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VARIANCE FUNCTIONS FOR m-GROUPED CYLINDRICALLY ROTATABLE DESIGNS OF TYPE 3

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Summary

Formulae for variance of difference between two estimated responses are derived for m-grouped first-order and 2-grouped second- and third-order cylindrically rotatable designs of type 3.

1. Introduction

Work of Box and Hunter [2], Herzberg [4], [5], Das and Dey [3], Box and Draper [1] is extended by deriving formulae for variance of difference between two estimated responses when design is m-grouped cylindrically rotatable of type 3.

Consider k-factor model of degree d with second-order distributional assumptions. Let $\hat{y}(\boldsymbol{x})$ be estimated response at point $\boldsymbol{x}$ and $V=N\sigma^{-2}\cdot\mathrm{Var}\{\hat{y}(\boldsymbol{x})-\hat{y}(\boldsymbol{z})\}$. Let $k=k_1+\cdots+k_m$, $I_i=\{1+J(i-1),\cdots,J(i)\}$ where $J(0)=0$ and $J(i)=k_1+\cdots+k_i$ $(i=1,\cdots,m)$. Consider factors partitioned into m groups with k_i factors x_j $(j\in I_i)$ in ith group. A design is m-grouped cylindrically rotatable of type 3 if $\mathrm{var}\{y(\boldsymbol{x})\}$ is a function of only $\rho_{xi}^2=\sum x_j^2$ $(i=1,\cdots,m)$ and for such a design V depends only on ρ_{xi}^2, ρ_{zi}^2 and $\theta_i=\cos^{-1}\sum x_jz_j/\rho_{xi}\rho_{zi}$ where summations are over $j\in I_i$. Let $r_{ij}^2=\rho_{xj}^{2i}+\rho_{zj}^{2i}-2\rho_{xj}^i p_{zj}^i\cos^i\theta_j$. Let λ_{2i}, λ_{4i}, λ_{4ip}, λ_{6i}, λ_{6ip}, λ_{6ipq} etc. denote nonzero moments of the designs. For e.g. λ_{6i}, λ_{6iip}, λ_{6ipq} are sixth moments involving all factors from 1 group, 2 groups and 3 groups, respectively (see Huda [6] for details).

2. Results

For a first-order m-grouped cylindrically rotatable design of type 3 the standardized variance V takes the following form

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$$V_1=\sum_{i=1}^{m} r_{1i}^2/\lambda_{2i} . \tag{1}$$

For higher order designs only the case $m=2$ is considered. Algebra involved for $m\geqq 3$ is rather complicated. For a second-order 2-grouped cylindrically rotatable design of type 3 the function V has the form

$$V_2=V_1+\sum_{i=1}^{2}\{r_{2i}^2/(2\lambda_{4i})+a_{4i}(\rho_{xi}^2-\rho_{zi}^2)^2\}+2a_{412}\prod_{i=1}^{2}(\rho_{xi}^2-\rho_{zi}^2) + \left(\prod_{i=1}^{2}\rho_{xi}^2+\prod_{i=1}^{2}\rho_{zi}^2-2\prod_{i=1}^{2}\rho_{xi}\rho_{zi}\cos\theta_i\right)/\lambda_{412} , \tag{2}$$

where

$$a_{41}=-B\{(k_2+2)\lambda_{41}\lambda_{42}+2k_2\lambda_{21}\lambda_{22}\lambda_{412}-(k_2+2)\lambda_{21}^2\lambda_{42}-k_2\lambda_2^2\lambda_{41}-k_2\lambda_{412}^2\}/(2\lambda_{41}) ,$$

$$a_{412}=B(\lambda_{21}\lambda_{22}-\lambda_{412}) ,$$

$$B=\left[\prod_{i=1}^{2}\{(k_i+2)\lambda_{4i}-k_i\lambda_{2i}^2\}-k_1k_2(\lambda_{412}-\lambda_{21}\lambda_{22})^2\right]^{-1}$$

and a_{42} may be obtained from a_{41} by symmetry.

