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EXISTENCE THEOREMS OF A MAXIMUM LIKELIHOOD ESTIMATE FROM A GENERALIZED CENSORED DATA SAMPLE

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Summary

A new type of random sample, called a generalized censored data sample, is defined. An approach to finding criteria for the existence of a maximum likelihood estimate from a finite generalized censored data sample is presented. This approach, named the probability contents boundary analysis, gives systematically a number of practical criteria, each of which is effective for various kinds of typical distribution families in statistical analysis.

1. Introduction

Let T' and T be given constants with $-\infty \leqq T' < T \leqq \infty$ and X be a random variable with values in $[T', T]$. Suppose that the distribution of X belongs to a family $\mathcal{P} = \{P_\theta;\ \theta \in \Theta\}$ of probability measures on $[T', T]$ such that $P_\theta(\{T'\}) = 0$ and $0 \leqq P_\theta(\{T\}) < 1$ for all $\theta \in \Theta$, where the parameter space Θ is an arbitrary nonempty set. Let $(X_1, \cdots, X_q)$ be a random sample from the distribution of X. In this paper we shall consider the situation where information available for X_i is only that its value lies in a subinterval C_i of $[T', T]$, whose extreme points are random variables with values in $[T', T]$, and where C_i, $1 \leqq i \leqq q$, has nonempty interior whenever $C_i \neq \{T\}$. The collection $C = \{C_1, \cdots, C_q\}$ is called a generalized censored (g.c.) data sample of size (q', q) (with respect to the family $\mathcal{P}$), where q'=number of C_k, $1 \leqq k \leqq q$, such that $T \in C_k$ and $P_\theta(C_k) \equiv P_\theta(\{T\})$ on Θ. For example, if $T'=0$ and $T=\infty$ and if X_i is the survival time of an individual i after an operation, then $C_i = (t, \infty]$ means that he is reported as alive at least time t, $C_i = \{\infty\}$ means that he is judged completely cured by his doctor and $P_\theta(\{\infty\})$ is the probability of cure (cf. Boag [1]). A grouped data sample, discussed at the end of Section 4, and a binary response data sample, dis-

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cussed in Section 6 are g.c. data samples of size $(0, q)$.

Maximum likelihood estimation from a g.c. data sample C of size (q', q) is the following maximizing problem:

$$\text{(Problem I)} \qquad \text{Find} \quad \sup\left\{c\prod_{i=1}^{q} P_\theta(C_i);\ \theta \in \Theta\right\},$$

where c is a positive constant independent of θ (see Moran [8] and Nabeya [9] for the validity for solving this problem). An optimal solution of Problem I is called a maximum likelihood estimate (MLE) of the unknown true parameter of the distribution of X.

The purpose of this paper is to present a new approach, named the probability contents boundary (p.c.b.) analysis, to finding criteria for the existence of an MLE. The p.c.b. analysis is to analyze the inner boundary of the set of probability contents of intervals, and is a unified and very general approach in comparison with the usual method of utilizing likelihood equations.

Practical criteria for the existence of an MLE from a grouped data sample (see (4.4) for its definition) are given in Kulldorff [7] where $\mathcal{P}$ is the family of one-parameter exponential distributions, or the family of normal distributions with location or scale parameter, or the family of one-parameter or two-parameter truncated exponential distributions, and in Carter et al. [3] where $\mathcal{P}$ is the family of Poisson distributions. They obtained these criteria by solving likelihood equations.

Practical criteria for the existence of an MLE from a g.c. data sample of size $(0, q)$ are given in Kariya and Nakamura [4] where $\mathcal{P}$ is the family of two-parameter normal distributions, and in Kariya [5] where $\mathcal{P}$ belongs to a class of families of one-parameter distributions on $\mathcal{R}$, the set of real numbers, satisfying some restrictions. These criteria are based on a method developed in Nakamura and Kariya [10]. The p.c.b. analysis proposed in this paper gives systematically criteria for the existence of an MLE which are effective for a good many families of multiparameter distributions as well as the criteria in [3]–[5] and [7].

