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BETWEENNESS FOR REAL VECTORS AND LINES, III
ALTERNATIVE CHARACTERIZATIONS OF BETWEENNESSES

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Summary

This paper introduces terminology enabling the discussion of center-related betweennesses defined with respect to centers other than the origin. Thus, it is then proved that the familiar $\mathcal{V}^v_\nu$-betweenness in $\mathcal{O}_{\mathcal{K}}$ which is origin-centered, is equivalent to a non-origin-centered $\mathcal{V}^v_0$-betweenness in one lower dimension. A new betweenness in $\mathcal{K}$, denoted by $\mathcal{V}^v_{\circledcirc,r}$ ($r>0$), is defined and studied, and it is shown that its restriction to $\mathcal{O}_{\mathcal{K}}$ is precisely $\mathcal{V}^v_\nu$-betweenness for a certain ν. Finally, by a method elaborated in the earlier papers, a new betweenness, $\tilde{\mathcal{V}}^v_{0,(3)}$, is induced on the upper open-hemisphere-plus-a-point in 3-space, and a characterization of it is obtained which is expected to facilitate later investigations of betweenness is complex spaces.

1. Introduction

This third article in our initial series of three articles on betweenness completes for now our restricted discussion of this notion for real vector spaces. Subsequent papers will move on to the case of complex vector spaces. And some of the results given here below are specifically with the intention of application to the complex case. Let it be noted that real and complex betweennesses are not walled away from each other, treatable only separately with independent definitions. Our previous two papers have shown how betweennesses are induced from one space to another, in either direction, when we have a function given. Well, there are mappings between real and complex vector spaces, and they can be so used to induce betweennesses in one space or the other. In particular, the betweenness relation on the subset of the surface of the real unit hemisphere, that we obtain here in Section 5, and more generally the form of mapping that is used to induce it,

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will later be used to generate betweenness relations for 1-dimensional linear manifolds in a complex unitary space.

We begin our present discussion with the establishment, in Section 2, of an alternative characterization of $\mathcal{V}^{\nu}_{0}$-betweenness in $\mathcal{O}_{\mathcal{K}}$, in the form of Theorem 2.1. This theorem substitutes its conditions i), ii) and iii) for the explicit condition of non-negativity of the coefficients a and b in the Definition I.5.2 of $\mathcal{V}^{\nu}_{0}$-betweenness. This new characterization is not merely of interest for itself: we will use it in these on-going studies (starting in Section 5 of the present paper, in fact) to help elaborate other betweennesses. It is with this in mind that we define, in Section 2 "other-centered" betweennesses in $\mathcal{K}$, and present Corollary 2.1.1.

Each of the ensuing two sections then presents another characterization of $\mathcal{V}^{\nu}_{0}$-betweenness in $\mathcal{O}_{\mathcal{K}}$. The theorem in Section 3 concerned with the case $\nu<\sqrt{\frac{1+\xi}{2}}$, gives a characterization in terms of w-centered $\mathcal{V}^{\nu}_{0}$-betweennesses. And the theorem in Section 4 finds that a certain newly-defined betweenness in $\mathcal{K}$—which we designated as $\mathcal{V}^{\nu}_{\circledcirc,r}$-betweenness—when restricted to $\mathcal{O}_{\mathcal{K}}$, is just $\mathcal{V}^{\nu}_{\nu}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ for a certain ν.

In Section 5 we develop another example of the procedure of inducing a betweenness in one space from a betweenness in another space through a function between the two spaces. (We have previously seen examples of this in both the papers I and II of this series.) Specifically, we induce from a non-θ-centered $\mathcal{V}^{\nu}_{0}$-betweenness on a certain sphere in 3-space the betweenness we have labeled $\tilde{\mathcal{V}}^{\nu}_{0,(3)}$, on the set $\tilde{\mathcal{O}}_{(3)}$ which is a modified open hemisphere of the unit sphere in the 3-space $\mathcal{K}_{(3)}$. This example has been chosen and worked out for the reason of its usefulness in the complex case—as we have remarked above; but it may have also the virtue of being strongly suggestive of new techniques for obtaining interesting betweennesses.

The notation and terminology in this paper will continue to follow the letter and spirit of those in the first two papers of the series. When specific new elements are introduced here below, they will either be defined or be clear in their meaning from the context.

2. On $\mathcal{V}^{\nu}_{0}$-betweenness

The principal burden of this section is to establish the following theorem:

THEOREM 2.1. *Let x, y and z be three elements of $\mathcal{O}_{\mathcal{K}}$, and let z be a linear combination of x and y. Then z is $\mathcal{V}^{\nu}_{0}$-between x and y if* *and only if the following three conditions are satisfied:*

$$\text{(2.1)}\qquad \begin{aligned} &\text{i)}\ \ \|z-x\|\leqq\|y-x\|\,, \\ &\text{ii)}\ \ \|z-y\|\leqq\|y-x\|\,; \\ &\text{iii)}\ \ \|z-x\|^2+\|x-y\|^2+\|y-z\|^2\leqq 8\,. \end{aligned}$$

PROOF. The hypothesis gives us that

$$z=ax+by \tag{2.2}$$

for some a and b. Thus it is to be shown that (2.1) is equivalent to the condition that a and b are both non-negative, or can be so chosen.

Let us begin by noting certain equalities. Since z is a unit vector, along with x and y, (2.2) gives us that

$$a^2+b^2+2ab(x, y)=1\,. \tag{2.3}$$

We get also from (2.2)

$$\begin{cases} (z, x)=a+b(x, y)\,, \\ (y, z)=a(x, y)+b\,. \end{cases} \tag{2.4}$$

Note that (2.3) can be presented in the form

$$a[a+b(x, y)]+b[a(x, y)+b]=1\,, \tag{2.5}$$

and also in the form

$$a[a+(b-1)(x, y)]+b[(a-1)(x, y)+b]+(x, y)[a+b+1]=1+(x, y)\,. \tag{2.6}$$

Either directly or by substituting from (2.4) into (2.5) we obtain another useful form:

$$a(z, x)+b(y, z)=1\,. \tag{2.7}$$

Finally, we observe that

$$\begin{cases} \|z-x\|^2\equiv 2[1-(z, x)]\,, \\ \|x-y\|^2\equiv 2[1-(x, y)]\,, \\ \|y-z\|^2\equiv 2[1-(y, z)]\,, \end{cases} \tag{2.8}$$

and consequently

$$\|z-x\|^2+\|x-y\|^2+\|y-z\|^2\equiv 6-2[(z, x)+(x, y)+(y, z)]\,. \tag{2.9}$$

Let us now proceed to prove the necessity assertion of the theorem. Suppose a and b are non-negative. We assert that it then follows that

$$\begin{cases} (z, x) \geqq (x, y), \\ (y, z) \geqq (x, y). \end{cases} \tag{2.10}$$

If $a=0$, the unit lengths of y and z imply that $b=1$ and so $z=y$. In this case (2.10) is immediately established. Similarly in the case $b=0$. Let us suppose, then, that $a>0$ and $b>0$. Consider the first relation in (2.10), and suppose it does not hold; that is, suppose that $(z, x) \ll (x, y)$. Since also $(y, z) \leqq 1 = (y, y)$, we have, by (2.7), using the positivity of a and b,

$$\begin{aligned} 1 &= a(z, x) + b(y, z) \\ &< a(x, y) + b(y, y) \\ &= (ax + by, y) \\ &= (z, y). \end{aligned} \tag{2.11}$$

This result is an obvious contradiction to $(y, z) < 1$. Therefore the first inequality in (2.10) does indeed hold. In a similar way the second inequality is proved.