Similarly, for a third-order 2-grouped cylindrically rotatable design of type 3 the variance function V has the form

$$\begin{aligned} V_3=V_2+\sum_{i=1}^{2}[&r_{3i}^2/(6\lambda_{6i})+(b_{0i}-\lambda_{2i}^{-1})r_{1i}^2+b_{1i}\{r_{1i}^4+2r_{2i}^2+(\rho_{zi}^2-\rho_{xi}^2)^2\}/2 \\ &+g_i\{\rho_{xi}^2\rho_{zi}^2r_{1i}^2+\prod_{j=2,4}(\rho_{xi}^j-\rho_{zi}^j)\}]+\sum_{i\neq j=1}^{2}[2c_i\{\rho_{xi}^2\rho_{xj}^2+\rho_{zi}^2\rho_{zj}^2 \\ &-\rho_{xi}\rho_{zi}(\rho_{xj}^2+\rho_{zj}^2)\cos\theta_i\}+2d_i\{\rho_{xi}^4\rho_{xj}^2+\rho_{zi}^4\rho_{zj}^2-\rho_{xi}\rho_{zi}(\rho_{xi}^2\rho_{zj}^2 \\ &+\rho_{zi}^2\rho_{xj}^2)\cos\theta_i\}+h_i(\rho_{xi}^2\rho_{xj}^4+\rho_{zi}^2\rho_{zj}^4-2\rho_{xi}\rho_{zi}\rho_{xj}^2\rho_{zj}^2\cos\theta_i) \\ &+(\rho_{xi}^4\rho_{xj}^2+\rho_{zi}^4\rho_{zj}^2-2\rho_{xi}^2\rho_{zi}^2\rho_{xj}\rho_{zj}\cos^2\theta_i\cos\theta_j)(2\lambda_{612i})^{-1}] , \end{aligned} \tag{3}$$

where

$$b_{01}=D_1\{(k_1+4)(k_2+2)\lambda_{61}\lambda_{6122}-k_2(k_1+2)\lambda_{6112}^2\} ,$$

$$b_{11}=-D_1\{(k_2+2)\lambda_{41}\lambda_{6122}-k_2\lambda_{412}\lambda_{6112}\} ,$$

$$c_1=-D_1\{(k_1+4)\lambda_{412}\lambda_{61}-(k_1+2)\lambda_{41}\lambda_{6112}\} ,$$

$$d_1=-D_1(\lambda_{21}\lambda_{6112}-\lambda_{41}\lambda_{412}) ,$$

$$\begin{aligned} g_1=-D_1(2\lambda_{61})^{-1}\{&(k_2+2)(\lambda_{21}\lambda_{61}\lambda_{6122}-\lambda_{41}^2\lambda_{6122})-k_2(\lambda_{21}\lambda_{6112}^2+\lambda_{412}^2\lambda_{61} \\ &-2\lambda_{41}\lambda_{412}\lambda_{6112})\} , \end{aligned}$$

$$\begin{aligned} h_1=-D_1(2k_2\lambda_{6122})^{-1}\{&2(k_1+2)\lambda_{41}^2\lambda_{6122}+k_2(k_1+4)\lambda_{21}\lambda_{61}\lambda_{6122} \\ &+2k_2(k_1+2)\lambda_{41}\lambda_{412}\lambda_{6112}-k_2(k_1+2)\lambda_{21}\lambda_{6112}^2 \\ &-(k_1+2)(k_2+2)\lambda_{41}^2\lambda_{6122}-k_2(k_1+4)\lambda_{412}^2\lambda_{61}\} , \end{aligned}$$

$$D_1=\{(k_1+4)(k_2+2)\lambda_{21}\lambda_{61}\lambda_{6122}+2k_2(k_1+2)\lambda_{41}\lambda_{412}\lambda_{6112}-k_2(k_1+2)\lambda_{21}\lambda_{6112}^2 -(k_1+2)(k_2+2)\lambda_{41}^2\lambda_{6122}-k_2(k_1+4)\lambda_{412}^2\lambda_{61}\}^{-1},$$

and b_{02}, b_{12}, c_2, d_2, g_2, h_2, D_2 may be obtained by symmetry.

3. Comments

Even in response surface designs difference between estimated responses at two points may be of greater interest than response at individual points. Hence results on variance of difference are presented here.

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