In the next section we give a general existence theorem of an MLE, which is fundamental for the p.c.b. analysis. In Section 3 structure of a set, called the probability contents inner boundary, is analyzed. Sections 4–6 apply our approach to more specific structure of the probability contents inner boundary. Examples 4.1–4.7, 5.1–5.6 and 6.1–6.3 in these sections cover many typical families of distributions which often appear in statistical analysis.

2. A general existence theorem of an MLE

We begin with rewriting the likelihood $\prod_{i=1}^{q} P_\theta(C_i)$ of the g.c. data sample $C=\{C_1,\cdots,C_q\}$ of size (q',q). It is assumed throughout this paper that $0\leqq q'<q$ and that:

(2.1) For each C_k with $C_k\neq\{T\}$, there exist two points x_{k1} and x_{k2} of $[T',T]$ such that $P_\theta(C_k\cap(T',T))\equiv P_\theta([x_{k1},x_{k2}))$ on Θ.

(2.2) There exists no C_k such that $T\notin C_k$ or $P_\theta(C_k)\not\equiv P_\theta(\{T\})$ on Θ and that $P_\theta(C_k)\equiv 0$ or 1 on Θ.

Put $H(\theta)=P_\theta(\{T\})$ and $F(x,\theta)=P_\theta([T',x))/(1-H(\theta))$. The assumption (2.1) enables us to express $P_\theta(C_k)$, $1\leqq k\leqq q$, in terms of $H(\theta)$, $F(x_{k1},\theta)$ and $F(x_{k2},\theta)$ even if $F(x,\theta)$ is not continuous at $x=x_{kj}$, $j=1,2$. The two points x and x' of $[T',T]$ are said to be equivalent (with respect to the family $\mathcal{P}$) if $F(x,\theta)\equiv F(x',\theta)$ on Θ. Because of (2.1), there exist a nonnegative integer m and a set $\{x_i\}$ of $(m+2)$ points of $[T',T]$ with the following properties:

(i) $x_0=T'<\cdots<x_{m+1}=T$.
(ii) The points x_i and x_j are not equivalent if $i\neq j$.
(iii) Every extreme point of $[x_{k1},x_{k2})$ is equivalent to some x_i.
(iv) Each point x_i, $1\leqq i\leqq m$, is equivalent to an extreme point of some $[x_{k1},x_{k2})$.

In case $m=0$, our observation is restricted to two possible types (e.g. an individual is not cured, or is cured). In the general case let $h_1(k)$ (resp. $h_2(k)$) denote the integer i such that x_i is equivalent to x_{k1} (resp. x_{k2}). Note that for each $\theta\in\Theta$ and for each C_k with $C_k\neq\{T\}$,

$$P_\theta(C_k)=(1-H(\theta))(F(x_{h_2(k)},\theta)-F(x_{h_1(k)},\theta)) \qquad \text{if } T\notin C_k,$$

$$P_\theta(C_k)=(1-H(\theta))(1-F(x_{h_1(k)},\theta))+H(\theta) \qquad \text{if } T\in C_k.$$

Because of (2.2), there is no C_k such that $T\notin C_k$ and $h_1(k)=h_2(k)$ or that $T\in C_k$, $P_\theta(C_k)\not\equiv H(\theta)$ on Θ and $(h_1(k),h_2(k))=(0,m+1)$.

In order to express the likelihood in terms of $F(x_1,\theta),\cdots,F(x_m,\theta)$ and $H(\theta)$, define nonnegative integers q_{ij} $(0\leqq i<j\leqq m+1)$ and q_i $(1\leqq i\leqq m)$ by

q_{ij}=number of C_k, $1\leqq k\leqq q$, such that $T\notin C_k$ and $(h_1(k),h_2(k))=(i,j)$,
q_i=number of C_k, $1\leqq k\leqq q$, such that $T\in C_k$, $P_\theta(C_k)\not\equiv H(\theta)$ on Θ and $(h_1(k),h_2(k))=(i,m+1)$.

