ based on the samples $\{X_{ij}\}$ is called second order asymptotically median unbiased (AMU) if

$$\lim_{n\to\infty}\sqrt{n}\left|P\{\hat{\theta}_n\leqq\theta\}-\frac{1}{2}\right|=\lim_{n\to\infty}\sqrt{n}\left|P\{\hat{\theta}_n\geqq\theta\}-\frac{1}{2}\right|=0$$

uniformly in some neighborhood of $(\theta, \tau_1, \cdots, \tau_m)$ (e.g. see [2]). From (3.9) and (3.24) it is easily seen that the estimators $\hat{\theta}^0_{ML}$ and $\hat{\theta}_{WM}$ are second order AMU. Hence we have established the following theorem.

THEOREM 3.1. *The bound of the asymptotic distributions of the all second order AMU estimators and the asymptotic expansion of the distribution of the estimators $\hat{\theta}^0_{ML}$ and $\hat{\theta}_{WM}$ up to the order $n^{-1/2}$ have the form of*

$$\Phi(t)-\frac{\alpha p_3 t^2}{p_2^{3/2}\sqrt{n}}\phi(t)\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right)$$

with α given by the following:

Table 3.1.

Estimator	α
The bound	1/6
$\hat{\theta}^0_{ML}$	1/2
$\hat{\theta}_{WM}$	1/2

Remark 3.2. By Theorem 3.1 we see that both of the estimators $\hat{\theta}^0_{ML}$ and $\hat{\theta}_{WM}$ are asymptotically equivalent up to the order $n^{-1/2}$ in spite that $\hat{\theta}^0_{ML}$ and $\hat{\theta}_{WM}$ are the estimators for known τ_i $(i=1,\cdots,m)$ and unknown τ_i's, respectively, but they are not second order asymptotically efficient in the sense that their asymptotic distributions do not attain the bound uniformly.

If $\tau_i=\tau$ $(i=1,\cdots,m)$ and it is known, then the MLE $\hat{\theta}^0_{ML}$ is the median of $\{X_{ij}\}$. In a similar way to Akahira and Takeuchi ([2], page 97) it follows that the asymptotic distribution of $\hat{\theta}^0_{ML}$ up to the order $n^{-1/2}$ is given by

$$P\left\{\frac{\sqrt{mn}}{\tau}(\hat{\theta}^0_{ML}-\theta)\leqq t\right\}=\Phi(t)-\frac{t^2}{2\sqrt{mn}}\phi(t)\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.25}$$

Hence it is seen that (3.25) is consistent with the asymptotic distribution of $\hat{\theta}^0_{ML}$ given in Theorem 3.1 in the same situation.

If τ_i $(i=1,\cdots,m)$ are known, then for each $i=1,\cdots,m$ the MLE $\hat{\theta}_i$ of θ is the median of X_{ij} $(j=1,\cdots,n)$, i.e. $\hat{\theta}_i=\operatorname{med}_{1\leq j\leq n} X_{ij}$. Then it follows from (3.25) that for each $i=1,\cdots,m$

$$(3.26)\qquad P\left\{\frac{\sqrt{n}}{\tau_i}(\hat{\theta}_i-\theta)\leq t\right\}=\Phi(t)-\frac{t^2}{2\sqrt{n}}\phi(t)\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right)$$
$$=F_{\hat{\theta}_i}(t)\qquad \text{(say)}.$$

From (3.26) we have the asymptotic density of $\hat{\theta}_i$ up to the order $n^{-1/2}$

$$(3.27)\qquad f_{\hat{\theta}_i}(t)=\frac{d}{dt}F_{\hat{\theta}_i}(t)=\phi(t)+\frac{1}{2\sqrt{n}}(t^3-2t)\phi(t)\operatorname{sgn} t+o\left(\frac{1}{\sqrt{n}}\right).$$

From (3.27) we obtain the asymptotic mean of $\hat{\theta}_i$

$$(3.28)\qquad E_\theta\left[\frac{\sqrt{n}}{\tau_i}(\hat{\theta}_i-\theta)\right]=\int_{-\infty}^{\infty} tf_{\hat{\theta}_i}(t)dt=o\left(\frac{1}{\sqrt{n}}\right).$$