From (2.10), via (2.8), we derive the fact that i) and ii) of (2.1) hold.

To prove that also (2.1), iii) holds, consider the equation, obtained from (2.4),

$$(z, x) + (x, y) + (y, z) = (a+b)(1 + (x, y)) + (x, y). \tag{2.12}$$

The non-negativity of a and b means that the right-hand side expression is $\geqq -1$. Therefore, the right-hand side of (2.9) is $\leqq 8$. And this is precisely the assertion iii) of (2.1). This completes the proof of necessity.

To establish the sufficiency assertion, suppose that (2.1) holds. Then i) and ii) of (2.1) imply, through (2.4), the inequalities

$$\begin{cases} a + (b-1)(x, y) \geqq 0, \\ (a-1)(x, y) + b \geqq 0. \end{cases} \tag{2.13}$$

And iii) of (2.1) implies, through (2.9), that the right-hand side of (2.1–) is $\geqq -1$; that is, that

$$(a+b+1)(1 + (x, y)) \geqq 0. \tag{2.14}$$

If $(x, y) = -1$, that is, if $y = -x$, then $z = (a-b)x = (b-a)y$; and so either $z = (a-b)x + 0 \cdot y$ or $z = 0 \cdot x + (b-a)y$ exhibits z as a non-negative linear combination of x and y, and the desired result is at hand in this case. In the complementary case, namely, $(x, y) \neq -1$, the inequality (2.14) yields

$$a+b+1 \geqq 0\,, \tag{2.15}$$

and we are to show that in this case (2.13) and (2.15) imply that $a \geqq 0$ and $b \geqq 0$.

We assert first that not both a and b are negative. Suppose they were. Then we should have the right-hand side of (2.6) greater than 0 while the first two terms on the left-hand side are not greater than 0. It would follow that the third term is necessarily positive, and therefore, by (2.15), that $(x, y) > 0$. On the other hand, either one of (2.13) implies that if a and b are negative then $(x, y) < 0$. Thus, a contradiction. And so, as asserted, a and b are not both negative.

We now go on to show that a is not negative. Suppose, to the contrary, that $a < 0$. Then we must have $b \geqq 0$. By (2.5) and (2.13) we get, since $a < 0$,

$$\begin{aligned} 1 &= a[a+b(x, y)]+b[a(x, y)+b] \\ &\leqq a(x, y)+b^2+ab(x, y) \\ &= b^2+(b+1)a(x, y)\,. \end{aligned} \tag{2.16}$$

This gives

$$(b+1)[a(x, y)+b-1] \geqq 0\,. \tag{2.17}$$

Since $b \geqq 0$, and therefore $b+1>0$, we get from this that

$$a(x, y)+b \geqq 1\,. \tag{2.18}$$

Since the left-hand side here is precisely (y, z), we have the implication that $(y, z)=1$, that is $z=y$. But this asserts that $a=0$, thus contradicting our assumption that $a<0$. Hence, indeed, we must have $a \geqq 0$.

Similarly it is proved that $b \geqq 0$. Therewith the sufficiency is demonstrated and the proof of the theorem is complete.

In Definition I.6.2 we defined the notion of B-betweenness in C when C is a subset of the space $\mathscr{X}$ in which the given B-betweenness is defined. According to that definition, if $\mathcal{O}'$ and $\mathcal{O}''$ are two spheres in our real unitary space $\mathcal{K}$, with equal radii but different centers, then $\mathcal{V}_0^v$-betweenness in $\mathcal{O}'$ and $\mathcal{V}_0^v$-betweenness in $\mathcal{O}''$ may fail to be congruent (—that is, may not admit an isometry of one sphere into the other that carries ordered between-triplets into ordered between-triplets—) even though the two spheres are themselves congruent. This terminological circumstance necessitates our making another definition for convenience, since significant occasions arise in which we want to discuss, for a given sphere, the betweenness relation on that sphere that is intrinsically identical with $\mathcal{V}_0^v$-betweenness on an origin-centered sphere of the same radius. It will suffice for now to make a definition that is suited to just this case:

DEFINITION 2.1. Let w be any particular point of the real unitary space $\mathcal{K}$. If x, y and z are three points in $\mathcal{K}$, we say that z is ***w*-centered-ひ^{ν}_{0}-between** x and y if z-w is ひ^{ν}_{0}-between x-w and y-w.

It follows, then, for example, that—with $\mathcal{O}_{w,r}$ denoting the sphere in $\mathcal{K}$ with center w and radius r—points x, y and z are such that z is w-centered-ひ^{ν}_{0}-between x and y in $\mathcal{O}_{w,r}$ if and only if z-w is ひ^{ν}_{0}-between x-w and y-w in $\mathcal{O}_{\theta,r}$.

We may now adapt Theorem 2.1 to the case of w-centered-ひ^{ν}_{0}-betweenness. Noting that z-w is a linear combination of x-w and y-w if and only if the end points of the four vectors, x, y, z and w, are coplanar, and that $z \in \mathcal{O}_{\theta,r}$ if and only if $\frac{z}{r} \in \mathcal{O}_{\mathcal{K}}$, we have the following corollary:

COROLLARY 2.1.1. *Let x, y and z be three elements of $\mathcal{O}_{w,r}$. Then z is w-centered-ひ^{ν}_{0}-between x and y if and only if the following four conditions are fulfilled:*

i) $\|z-x\| \leqq \|y-x\|$,

ii) $\|z-y\| \leqq \|y-x\|$,

iii) $\|z-x\|^2+\|x-y\|^2+\|y-z\|^2 \leqq 8r^2$,

iv) *the endpoints of the four vectors, x, y, z and w, are coplanar.*

3. ひ^{ν}_{ν}-betweenness in terms of ひ^{ν}_{0}-betweenness

In the pondering on the geometry of ひ^{ν}_{ν}-betweenness, it suggests itself readily that if a point z is ひ^{ν}_{ν}-between x and y in $\mathcal{O}_{\mathcal{K}}$, then there is a lower-dimensional sphere on which x, y and z all lie and in which, relative to its center, z is ひ^{ν}_{0}-between x and y. This turns out to be exactly true, the precise statement being that of the following theorem. In this statement the notation $\mathcal{O}_{w,r}(\mathcal{L})$ designates the sphere with center w and radius r in the linear variety $w+\mathcal{L}$, where $\mathcal{L}$ is a sub-manifold of $\mathcal{K}$.