Then the likelihood can be rewritten as

$$\prod_{i=1}^{q}P_\theta(C_i)=(1-H(\theta))^{q\cdot\cdot}H(\theta)^{q'}\prod_{0\leqq i<j\leqq m+1}[F(x_j,\theta)-F(x_i,\theta)]^{q_{ij}}$$
$$\cdot\prod_{i=1}^{m}[1-F(x_i,\theta)+F(x_i,\theta)H(\theta)]^{q_i},$$

where $q..=\sum_{0\leqq i<j\leqq m+1}q_{ij}$. Note that $q=q..+q.+q'$ with $q.=\sum_{i=1}^{m}q_i$. This

function or the log-likelihood $\log \prod_{i=1}^{q} P_\theta(C_i)$ is determined by "probability contents" of the intervals $[x_i, x_j)$, the intervals $[x_i, T]$ and the set $\{T\}$. Let $\bar{\mathcal{R}}$ (resp. $\bar{\mathcal{S}}$) denote the set of extended real numbers (resp. the closure of a subset $\mathcal{S}$ of $\mathcal{R}^r$, Euclidean r-space), and put $\mathcal{Z}=\{(z_1,\cdots, z_{m+1})\in\mathcal{R}^{m+1};\ 0<z_1<\cdots<z_m<1 \text{ and } 0<z_{m+1}<1\}$. We define the function $L:\mathcal{Z}\to\bar{\mathcal{R}}$ and the mapping $\boldsymbol{Z}:\Theta\to\bar{\mathcal{Z}}$ by

$$(2.3)\quad L(\boldsymbol{z})=\sum_{0\leq i<j\leq m} q_{ij}\log(z_j-z_i)+\sum_{i=1}^{m} q_{i\,m+1}\log(1-z_i) + \sum_{i=1}^{m} q_i\log(1-z_i+z_iz_{m+1})+q_{..}\log(1-z_{m+1})+q'\log z_{m+1}\ ,$$

$$\boldsymbol{Z}(\theta)=\begin{cases} H(\theta) & \text{in case } m=0\,,\\ (F(x_1,\theta),\cdots,F(x_m,\theta),H(\theta)) & \text{in case } m\geqq 1\,,\end{cases}$$

where $z_0=0$ and the sum over the empty set ϕ is equal to 0. Note that $\mathcal{Z}\subset[0,1]^{m+1}$, the $(m+1)$-fold cartesian product of $[0,1]$. To relate the function $L(\boldsymbol{z})$ to the log-likelihood, we regard $\bar{\mathcal{R}}$ as a metric space with the distance:

$$\operatorname{dist}(t,t')=|\arctan t-\arctan t'|\,,\qquad t,t'\in\bar{\mathcal{R}}\,,$$

where $\arctan(-\infty)=-\pi/2$ and $\arctan(\infty)=\pi/2$. Since $\bar{\mathcal{R}}$ is compact, $L(\boldsymbol{z})$ can be extended to a continuous function $\tilde{L}(\boldsymbol{z})$ on $\bar{\mathcal{Z}}$ to $\bar{\mathcal{R}}$ by $\tilde{L}(\boldsymbol{z})=\lim_n L(\boldsymbol{z}_n)$, where the symbol "$\lim_n$" denotes "$\lim_{n\to\infty}$" and $\{\boldsymbol{z}_n\}$ is a sequence in $\mathcal{Z}$ with its limit $\boldsymbol{z}$ (cf. Bourbaki [2], Chapter 1). In computing these functions, the following rules are used: $\log 0=-\infty$, $0\cdot\log 0=0$, $(-\infty)+(-\infty)=-\infty$ and $t\cdot(-\infty)=-\infty$ for all $t>0$. The log-likelihood is equal to the function $\tilde{L}(\boldsymbol{Z}(\theta))$ up to a constant. Now, Problem I is reformulated as follows:

(Problem II) Find $M=\sup\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in\boldsymbol{Z}(\Theta)\}$.

An optimal solution of Problem II is also called an MLE. Problems I and II are equivalent in the sense that the existence of an MLE of Problem I implies the existence of an MLE of Problem II, and vice versa.

In relation to Problem II, we consider the following two maximizing problems with the objective function $\tilde{L}(\boldsymbol{z})$:

(2.4) Find $M_c=\sup\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in\overline{\boldsymbol{Z}(\Theta)}\}$.