Hence it is seen by (3.28) that the asymptotic variance of $\hat{\theta}_i$ is given by

$$(3.29)\qquad V_\theta\left(\frac{\sqrt{n}}{\tau_i}(\hat{\theta}_i-\theta)\right)=\int_{-\infty}^{\infty} t^2f_{\hat{\theta}_i}(t)dt=1+\frac{2\sqrt{2}}{\sqrt{n\pi}}+o\left(\frac{1}{\sqrt{n}}\right).$$

Since

$$\hat{\theta}^0=\sum_{i=1}^m\frac{1}{\tau_i^2}\hat{\theta}_i\Big/\sum_{i=1}^m\frac{1}{\tau_i^2}=\frac{1}{p_2}\sum_{i=1}^m\frac{1}{\tau_i^2}\hat{\theta}_i\,,$$

it follows from (3.29) that the asymptotic variance of $\hat{\theta}^0$ is given by

$$(3.30)\qquad V_\theta(\sqrt{p_2 n}(\hat{\theta}^0-\theta))=1+\frac{2\sqrt{2}}{\sqrt{n\pi}}+o\left(\frac{1}{\sqrt{n}}\right).$$

Since $\hat{\theta}_i$ $(i=1,\cdots,m)$ are independent, it follows from (3.26) that the limiting distribution of $\sqrt{p_2 n}(\hat{\theta}^0-\theta)$ is the standard normal distribution in the first order. On the other hand we have from (3.9)

$$(3.31)\qquad V_\theta(\sqrt{p_2 n}(\hat{\theta}^0_{ML}-\theta))=1+\frac{p_3}{p_2^{3/2}}\cdot\frac{2\sqrt{2}}{\sqrt{n\pi}}+o\left(\frac{1}{\sqrt{n}}\right).$$

Note that $1/\sqrt{m}\leq p_3/p_2^{3/2}\leq 1$ since

$$\left\{\left(\sum_{i=1}^m\frac{1}{\tau_i^2}\right)\Big/m\right\}^{1/2}\leq\left\{\left(\sum_{i=1}^m\frac{1}{\tau_i^3}\right)\Big/m\right\}^{1/3};\qquad \left(\sum_{i=1}^m\frac{1}{\tau_i^3}\right)^2\leq\left(\sum_{i=1}^m\frac{1}{\tau_i^2}\right)^3.$$

Then we obtain by (3.30) and (3.31)

$$(3.32)\qquad \lim_{n\to\infty}\sqrt{n}\,[V_\theta(\sqrt{p_2 n}(\hat{\theta}^0-\theta))-V_\theta(\sqrt{p_2 n}(\hat{\theta}^0_{ML}-\theta))]$$
$$=\frac{2\sqrt{2}}{\sqrt{\pi}}\left(1-\frac{p_3}{p_2^{3/2}}\right)\geq\frac{2\sqrt{2}}{\sqrt{\pi}}\left(1-\frac{1}{\sqrt{m}}\right)\geq 0\,.$$

Hence we have established the following theorem.

THEOREM 3.2. *If τ_i $(i=1,\cdots,m)$ are known, then the MLE $\hat{\theta}^0_{ML}$ is asymptotically better than $\hat{\theta}^0$ up to the order $n^{-1/2}$ in the sense of* (3.32).

Remark 3.3. By Remark 3.2 and Theorem 3.2 we see that the weighted median $\hat{\theta}_{WM}$ by the weights $1/\hat{\tau}_i(\hat{\theta}_{WM})$ $(i=1,\cdots,m)$ is also asymptotically better than $\hat{\theta}^0$ up to the order $n^{-1/2}$.

Next we shall obtain the asymptotic variance of the weighted mean $\hat{\theta}^*$ of $\hat{\theta}_i$ $(i=1,\cdots,m)$ by the weights $1/\hat{\tau}_i^2(\hat{\theta}_i)$ $(i=1,\cdots,m)$, where $\hat{\tau}_i(\hat{\theta}_i)=\sum_{j=1}^n |X_{ij}-\hat{\theta}_i|$ with $\hat{\theta}_i=\operatorname{med}_{1\le j\le n} X_{ij}$ $(i=1,\cdots,m)$. Putting $\Delta_i=\hat{\tau}_i-\tau_i$ $(i=1,\cdots,m)$, we have

$$(3.33)\qquad \sum_{i=1}^m \frac{1}{\hat{\tau}_i^2}=\sum_{i=1}^m \frac{1}{\tau_i^2\{1+(1/\tau_i)(\hat{\tau}_i-\tau_i)\}^2}=\sum_{i=1}^m \frac{1}{\tau_i^2}\left(1-\frac{2\Delta_i}{\tau_i}+o_p(\Delta_i)\right).$$