THEOREM 3.1. *Let x and y be vectors in $\mathcal{O}_{\mathcal{K}}$. Let $\nu>0$ and let と denote the inner product (x, y). Let the condition*

$$\nu<\sqrt{\frac{1+\text{と}}{2}} \tag{3.1}$$

hold.

Then, a necessary and sufficient condition that z be ひ^{ν}_{ν}-between x and y in $\mathcal{O}_{\mathcal{K}}$ is that, for some $\alpha \in \left[0, \frac{2\nu^2}{1+\text{と}}\right]$, there is an element $v \perp$

x, y, with

$$\|v\|=\sqrt{\frac{\alpha(1-\alpha)}{2}(1+\text{と})} \tag{3.2}$$

such that, on defining

$$w=\alpha\left(\frac{x+y}{2}\right)+v\,, \tag{3.3}$$

we have that the element z is w-centered-$\mho^{\nu}_{0}$-between the elements x and y in $\mathcal{O}_{w,\sqrt{1-\|w\|^2}}([\![\{w\}]\!]^{\perp})$.

PROOF. Notice that, by virtue of (3.1), the stated α-interval is a proper sub-interval of $[0, 1]$. Notice also that from (3.2) and (3.3) we have

$$(x, w)=(y, w)=\frac{\alpha}{2}(1+\text{と})=\|w\|^2\,, \tag{3.4}$$

and that from this it follows that x-w and y-w are both orthogonal to w, and both of length $\sqrt{1-\|w\|^2}$; thus, that x and y are elements of $\mathcal{O}_{w,\sqrt{1-\|w\|^2}}([\![\{w\}]\!]^{\perp})$.

We shall first prove necessity. Suppose z is $\mho^{\nu}_{\nu}$-between x and y in $\mathcal{O}_{\mathcal{K}}$. According to Theorem I.6.2, we then have that z is of the form

$$z=ax+by+u\,, \qquad u\perp x, y\,, \tag{3.5}$$

with

$$\rho\|u\|\leqq\nu(a+b-1)\,, \tag{3.6}$$

where

$$\rho=\sqrt{1-\frac{2\nu^2}{1+\text{と}}}\;. \tag{3.7}$$

Furthermore, if $y\neq x$, the coefficients a and b are non-negative; and if $y=x$, then $a+b=1$ and $u=\theta$.

Consider first the case of $u=\theta$. In this circumstance z is a non-negative linear combination of x and y, and so is $\mho^{\nu}_{0}$-between x and y in $\mathcal{O}_{\mathcal{K}}=\mathcal{O}_{\theta,1}(\mathcal{K})$. This is exactly the condition asserted by the theorem with $\alpha=0$.

Suppose now that $u\neq\theta$. Then we take

$$\alpha=\frac{1}{1+\dfrac{\kappa^2(1+\text{と})}{2\|u\|^2}} \tag{3.8}$$

where for brevity we have set

$$\kappa = a + b - 1\,. \tag{3.9}$$

It follows from (3.6) and (3.7) that this α is $\leqq \frac{2\nu^2}{1+ヒ}$, and consequently from (3.1) that $\alpha < 1$. Hence, we may next define

$$\gamma = \frac{\alpha\kappa}{2(1-\alpha)}\,, \tag{3.10}$$

and then

$$v = -\frac{1}{\kappa + 2\gamma} u\,. \tag{3.11}$$

With these definitions we find, on defining w as in (3.3), that our z of (3.5) may be put in the following form:

$$z = w + (a+\gamma)(x-w) + (b+\gamma)(y-w)\,. \tag{3.12}$$

The coefficients $a+\gamma$ and $b+\gamma$ are non-negative, since a and b are, and therefore (3.12) asserts that z is w-centered-$\mho_0^\nu$-between x and y. But it must be shown additionally that x, y and z are elements of $\mathcal{O}_{w,\sqrt{1-\|w\|^2}}(\llbracket\{w\}\rrbracket^\perp)$. This has already been done for x and y, assuming that (3.2) holds. That (3.2) does indeed hold under (3.8)–(3.11), inclusive, we may see by first using (3.10) and (3.11) to obtain

$$\|v\|^2 = \frac{(1-\alpha)^2}{\kappa^2}\|u\|^2\,, \tag{3.13}$$

and then applying to this the expression for $\|u\|^2$ that is given by solving (3.8), namely,

$$\|u\|^2 = \frac{\kappa^2(1+ヒ)}{2}\cdot\frac{\alpha}{1-\alpha}\,. \tag{3.13}$$

To show next that z also belongs to $\mathcal{O}_{w,\sqrt{1-\|w\|^2}}(\llbracket\{w\}\rrbracket^\perp)$, we note first that $z-w$ belongs to $\llbracket\{w\}\rrbracket^\perp$ since $x-w$ and $y-w$ do, and $z-w$ is, by (3.12), a linear combination of $x-w$ and $y-w$. Secondly, we establish the fact that $\|z-w\|^2 = 1 - \|w\|^2$ by the following sequence of calculations:

$$\begin{aligned} \|z-w\|^2 &= \|(z-u)+u-w\|^2 \\ &= \|z-u\|^2 + \|u\|^2 + \|w\|^2 - 2(z-u, w) - 2(u, w) \\ &= 1 - \|u\|^2 + \|u\|^2 + \|w\|^2 - \alpha(ax+by, x+y) - 2(u, v) \\ &= 1 + \|w\|^2 - \alpha(a+b)(1+ヒ) + \frac{2}{\kappa+2\gamma}\|u\|^2 \\ &= 1 + \|w\|^2 - \alpha(\kappa+1)(1+ヒ) + \frac{2(1-\alpha)}{\kappa}\left[\frac{\kappa^2(1+ヒ)}{2}\cdot\frac{\alpha}{1-\alpha}\right] \end{aligned} \tag{3.15}$$

$$=1+\|w\|^2-\alpha(1+\text{と})$$
$$=1-\|w\|^2 .$$

This completes the proof of necessity.

We now prove the sufficiency assertion. Suppose the conditions of the theorem are fulfilled. Then we have, for some non-negative numbers a_1 and b_1,

$$z-w=a_1(x-w)+b_1(y-w) , \tag{3.16}$$

and also

$$\|z-w\|^2=1-\|w\|^2 . \tag{3.17}$$

The evaluations (3.4) hold, and from them and (3.16) we obtain that $(z, w)=\|w\|^2$; and therefore (3.17) yields $\|z\|=1$. Thus, $z \in \mathcal{O}_{\mathcal{K}}$.