(2.5) Find $M_b=\sup\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in\partial\boldsymbol{Z}(\Theta)\}$,
where $\partial\boldsymbol{Z}(\Theta)=\overline{\boldsymbol{Z}(\Theta)}-\boldsymbol{Z}(\Theta)$.

Supremum (resp. infimum) of a function over the empty set is defined to be $-\infty$ (resp. ∞), and for subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{R}^r$, $\mathcal{S}_1-\mathcal{S}_2$ denotes the difference between $\mathcal{S}_1$ and $\mathcal{S}_2$. The set $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$ will be called the probability contents inner boundary (p.c.i.b.) of the g.c. data sample C for the family $\mathcal{P}$. Note that the p.c.i.b. $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$ is not necessarily the boundary of $\boldsymbol{Z}(\boldsymbol{\Theta})$.

The values M, M_c and M_b satisfy the inequalities:

LEMMA 2.1. $M_b \leqq M = M_c$.

PROOF. Because of the relation $\max(M, M_b) \leqq M_c$, it suffices to prove that $M_c \leqq M$ in case $M_c > -\infty$. Let $t < M_c$ and let $\hat{z}$ be an optimal solution of Problem (2.4). Since $\tilde{L}(z)$ is continuous on $\overline{\boldsymbol{Z}(\boldsymbol{\Theta})}$, there exists a neighborhood $\mathcal{S}$ of $\hat{z}$ such that $\tilde{L}(z) > t$ for all $z \in \mathcal{S}$. Since $\boldsymbol{Z}(\boldsymbol{\Theta}) \cap \mathcal{S} \neq \phi$, $t < M$. By letting $t \to M_c$, we have $M_c \leqq M$.

Now we show a general existence theorem of an MLE of Problem II, which is equivalent to Problem I.

THEOREM 2.1. *An MLE of Problem* II *exists if and only if there exists* $z \in \boldsymbol{Z}(\boldsymbol{\Theta})$ *such that* $\tilde{L}(z) \geqq M_b$.

PROOF. Lemma 2.1 proves the "only if" part of the theorem. Let z be that of the condition. If $M = \tilde{L}(z)$, then z is an MLE of Problem (2.4). Assume that $\tilde{L}(z) < M$ and let $\hat{z}$ be the same as in the proof of Lemma 2.1. By Lemma 2.1, $\tilde{L}(\hat{z}) = M_c = M$. From this and $M_b \leqq \tilde{L}(z) < M$, the inequality $M_b < \tilde{L}(\hat{z})$ follows. Hence $\hat{z} \in \boldsymbol{Z}(\boldsymbol{\Theta})$ and $\hat{z}$ is an MLE of Problem II.

COROLLARY 2.1. *If* $M_b = -\infty$, *then an MLE of Problem* I *exists.*

If $\boldsymbol{Z}(\boldsymbol{\Theta})$ is closed, then $M_b = -\infty$. The converse statement is not always true.

3. Analysis of the structure of $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$

A criterion for the existence of an MLE is obtained by seeking a sufficient condition for Theorem 2.1. For this reason, it is important to find the value of M_b determined by $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$. The p.c.i.b. $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$ depends on the family $\mathcal{P}$. We can analyze, however, the structure of $\partial\boldsymbol{Z}(\boldsymbol{\Theta})$ under fairly general assumptions on $\mathcal{P}$. Now we assume that the parameter space $\boldsymbol{\Theta}$ is a nonempty subset of the product of a Hausdorff space $\boldsymbol{\Theta}'$ and the interval $[0, 1)$, and that $F(x, \theta)$ ($\theta = (\theta', \eta) \in \boldsymbol{\Theta}$) is independent of η and $H(\theta) = \eta$ for all $\theta = (\theta', \eta) \in \boldsymbol{\Theta}$. We intentionally mix up $F(x, \theta)$ and $F(x, \theta')$ to avoid an additional notation. Put $\boldsymbol{F}(\theta') =(F(x_1, \theta'), \cdots, F(x_m, \theta'))$ and $\partial \boldsymbol{F}(\Theta')=\overline{\boldsymbol{F}(\Theta')}-\boldsymbol{F}(\Theta')$. In order to state above assumptions and results, define the point-to-set mappings $\mathcal{H}(\theta')$ ($\theta' \in \Theta'$) and $\boldsymbol{h}(z')$ ($z' \in \overline{\boldsymbol{F}(\Theta')}$) by

$$\mathcal{H}(\theta')=\{\eta \in [0, 1);\ (\theta', \eta) \in \Theta\}\ ,$$

$$\boldsymbol{h}(z')=\{\eta \in [0, 1];\ (z', \eta) \in \overline{\boldsymbol{Z}(\Theta)}\}\ .$$