By (3.33) we obtain

$$(3.34)\qquad \begin{aligned}\hat{\theta}^*&=\sum_{i=1}^m \frac{1}{\hat{\tau}_i^2}\hat{\theta}_i\Big/\sum_{i=1}^m \frac{1}{\hat{\tau}_i^2}\\ &=\frac{1}{p_2}\left(\sum_{i=1}^m \frac{\hat{\theta}_i}{\tau_i^2}-2\sum_{i=1}^m \frac{\hat{\theta}_i\Delta_i}{\tau_i^3}+\frac{2}{p_2}\sum_{i=1}^m \frac{\hat{\theta}_i}{\tau_i^2}\sum_{i=1}^m \frac{\Delta_i}{\tau_i^3}\right)+o_p\left(\sum_{i=1}^m \Delta_i\right).\end{aligned}$$

Without loss of generality we assume that $\theta=0$. Since by (3.34)

$$(3.35)\qquad \begin{aligned}\sqrt{n}\,\hat{\theta}^*=\frac{1}{p_2}&\left(\sum_{i=1}^m \frac{\sqrt{n}\,\hat{\theta}_i}{\tau_i^2}-\frac{2}{\sqrt{n}}\sum_{i=1}^m \frac{\sqrt{n}\,\hat{\theta}_i\sqrt{n}\,\Delta_i}{\tau_i^3}\right.\\ &\left.+\frac{2}{p_2\sqrt{n}}\sum_{i=1}^m \frac{\sqrt{n}\,\hat{\theta}_i}{\tau_i^2}\sum_{i=1}^m \frac{\sqrt{n}\,\Delta_i}{\tau_i^3}\right)+o_p\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

it follows from (3.28) that

$$(3.36)\qquad \begin{aligned}E_0[\sqrt{n}\,\hat{\theta}^*]=\frac{1}{p_2}&\left\{\sum_{i=1}^m \frac{1}{\tau_i^2}E_0[\sqrt{n}\,\hat{\theta}_i]-\frac{2}{\sqrt{n}}\sum_{i=1}^m \frac{1}{\tau_i^3}E_0[\sqrt{n}\,\hat{\theta}_iE_0[\sqrt{n}\,\Delta_i|\hat{\theta}_i]]\right.\\ &\left.+\frac{2}{p_2\sqrt{n}}\sum_{i=1}^m\sum_{j=1}^m \frac{1}{\tau_i^2\tau_j^3}E_0[\sqrt{n}\,\hat{\theta}_iE_0[\sqrt{n}\,\Delta_i|\hat{\theta}_i]]\right\}\\ &+o\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

where $E[\cdot|\hat{\theta}_i]$ denotes the asymptotic conditional mean given $\hat{\theta}_i$. After some manipulation we have by (3.36)

$$(3.37)\qquad E_0[\sqrt{n}\,\hat{\theta}^*]=o\left(\frac{1}{\sqrt{n}}\right)$$

(for detail see Akahira and Takeuchi [5]). Since by (3.35)

$$E_0[n\hat{\theta}^{*2}]=\frac{1}{p_2^2}\left\{E_0\left[\left(\sum_{i=1}^{m}\frac{\sqrt{n}\,\hat{\theta}_i}{\tau_i^2}\right)^2\right]-\frac{4}{\sqrt{n}}E_0\left[\sum_{i=1}^{m}\frac{\sqrt{n}\,\hat{\theta}_i}{\tau_i}\right.\right.$$
$$\cdot E_0\left[\sum_{i=1}^{m}\frac{\sqrt{n}\,\hat{\theta}_i\sqrt{n}\,\Delta_i}{\tau_i^3}\middle|\hat{\theta}_i\right]\Bigg]+\frac{4}{p_2\sqrt{n}}E_0\left[\left(\sum_{i=1}^{m}\frac{\sqrt{n}\,\hat{\theta}_i}{\tau_i^2}\right)^2\right.$$
$$\left.\left.\cdot E_0\left[\sum_{i=1}^{m}\frac{\sqrt{n}\,\Delta_i}{\tau_i}\middle|\hat{\theta}_i\right]\right]\right\}+o\left(\frac{1}{\sqrt{n}}\right),$$

it follows that

$$E_0[n\hat{\theta}^{*2}]=\frac{1}{p_2}\left\{1+\frac{2\sqrt{2}}{\sqrt{n\pi}}\right\}+O\left(\frac{1}{n}\right). \tag{3.38}$$