We now want to show that (3.5) and (3.6) hold for some a, b and u. Substituting from (3.3) into (3.16) gives us the (3.5)-form:

$$z=\left(a_1-\frac{\kappa_1\alpha}{2}\right)x+\left(b_1-\frac{\kappa_1\alpha}{2}\right)y-\kappa_1 v , \tag{3.18}$$

wherein we have set

$$\kappa_1=a_1+b_1-1 . \tag{3.19}$$

It remains, then, only to be shown that

$$\rho\|-\kappa_1 v\|\leqq\nu\left[\left(a_1-\frac{\kappa_1\alpha}{2}\right)+\left(b_1-\frac{\kappa_1\alpha}{2}\right)-1\right]. \tag{3.20}$$

On applying (3.2) and (3.7), and making certain rearrangements, we express this inequality in the form

$$|\kappa_1|\sqrt{\left(\frac{1+\text{と}}{2\nu^2}-1\right)\alpha(1-\alpha)}\leqq\kappa_1(1-\alpha) . \tag{3.21}$$

From (3.16) and (3.17)—using the fact that $\|x-w\|=\|y-w\|=\sqrt{1-\|w\|^2}$, as already seen from (3.4)—we find

$$\sqrt{1-\|w\|^2}=\|a_1(x-w)+b_1(y-w)\|\leqq(a_1+b_1)\sqrt{1-\|w\|^2} , \tag{3.22}$$

from which it follows that

$$\kappa_1\geqq 0 . \tag{3.23}$$

In light of this result, (3.21) becomes

$$\kappa_1\sqrt{\left(\frac{1+\text{と}}{2\nu^2}-1\right)\alpha(1-\alpha)}\leqq\kappa_1(1-\alpha) . \tag{3.24}$$

We now see that in the cases $\kappa_1=0$, $\alpha=0$, $\alpha=1$, this inequality, and hence the inequality (3.20), holds. In all other cases (3.24), and therefore (3.21), is equivalent to

$$\alpha \leqq \frac{2\nu^2}{1+\text{と}}, \tag{3.25}$$

which is seen to hold by virtue of the theorem's assertion that $\alpha \in \left[0, \frac{2}{1+\text{と}}\right]$.

This completes the proof of sufficiency and of the theorem.

4. Induction of ひ_ν^v-betweenness in $\mathcal{O}_{\mathcal{K}}$ by a non-ひ_ν^v-betweenness in $\mathcal{K}$

We are here going to define a type of betweenness relation in $\mathcal{K}$ that we have not discussed before; and we will show that its restriction to $\mathcal{O}_{\mathcal{K}}$ is precisely ひ_ν^v-betweenness for a certain ν. We shall give the definition of this new betweenness in terms of a Theorem I.5.1 representation for it, in which, moreover, the relation is of membership form (see Definition I.5.1). Thus, if Ω is the pertinent class of subsets of $\mathcal{K}$, the spread of the pair of points, x, y, in $\mathcal{K}$ will be the intersection of all $\omega \in \Omega$ each of which contains both x and y. Having recalled these facts, and noting that Ω, in the present, typical case, is the collection of all closed balls in $\mathcal{K}$ of a given, fixed radius, r (>0), we can proceed to give our definition:

DEFINITION 4.1. Let r be a fixed positive number. Then, for a point x and a point y, in $\mathcal{K}$, we define $z \in \mathcal{K}$ to be $\text{ひ}_{\circledcirc,r}^v$-between x and y if z lies in every closed ball of radius r that contains x and y.

We see immediately that if $\|x-y\|>2r$, then every $z \in \mathcal{K}$ is $\text{ひ}_{\circledcirc,r}^v$-between x and y. If $\|x-y\|=2r$, the ball having the segment $\overline{x, y}$ as one of its diameters is the set of all z's each of which is $\text{ひ}_{\circledcirc,r}^v$-between x and y. The case of $\|x-y\|<2r$ is the one of principal interest, and we now go on to obtain a characterization in this case. The subcase of $y=x$ is, of course, trivial; it is in the situation of $y \neq x$ that extended argumentation will be needed to obtain the desired characterization, namely, of the set

$$\tau_{\circledcirc,r}^v(\{x, y\}) = \left\{ z \in \mathcal{K} \,\middle|\, \begin{bmatrix} \|x-u\| \leqq r, \\ \|y-u\| \leqq r \end{bmatrix} \Longrightarrow \|z-u\| \leqq r \right\}. \tag{4.1}$$

This set is the $\text{ひ}_{\circledcirc,r}^v$-spread of the set $\{x, y\}$. Our approach to the characterization of this spread is going to be as follows: we shall first obtain a characterization of the alternative set

$$\tau'(\{x, y\}) = \left\{ z \in \mathcal{K} \,\middle|\, \begin{bmatrix} \|x-u\| = r\,, \\ \|y-u\| = r \end{bmatrix} \Longrightarrow \|z-u\| \leqq r \right\}, \tag{4.2}$$

and then we will prove that these two sets are identical.

Relating to (4.2), then, let us set

$$w \overset{\text{def.}}{=} y - x \tag{4.3}$$

and write

$$u = x + \beta w + \xi\,, \qquad \xi \perp w\,. \tag{4.4}$$

The conditions

$$\begin{cases} \|x-u\| = r\,, \\ \|y-u\| = r \end{cases} \tag{4.5}$$

may then be put in the respective forms

$$\begin{cases} \beta^2\|w\|^2 + \|\xi\|^2 = r^2\,, \\ (1-\beta)^2\|w\|^2 + \|\xi\|^2 = r^2\,. \end{cases} \tag{4.6}$$

These two equations clearly imply that $\beta = \frac{1}{2}$ and

$$\|\xi\|^2 = r^2 - \frac{\|w\|^2}{4}\,. \tag{4.7}$$

More completely, we see that (4.5) is satisfied if and only if

$$u = x + \frac{1}{2}w + \xi\,, \qquad \xi \perp w\,, \tag{4.8}$$

with ξ satisfying (4.7).

For such a u we now want to examine the inequality

$$\|z-u\| \leqq r\,. \tag{4.9}$$

We put

$$z = x + \alpha w + v\,, \qquad v \perp w\,, \tag{4.10}$$

and evaluate (using (4.7)):

$$\begin{aligned} \|z-u\|^2 &= \left\| \left(\alpha - \frac{1}{2}\right) w + v - \xi \right\|^2 \\ &= r^2 + \|v\|^2 - 2(v, \xi) - \alpha(1-\alpha)\|w\|^2\,. \end{aligned} \tag{4.11}$$

Thus, the inequality (4.9) takes the form

$$\|v\|^2-2(v,\xi)-\alpha(1-\alpha)\|w\|^2\leqq 0\,. \tag{4.12}$$

The task now is to distinguish those pairs $\langle\alpha, v\rangle$ which satisfy (4.12) for all ξ orthogonal to w and with norm given by (4.7). To this end, observe first that if $\langle\alpha, v\rangle$ is such a pair with $v\neq\theta$ then it fulfills (4.12) in particular for ξ chosen as

$$-\frac{v}{\|v\|}\sqrt{r^2-\frac{\|w\|^2}{4}}\,. \tag{4.13}$$

This ξ makes the first two signed terms in (4.12) both positive. We see then that α cannot be either <0 or >1; for, if it were, the last signed term also would be positive and we should have a contradiction to the inequality sign in (4.12). We get this contradiction immediately if $v=\theta$. Thus, we have the first portion of result: *for any solution pair* $\langle\alpha, v\rangle$ *we have* $0\leqq\alpha\leqq 1$.