The former depends only on $\mathcal{P}$ and the latter on $\mathcal{P}$ and C. We say that $\mathcal{H}(\theta')$ is upper semicontinuous on Θ' if, for every $\theta'_0 \in \Theta'$ and for every open set $\mathcal{O}$ in $\mathcal{R}$ containing $\mathcal{H}(\theta'_0)$, there exists a neighborhood $\mathcal{V}$ of θ'_0 such that $\mathcal{H}(\theta')\subset\mathcal{O}$ for all $\theta' \in \mathcal{V}$. The upper semicontinuity of $\mathcal{H}(\theta')$ implies that, for every sequence $\{\theta_n=(\theta'_n, \eta_n)\}$ in Θ such that $\lim_n \theta'_n=\theta'_0 \in \Theta'$ and $\lim_n \eta_n=\eta$, $\eta \in \overline{\mathcal{H}(\theta'_0)}$.

In order to study the structure of $\partial \boldsymbol{Z}(\Theta)$, define

$$\mathcal{B}=\cup\{\{z'\}\times \boldsymbol{h}(z');\ z' \in \partial \boldsymbol{F}(\Theta')\}\ .$$

We use the convention that the union over the empty set is the empty set. It should be noted that $\mathcal{B}\subset\partial \boldsymbol{Z}(\Theta)$.

THEOREM 3.1. *Let the following condition on $\mathcal{P}$ be satisfied:*

(H.1) *$\overline{\mathcal{H}(\theta')}$ ($\theta' \in \Theta'$) is independent of θ' and $\overline{\mathcal{H}(\theta')}\cap[0, 1)=\mathcal{H}(\theta')$ for all $\theta' \in \Theta'$.*

Then $\partial \boldsymbol{Z}(\Theta)\subset\{1\}$ in case $m=0$ and $\partial \boldsymbol{Z}(\Theta)\subset\mathcal{B}\cup(\boldsymbol{F}(\Theta')\times\{1\})$ in case $m\geqq1$.

PROOF. Let $m\geqq1$ and $\{\theta_n=(\theta'_n, \eta_n)\}$ be a sequence in Θ such that the sequence $\{\boldsymbol{Z}(\theta_n)\}$ converges to $z=(z_1, \cdots, z_{m+1}) \in \partial \boldsymbol{Z}(\Theta)$, i.e., $\lim_n \boldsymbol{F}(\theta'_n) =(z_1, \cdots, z_m)=z'$ and $\lim_n \eta_n=z_{m+1}$. If $z' \in \partial \boldsymbol{F}(\Theta')$, then $z \in \mathcal{B}$. Consider the case $z' \in \boldsymbol{F}(\Theta')$ and assume $z_{m+1}<1$. Choose $\theta' \in \Theta'$ such that $\boldsymbol{F}(\theta') =z'$. It follows from (H.1) that $z_{m+1} \in \overline{\mathcal{H}(\theta')}\cap[0, 1)=\mathcal{H}(\theta')$. Hence $z \in \boldsymbol{Z}(\Theta)$. This contradicts $z \in \partial \boldsymbol{Z}(\Theta)$. Thus $z_{m+1}=1$ in case $z' \in \boldsymbol{F}(\Theta')$. Similarly we can prove our assertion in the case $m=0$.

Hereafter, unless otherwise stated, we assume $m\geqq1$. In order to discuss a weaker condition than (H.1), put

$$F_x^{-1}([u, u'])=\{\theta' \in \Theta';\ u\leqq F(x, \theta')\leqq u'\}$$

for each $x \in \mathcal{R}$ and for each pair (u, u') with $0\leqq u\leqq u'\leqq1$, and let $\mathcal{I}(p)$ denote the set of all p-tuples $(i_1, \cdots, i_p)$ of integers with $1\leqq i_1<\cdots< i_p\leqq m$, where p is an integer with $1\leqq p\leqq m$.