From (3.37) and (3.38) we have

$$V_0(\sqrt{p_2 n}\hat{\theta}^*)=1+\frac{2\sqrt{2}}{\sqrt{n\pi}}+O\left(\frac{1}{n}\right).$$

It is noted from (3.26) and (3.35) that the limiting distribution of $\sqrt{p_2 n}\hat{\theta}^*$ is the standard normal distribution in the first order. Hence we have established the following theorem.

THEOREM 3.3. *The weighted mean $\hat{\theta}^*$ of $\hat{\theta}_i$ $(i=1,\cdots,m)$ by the weights $1/\hat{\tau}_i^2(\hat{\theta}_i)$ $(i=1,\cdots,m)$ is asymptotically equivalent to the weighted mean $\hat{\theta}^0$ by the weights $1/\tau_i^2$ $(i=1,\cdots,m)$ up to the order $n^{-1/2}$ in the sense that*

$$\lim_{n\to\infty}\sqrt{n}\,|V_\theta(\sqrt{p_2 n}(\hat{\theta}^*-\theta))-V_\theta(\sqrt{p_2 n}(\hat{\theta}^0-\theta))|=0 .$$

From Remark 3.3 and Theorem 3.3 we also have the following:

COROLLARY 3.1. *The weighted median $\hat{\theta}_{WM}$ by the weights $1/\hat{\tau}_i(\hat{\theta}_{WM})$ $(i=1,\cdots,m)$ is asymptotically better than the weighted mean $\hat{\theta}^*$ up to the order $n^{-1/2}$ in the sense that*

$$\lim_{n\to\infty}\sqrt{n}\,[V_\theta(\sqrt{p_2 n}(\hat{\theta}_{WM}-\theta))-V_\theta(\sqrt{p_2 n}(\hat{\theta}^*-\theta))]\geqq 0 .$$

As is seen from the above discussion, the second order asymptotic comparison of the asymptotically efficient estimators $\hat{\theta}_{ML}^0$, $\hat{\theta}_{WM}$, $\hat{\theta}^0$ and $\hat{\theta}^*$ is given by

$$\hat{\theta}_{WM}\sim\hat{\theta}_{ML}^0\succ\hat{\theta}^0\sim\hat{\theta}^* ,$$

where "$a\sim b$" ("$a\succ b$") means that a is asymptotically equivalent to (better than) b up to the order $n^{-1/2}$.

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REFERENCES

[1] Akahira, M. (1983). Asymptotic deficiencies of estimators for pooled samples from the same distribution. Probability Theory and Mathematical Statistics, *Lecture Notes in Mathematics*, **1021**, 6-14, Springer-Verlag, Berlin.

[2] Akahira, M. and Takeuchi, K. (1981). *Asymptotic Efficiency of Statistical Estimators: Concepts and Higher Order Asymptotic Efficiency*, Lecture Notes in Statistics, **7**, Springer-Verlag, New York.

[3] Akahira, M. and Takeuchi, K. (1982). On asymptotic deficiency of estimators in pooled samples in the presence of nuisance parameters, *Statistics and Decisions*, **1**, 17-38.

[4] Akahira, M. and Takeuchi, K. (1985). Estimation of a common parameter for pooled samples from the uniform distributions, *Ann. Inst. Statist. Math.*, **37**, 131-140.

[5] Akahira, M. and Takeuchi, K. (1985). *Non-Regular Statistical Estimation*, Monograph.

[6] Akai, T. (1982). A combined estimator of a common parameter, *Keio Science and Technology Reports*, **35**, 93-104.

[7] Bhattacharya, C. G. (1981). Estimation of a common location, *Commun. Statist.*, A**10**(10), 955-961.

[8] Cohen, A. (1976). Combining estimates of location, *J. Amer. Statist. Ass.*, **71**, 172-175.

[9] Takeuchi, K. and Akahira, M. (1983). Loss of information of the order statistics and related estimators in the double exponential distribution case, submitted for publication.