In we take $\alpha=0$ or 1 in (4.12), and note that the resulting inequality must hold with ξ replaced by (4.13), then we see that there is no solution of (4.12) in these cases with $\|v\|>0$. Thus, $\alpha=0$ or 1 implies $v=\theta$.

If $0<\alpha<1$, then the quantity $\alpha(1-\alpha)\|w\|^2$ is positive. If $\tilde{v}$ is any unit vector orthogonal to w, and v is a multiple of $\tilde{v}$, the maximum value—for variations in ξ—of the sum of the first two terms of (4.12) is taken on for ξ equal to the quantity (4.13), and is

$$\|v\|^2+2\|v\|\sqrt{r^2-\frac{\|w\|^2}{4}}\,. \tag{4.14}$$

We see that for $\|v\|$ sufficiently small this quantity does not exceed $\alpha(1-\alpha)\|w\|^2$, and so (4.12) is satisfied. Thus, *for every* $\alpha\in(0,1)$ *there is a pair* $\langle\alpha, v\rangle$ *with* $v\neq\theta$ *which satisfies* (4.12) *for all pertinent* ξ.

Let $\langle\alpha, v\rangle$, with $v\neq\theta$, satisfy (4.12) for all pertinent ξ. If $k\in[0,1]$, note that

$$\|kv\|^2-2(kv,\xi)\equiv k\{[\|v\|^2-2(v,\xi)]-(1-k)\|v\|^2\}\,, \tag{4.15}$$

and that therefore also $\langle\alpha, kv\rangle$ satisfies (4.12) for all pertinent ξ. In other words, *for each* $\alpha\in[0,1]$, *the following holds: for each unit vector* $\tilde{v}$ *orthogonal to* w, *there is a real number, say* $\gamma_{\tilde{v}}$, *such that* $\langle\alpha,\gamma_{\tilde{v}}\rangle$ *fulfills* (4.12) *for all pertinent* ξ *if and only if* $\gamma\in[0,\gamma_{\tilde{v}}]$. *It is clear from* (4.12) *that* $\gamma_{\tilde{v}}$ *is finite.*

The characterization we are looking for now requires only that we find the number $\gamma_{\tilde{v}}$ for each α and each $\tilde{v}$. We have already determined that $\gamma_{\tilde{v}}\equiv 0$ for $\alpha=0$ and 1. We will find that also for $\alpha\in(0,1)$, $\gamma_{\tilde{v}}$ depends only on α, and not on $\tilde{v}$.

Let us rewrite (4.12) as follows:

$$\|v-\xi\|^2 \leqq r^2 - \frac{1}{4}\|w\|^2 + \alpha(1-\alpha)\|w\|^2 . \tag{4.16}$$

For given α and v, the left-hand side is maximized in ξ by (4.13); substituting that value for ξ into (4.16) and rearranging terms after square-rooting, we get

$$\|v\| \leqq \sqrt{\left(r^2 - \frac{1}{4}\|w\|^2\right) + \alpha(1-\alpha)\|w\|^2} - \sqrt{r^2 - \frac{1}{4}\|w\|^2} . \tag{4.17}$$

The right-hand side here is thus an upper bound of $\|v\|$ for all v such that $\langle \alpha, v\rangle$ is admissible (i.e., satisfies (4.12) for all pertinent ξ). It follows that this right-hand quantity is $\geqq \gamma_{\bar{v}}$. On the other hand, (4.17) implies (4.16) for every ξ orthogonal to w and of the square norm of (4.7), and this means that the right-hand side quantity in (4.17) is $\leqq \gamma_{\bar{v}}$. Hence the quantity in question is $=\gamma_{\bar{v}}$. With this we have completed our desired characterization, which is, namely, the following: *with* $w=y-x$ *and* $m_{w,\alpha}$ *denoting the quantity on the right-hand side of* (4.17), *we have*

$$\tau'(\{x, y\}) = \left\{ x+\alpha w+v \in \mathcal{K} \,\middle|\, \begin{matrix} 0\leqq\alpha\leqq 1;\ v\perp w; \\ \|v\|\leqq m_{w,\alpha} \end{matrix} \right\} . \tag{4.18}$$

An indicated above, we now want to prove that the set (4.1) is identical to the set (4.2), that is, to the set (4.18). Clearly, from (4.1) and (4.2) we have that $\tau^{v}_{\circledcirc,r}(\{x, y\}) \subseteq \tau'(\{x, y\})$. It remains then only to show the converse of this.

Let $z=x+\alpha w+v$ be an element of $\tau'(\{x, y\})$—expressed in the characteristic form of (4.18)—and let

$$u = x+\beta w+\xi , \qquad \xi\perp w , \tag{4.19}$$

be an element of $\mathcal{K}$ such that

$$\begin{cases} \|x-u\|\leqq r , \\ \|y-u\|\leqq r . \end{cases} \tag{4.20}$$

Notice that (4.19) and (4.20) give

$$\begin{cases} \beta^2\|w\|^2 + \|\xi\|^2 \leqq r^2 , \\ (1-\beta)^2\|w\|^2 + \|\xi\|^2 \leqq r^2 . \end{cases} \tag{4.21}$$

Summing these two inequalities and dividing by 2, we get

$$\frac{1}{2}[\beta^2 + (1-\beta)^2]\|w\|^2 + \|\xi\|^2 \leqq r^2 . \tag{4.22}$$

The quadratic function $\frac{1}{2}[\beta^2+(1-\beta)^2]$ evidently takes its minimum value at $\beta=\frac{1}{2}$. That minimum value is $\frac{1}{4}$, and so the left-hand side of (4.22) is $\geqq\frac{1}{4}\|w\|^2+\|\xi\|^2$, and we therefore obtain

$$\|\xi\|^2\leqq r^2-\frac{\|w\|^2}{4}. \tag{4.23}$$

If we set, for brevity,

$$\begin{cases} A=r^2-\dfrac{\|w\|^2}{4}, \\ B=A+\alpha(1-\alpha)\|w\|^2, \end{cases} \tag{4.24}$$

then (4.17) can be written

$$\|v\|\leqq\sqrt{B}-\sqrt{A}, \tag{4.25}$$

and we have also

$$|(v,\xi)|\leqq\|v\|\cdot\|\xi\|\leqq(\sqrt{B}-\sqrt{A})\sqrt{A}. \tag{4.26}$$

With these preliminaries we can now proceed to show what is to be shown, namely, that $\|z-u\|\leqq r$. Indeed,

$$\begin{aligned}
\|z-u\|^2&=\|(\alpha-\beta)w+(v-\xi)\|^2 \\
&=(\alpha-\beta)^2\|w\|^2+\|v-\xi\|^2 \\
&=[\alpha(1-\beta)^2+(1-\alpha)\beta^2-\alpha(1-\alpha)]\|w\|^2+\|v\|^2+\|\xi\|^2-2(v,\xi) \\
&\leqq r^2-\|\xi\|^2-\alpha(1-\alpha)\|w\|^2+\|v\|^2+\|\xi\|^2-2(v,\xi) \qquad \text{(by (4.21))} \\
&=r^2-\alpha(1-\alpha)\|w\|^2+\|v\|^2-2(v,\xi) \\
&\leqq r^2-\alpha(1-\alpha)\|w\|^2+A+B-2\sqrt{AB}-2(v,\xi) \qquad \text{(by (4.25))} \\
&=r^2+2A-2\sqrt{AB}-2(v,\xi) \qquad \text{(by (4.24))} \\
&\leqq r^2-2\sqrt{A}(\sqrt{B}-\sqrt{A})+2|(v,\xi)| \\
&\leqq r^2 \qquad \text{(by (4.26))}.
\end{aligned} \tag{4.27}$$

Thus it is proved that $\tau'(\{x, y\})\subseteq\tau^{v}_{\circledS,r}(\{x, y\})$, and hence that these two sets are identical.

We have covered all cases, and a full characterization theorem can now be stated:

THEOREM 4.1. *Let r be a fixed positive number, and let x and y be points of $\mathcal{K}$ (possibly the same point). In the case $y=x$, a point z is $\mho^{v}_{\circledS,r}$-between x and y if and only if $z=x=y$.*

If $\|x-y\|>2r$, *every* $z\in\mathcal{K}$ *is* $\mathcal{U}^{v}_{\circledcirc,r}$*-between* x *and* y. *If* $\|x-y\|=2r$, z *is* $\mathcal{U}^{v}_{\circledcirc,r}$*-between* x *and* y *if and only if* $\left\|z-\frac{x+y}{2}\right\|\leqq r$.

If $\|x-y\|\leqq 2r$, *then* z *is* $\mathcal{U}^{v}_{\circledcirc,r}$*-between* x *and* y *if and only if it is of the form*

$$z=x+\alpha w+v \tag{4.28}$$

where

$$w\equiv y-x$$

and

$$\begin{cases} 0\leqq\alpha\leqq 1\,, \\ v\perp w\,, \\ \|v\|\leqq m_{w,\alpha}\equiv\sqrt{\left(r^2-\frac{1}{4}\|w\|^2\right)+\alpha(1-\alpha)\|w\|^2}-\sqrt{r^2-\frac{1}{4}\|w\|^2}\,. \end{cases} \tag{4.29}$$

We now go on to study the restriction of $\mathcal{U}^{v}_{\circledcirc,r}$-betweenness to $\mathcal{O}_{\mathcal{K}}$. Thus, for given x and y, with $\|x\|=\|y\|=1$, we want to describe the elements z that are $\mathcal{U}^{v}_{\circledcirc,r}$-between x and y, and for which also $\|z\|=1$.

From the first statements in Theorem 4.1 we immediately have: for x, y and $z\in\mathcal{O}_{\mathcal{K}}$, if $y=x$, then z is $\mathcal{U}^{v}_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ if and only if $z=x=y$; and if $\|x-y\|>2r$, then every $z\in\mathcal{O}_{\mathcal{K}}$ is $\mathcal{U}^{v}_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$. The remaining case to be considered, namely, $y\neq x$ and $\|x-y\|\leqq 2r$, is characterized by (4.28) and (4.29) along with the conditions

$$\begin{cases} \|x\|=\|y\|=1\,, \\ \|z\|=1\,. \end{cases} \tag{4.30}$$

Let us now proceed to render this set of conditions more explicit.

We express the second of (4.30) using the form (4.28) and applying the first of (4.30); we get, on writing $v=\|v\|\cdot\tilde{v}$—where $\tilde{v}$ thus is a unit vector orthogonal to w—

$$\|v\|^2+2(\tilde{v},x)\|v\|-\alpha(1-\alpha)\|w\|^2=0\,. \tag{4.31}$$

If $\alpha=0$ or 1, we know immediately from (4.29) that $\|v\|=0$. If $0<\alpha<1$, the graph of the quadratic function on the left of (4.31) opens upward and is negative at the 0-value of the variable. Therefore, the equation (4.31) has one and only one positive root, which is thus the unique solution for the non-negative quantity $\|v\|$; this solution is:

$$\|v\|=-(\tilde{v},x)+\sqrt{(\tilde{v},x)^2+\alpha(1-\alpha)\|w\|^2}\,. \tag{4.32}$$

Now we must describe, in the case of $0<\alpha<1$, the collection of those $\tilde{v}$'s for which the solution (4.32) satisfies the last inequality in (4.29). If we define a function f of a real variable t by

$$f(t)=-t+\sqrt{t^2+\kappa}\ , \tag{4.33}$$

where $\kappa\equiv\alpha(1-\alpha)\|w\|^2$, then we see that the question at hand is: for a given α, for which vectors $\tilde{v}$ do we have

$$f((\tilde{v}, x))\leqq f\left(\sqrt{r^2-\frac{1}{4}\|w\|^2}\right)\ ? \tag{4.34}$$

With $0<\alpha<1$, we have $\kappa>0$, and the function f just defined is seen to be positive for all t and to have a negative derivative everywhere in t. Thus, f is a positive function, everywhere strictly decreasing. It follows that (4.34) holds if and only if the inequality

$$(\tilde{v}, x)\geqq\sqrt{r^2-\frac{1}{4}\|w\|^2} \tag{4.35}$$

holds.

There is more to be said. Since $\tilde{v}$ is orthogonal to w there is an upper bound to the quantity $(\tilde{v}, x)$ which is <1. We shall find this upper bound now. Let x be written in the form

$$x=hw+\zeta\ , \qquad \zeta\perp w\ , \tag{4.36}$$

so that

$$h^2\|w\|^2+\|\zeta\|^2=1\ . \tag{4.37}$$

Inner producting (4.36) with w, we get

$$(x, w)=h\|w\|^2\ . \tag{4.38}$$

Also:

$$\begin{aligned}(x, w)&=(x, y-x)=(x, y)-1\\ &=-\frac{1}{2}[\|x\|^2-2(x, y)+\|y\|^2]\\ &=-\frac{1}{2}\|w\|^2\ .\end{aligned} \tag{4.39}$$

Thus, we get $h=-\dfrac{1}{2}$ and (4.37) becomes

$$\frac{1}{4}\|w\|^2+\|\zeta\|^2=1\ . \tag{4.40}$$

We have, since $\tilde{v}\perp w$,

$$(\tilde{v}, x) = (\tilde{v}, \zeta)\,. \tag{4.41}$$

It is clear now that this quantity is maximized for $\tilde{v} = \zeta/\|\zeta\|$. Thus,

$$(\tilde{v}, x) \leqq \left(\frac{\zeta}{\|\zeta\|}, \zeta\right) = \|\zeta\|\,. \tag{4.42}$$

Evaluating $\|\zeta\|$ from (4.40), we finally have:

$$(\tilde{v}, x) \leqq \sqrt{1 - \frac{1}{4}\|w\|^2}\,. \tag{4.43}$$

Comparing (4.35) and (4.43) we obtain a result that is, in fact, geometrically clear. Notice that for $r > 1$ there is no $\tilde{v}$ that satisfies both (4.35) and (4.43), and hence there is no $z \in \mathcal{O}_{\mathcal{K}}$, of the form (4.28) with $0 < \alpha < 1$, which is $\mho^{v}_{\circledcirc, r}$-between x and y. That is, in the case $r > 1$, x and y are the only points each of which is $\mho^{v}_{\circledcirc, r}$-between x and y. In the case $r = 1$ it is clear from the above discussion that the only possibility for $\tilde{v}$ is $\zeta/\|\zeta\|$. From (4.36)—with $h = -\frac{1}{2}$—ζ is found to be $\frac{(x+y)}{2}$, and consequently v becomes $\frac{(x+y)}{\|x+y\|}$. If this is used, together with (4.32), to find v, and then the right-hand side of (4.28) is evaluated, it is found that for every α in $(0, 1) z$ is a positive linear combination of x and y.

We are now able to state:

THEOREM 4.2. *Let r be a fixed positive number, and let x and y be points of $\mathcal{O}_{\mathcal{K}}$ (possibly the same point). In the case $y = x$, a point $z \in \mathcal{O}_{\mathcal{K}}$ is $\mho^{v}_{\circledcirc, r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ if and only if $z = x = y$.*

If $\|x - y\| > 2r$, every $z \in \mathcal{O}_{\mathcal{K}}$ is $\mho^{v}_{\circledcirc, r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$.

If $\|x - y\| \leqq 2r$, then $z \in \mathcal{O}_{\mathcal{K}}$ is $\mho^{v}_{\circledcirc, r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ if and only if it is of the form

$$z = x + \alpha w + v\,. \tag{4.44}$$

where

$$w \equiv y - x$$

and

$$\begin{cases} 0 \leqq \alpha \leqq 1\,, \\ v \perp w\,, \end{cases} \tag{4.45}$$

with $v = 0$ if $\alpha = 0$ or 1, and with, in the case of $0 < \alpha < 1$, $v \neq 0$ and specifically

$$\left(\frac{v}{\|v\|}, x\right) \geqq \sqrt{r^2 - \frac{1}{4}\|w\|^2} \tag{4.46}$$

and

$$\|v\| = -\left(\frac{v}{\|v\|}, x\right) + \sqrt{\left(\frac{v}{\|v\|}, x\right)^2 + \alpha(1-\alpha)\|w\|^2}\ . \tag{4.47}$$

By virtue of the orthogonality of v and w, v also satisfies the inequality

$$\left(\frac{v}{\|v\|}, x\right) \leqq \sqrt{1 - \frac{1}{4}\|w\|^2}\ . \tag{4.48}$$

Consequently, if $r>1$ there are no points other than x and y that are $\mathcal{V}^{v}_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$. If $r=1$ the points that are $\mathcal{V}^{v}_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ are simply the points of $\mathcal{O}_{\mathcal{K}}$ that are $\mathcal{V}^{v}_{0}$-between x and y, that is, the non-negative linear combinations of x and y in $\mathcal{O}_{\mathcal{K}}$.

We shall now prove the result that was announced in the Introduction; namely, that, for $r<1$, $\mathcal{V}^{v}_{\circledcirc,r}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ is identical with $\mathcal{V}^{v}_{\nu}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ for a certain ν. This result may be established using the theorem we have just proved in conjunction with the characterization of $\mathcal{V}^{v}_{\nu}$-betweenness given by Theorem I.6.2. However, we shall give another proof here, relying on one of the original definitional characterizations of $\mathcal{V}^{v}_{\nu}$-betweenness.

From Definition 4.1 it follows that z is $\mathcal{V}^{v}_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ if and only if

$$\left.\begin{array}{l} \|x-u\| \leqq r\,, \\ \|y-u\| \leqq r\,, \\ \|x\| = \|y\| = \|z\| = 1 \end{array}\right\} \Longrightarrow \|z-u\| \leqq r\,. \tag{4.49}$$

Expanding the norms of the differences here gives us the following implication form equivalent to the form (4.49):

$$\left.\begin{array}{l} (x, u) \geqq \frac{1}{2}[\|u\|^2 + 1 - r^2]\,, \\ (y, u) \geqq \frac{1}{2}[\|u\|^2 + 1 - r^2]\,, \\ \|x\| = \|y\| = \|z\| = 1 \end{array}\right\} \Longrightarrow (z, u) \geqq \frac{1}{2}[\|u\|^2 + 1 - r^2]\,. \tag{4.50}$$

The premise here is self-denying if $u=\theta$, and so (4.50) is equivalent to the implication form which includes the condition $\|u\| \neq 0$ on the left-hand side. If we insert this condition, then set $\|u\| = \delta$ and $u/\|u\| = \tilde{u}$, the implication (4.50) can be equivalently presented as

$$\text{(4.51)}\qquad \left.\begin{array}{l}(x, \tilde{u}) \geqq \dfrac{1}{2\delta}[\delta^2+1-r^2]\,, \\ (y, \tilde{u}) \geqq \dfrac{1}{2\delta}[\delta^2+1-r^2]\,, \\ \|x\|=\|y\|=\|z\|=\|\tilde{u}\|=1\,, \\ \delta>0 \end{array}\right\} \Longrightarrow (z, \tilde{u}) \geqq \frac{1}{2\delta}[\delta^2+1-r^2]\,.$$

Now, the quantity $\frac{1}{2\delta}[\delta^2+1-r^2]$, as a function of $\delta>0$, takes on all values $<\infty$ and greater than or equal to its minimum value of $\sqrt{1-r^2}$. It follows that (4.51) can be replaced by the equivalent statement

$$\text{(4.52)}\qquad \left.\begin{array}{l}(x, \tilde{u}) \geqq \kappa\,, \\ (y, \tilde{u}) \geqq \kappa\,, \\ \|x\|=\|y\|=\|z\|=\|\tilde{u}\|=1\,, \\ \sqrt{1-r^2} \leqq \kappa \leqq 1 \end{array}\right\} \Longrightarrow (z, \tilde{u}) \geqq \kappa\,.$$

According to Section 6 of [1], this translates directly into the statement

(4.53) *z is $\mho^v_\kappa$-between x and y in $\mathcal{O}_{\mathcal{K}}$ for every* $\kappa \in [\sqrt{1-r^2}, 1]$.

But we have the fact that if $\nu_1<\nu_2$ and z is $\mho^v_{\nu_1}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ then z is $\mho^v_{\nu_2}$-between x and y in $\mathcal{O}_{\mathcal{K}}$. (This may be proved using Theorem I.6.2, for example.) Because of this, the statement (4.53) is equivalent to the simpler statement

(4.54) *z is $\mho^v_{\sqrt{1-r^2}}$-between x and y in $\mathcal{O}_{\mathcal{K}}$.*

We have thus derived the equivalence of the statements (4.49) and (4.54). That is, we have shown that z is $\mho^v_{\circledcirc,r}$-between x and y in $\mathcal{O}_{\mathcal{K}}$ if and only if (4.54) holds. This is the result we sought to establish; we state it as

THEOREM 4.3. *Let $r \in (0, 1]$. Then $\mho^v_{\circledcirc,r}$-betweenness in $\mathcal{O}_{\mathcal{K}}$ is identical with $\mho^v_{\sqrt{1-r^2}}$-betweenness in $\mathcal{O}_{\mathcal{K}}$.*

5. Definition and characterization of $\tilde{\mho}^v_{0,(3)}$-betweenness

In this section we are going to induce a betweenness relation on the set in $\mathcal{K}_{(3)}$ ($\equiv$3-dimensional real unitary space) which is an open hemisphere of the unit sphere plus a point of the boundary. The corresponding procedure could be carried out also in higher dimensions, but the calculations are already involved enough in 3-dimesional space and we rest content with the discussion of this case. The betweenness we are going to define may seem quite arbitrarily chosen. But that is most decidedly not the case. The fact is—as will be shown in a subsequent publication—that this apparently overly complicated example of an "off-the-beaten-path" betweenness relation actually very simply induces our familiar $\mathfrak{V}_0$-betweenness in the set of 1-dimensional linear manifolds of a complex 2-space. The demonstration of this fact will be facilitated by the characterization we develop here below.

Let e_1, e_2 and e_3 be three mutually orthogonal unit vectors in $\mathcal{K}_{(3)}$. We define a one-to-one function t on $\mathcal{O}_{1/2\cdot e_3, 1/2}$ (which, in the notation introduced in Section 2, is the sphere (surface) of center $\frac{1}{2}e_3$ and radius $\frac{1}{2}$) to the set $\tilde{\mathcal{O}}_{(3)}=\{x \in \mathcal{O}_{(3)} | (x, e_3)>0\} \cup \{e_1\}$, where $\mathcal{O}_{(3)}$ is the unit sphere (surface) in 3-space. This function t is as follows:

$$t(x)=\begin{cases} \dfrac{x}{\|x\|}, & x\neq\theta, \\ e_1, & x=0. \end{cases} \qquad x \in \mathcal{O}_{1/2\cdot e_3, 1/2}. \tag{5.1}$$

The inverse of this function is

$$t^{-1}(y)=(y, e_3)y, \qquad y \in \tilde{\mathcal{O}}_{(3)}. \tag{5.2}$$

We may now state our definition (availing ourselves of Definition 2.1 above):

DEFINITION 5.1. Let x, y and z be three points in the set $\tilde{\mathcal{O}}_{(3)}$. Then z is said to be $\tilde{\mathfrak{V}}^v_{0,(3)}$-between x and y $t^{-1}(z)$ is $\frac{1}{2}e_3$-centered-$\mathfrak{V}^v_0$-between $t^{-1}(x)$ and $t^{-1}(y)$.

Concerning this new betweenness, we shall establish the following theorem:

THEOREM 5.1. *Let x, y and z be three points in the set $\tilde{\mathcal{O}}_{(3)}$. Then z is $\mathfrak{V}^v_{0,(3)}$-between x and y if and only if the following four conditions are fulfilled:*

i) $2(x, e_3)(z, e_3)(x, z)-(z, e_3)^2 \geqq 2(x, e_3)(y, e_3)(x, y)-(y, e_3)^2$,

ii) $2(y, e_3)(z, e_3)(y, z)-(z, e_3)^2 \geqq 2(x, e_3)(y, e_3)(x, y)-(x, e_3)^2$,

iii) $(x, e_3)(y, e_3)(x, y)+(y, e_3)(z, e_3)(y, z)+(z, e_3)(x, e_3)(z, x)$
$-(x, e_3)^2-(y, e_3)^2-(z, e_3)^2+1 \geqq 0$,

iv) $(z, e_3)(x, e_3)[2(y, e_3)^2-1]\cdot[(x, e_1)(z, e_2)-(z, e_1)(x, e_2)]$
$+(x, e_3)(y, e_3)[2(z, e_3)^2-1]\cdot[(y, e_1)(x, e_2)-(x, e_1)(y, e_2)]$
$+(y, e_3)(z, e_3)[2(x, e_3)^2-1]\cdot[(z, e_1)(y, e_2)-(y, e_1)(z, e_2)]=0$.

PROOF. According to Corollary 2.1.1, z is $\tilde{\mathfrak{O}}^{v}_{0,(3)}$-between x and y if and only if these following four conditions are satisfied:

$$\|t^{-1}(z)-t^{-1}(x)\|\leqq\|t^{-1}(y)-t^{-1}(x)\| , \tag{5.3}$$

$$\|t^{-1}(z)-t^{-1}(y)\|\leqq\|t^{-1}(y)-t^{-1}(x)\| , \tag{5.4}$$

$$\|t^{-1}(z)-t^{-1}(x)\|^2+\|t^{-1}(x)-t^{-1}(y)\|^2+\|t^{-1}(y)-t^{-1}(z)\|^2\leqq 2 , \tag{5.5}$$

(5.6) *the endpoints of the four vectors,* $t^{-1}(x)$, $t^{-1}(y)$, $t^{-1}(z)$ *and* $\frac{1}{2}e_3$, *are coplanar.*

We are now going to show that these conditions, in the order written, are equivalent statements of the four conditions listed in Theorem 5.1.

Consider (5.3). We have:

$$\begin{aligned}\|t^{-1}(y)-t^{-1}(x)\|^2-\|t^{-1}(z)-t^{-1}(x)\|^2 &=\|(y, e_3)y-(x, e_3)x\|^2-\|(z, e_3)z-(x, e_3)x\|^2\\ &=[2(x, e_3)(z, e_3)(x, z)-(z, e_3)^2]-[2(x, e_3)(y, e_3)(x, y)-(y, e_3)^2] ,\end{aligned} \tag{5.7}$$

and we see immediately that i) of the theorem and (5.3) are equivalent. The equivalence of ii) and (5.4) is similarly shown. And the same procedure of developing the terms in the left-hand side of (5.5) shows that this inequality is an equivalent statement to iii).

To demonstrate the last equivalence let us note that (5.6) is equivalent to the statement

(5.8) *the three vectors* $t^{-1}(z)-t^{-1}(x)$, $t^{-1}(z)-t^{-1}(y)$ *and* $t^{-1}(z)-\frac{1}{2}e_3$ *are coplanar*.

This, in turn, we know to be equivalent to the vanishing of a determinant, namely, the determinant of the components of these three vectors with respect to the three vectors in an orthonormal basis. In the case of our present basis $\{e_1, e_2, e_3\}$, the determinantal condition is

$$\begin{bmatrix} (z, e_3)(z, e_1)-(x, e_3)(x, e_1) & (z, e_3)(z, e_2)-(x, e_3)(x, e_2) & (z, e_3)^2-(x, e_3)^2 \\ (z, e_3)(z, e_1)-(y, e_3)(y, e_1) & (z, e_3)(z, e_2)-(y, e_3)(y, e_2) & (z, e_3)^2-(y, e_3)^2 \\ (z, e_3)(z, e_1) & (z, e_3)(z, e_2) & (z, e_3)^2-\frac{1}{2} \end{bmatrix} =0 . \tag{5.9}$$

And now if the determinant is expanded, cancellations made and terms gathered, we find that (5.9) is identical with iv) of the theorem. This completes the proof of the theorem.

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