THEOREM 3.2. *The relation $\partial \boldsymbol{Z}(\Theta)\subset\mathcal{B}\cup(\boldsymbol{F}(\Theta')\times\{1\})$ holds if $\boldsymbol{F}(\theta')$ is continuous on Θ' and if there exists a positive integer $p\,(\leqq m)$ such that the following three conditions are satisfied:*

$(\mathrm{F.1})_p$ *For every set of pairs* (u_j, u_j'), $1 \leqq j \leqq p$, *with* $0 < u_j \leqq u_j' < u_{j+1} < 1$ *and* $(i_1, \cdots, i_p) \in \mathcal{I}(p)$, *the set* $\bigcap_{j=1}^{p} F_{x_{i_j}}^{-1}([u_j, u_j'])$ *is compact.*

$(\mathrm{F.2})_p$ *For each* $\theta' \in \Theta'$, *there exists* $(i_1, \cdots, i_p) \in \mathcal{I}(p)$ *such that* $0 < F(x_{i_1}, \theta') < \cdots < F(x_{i_p}, \theta') < 1$.

(H.2) $\mathcal{H}(\theta')$ *is upper semicontinuous on* Θ' *and* $\overline{\mathcal{H}(\theta')} \cap [0, 1) = \mathcal{H}(\theta')$ *for all* $\theta' \in \Theta'$.

PROOF. Let $\{\theta_n = (\theta_n', \eta_n)\}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ be the same as in the proof of Theorem 3.1. If $\boldsymbol{z}' \in \partial \boldsymbol{F}(\Theta')$, then $\boldsymbol{z} \in \mathcal{B}$. We show $z_{m+1} = 1$ in case $\boldsymbol{z}' \in \boldsymbol{F}(\Theta')$. By $(\mathrm{F.2})_p$, we can find $(i_1, \cdots, i_p) \in \mathcal{I}(p)$ such that $0 < z_{i_1} < \cdots < z_{i_p} < 1$. Take $\delta > 0$ so that $2\delta < \min(z_{i_1}, 1 - z_{i_p}, \min_{2 \leqq k \leqq p}(z_{i_k} - z_{i_{k-1}}))$. Then $\theta_n' \in \bigcap_{k=1}^{p} F_{x_{i_k}}^{-1}([z_{i_k} - \delta, z_{i_k} + \delta])$ for infinitely many n, since $\lim_n \boldsymbol{F}(\theta_n') = \boldsymbol{z}'$. Because of $(\mathrm{F.1})_p$, the sequence $\{\theta_n'\}$ has a cluster point θ_0' in Θ'. Without loss of generality, we may assume that the sequence $\{\theta_n'\}$ converges to θ_0'. We have $\boldsymbol{F}(\theta_0') = \boldsymbol{z}'$ by the continuity of $\boldsymbol{F}$. It follows from the upper semicontinuity of $\mathcal{H}(\theta')$ that $z_{m+1} \in \overline{\mathcal{H}(\theta_0')}$. The rest of the proof can be carried out by the same argument as in the proof of Theorem 3.1.

Remark 3.1. (i) The condition $(\mathrm{F.1})_p$ (resp. $(\mathrm{F.2})_{p+1}$) implies the condition $(\mathrm{F.1})_{p+1}$ (resp. $(\mathrm{F.2})_p$). The converse is not always true.
(ii) It can be seen that the condition (H.1) implies the upper semicontinuity of $\mathcal{H}(\theta')$. Hence the condition (H.1) implies the condition (H.2).

Suppose that $\partial \boldsymbol{Z}(\Theta) \subset \mathcal{B} \cup (\boldsymbol{F}(\Theta') \times \{1\})$ and $q_{..} \neq 0$. Then $M_b = \sup \{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z} \in \mathcal{B}\}$. The set $\mathcal{B}$ is determined if the structure of $\partial \boldsymbol{F}(\Theta')$ is found. We shall be concerned with the set $\partial \boldsymbol{F}(\Theta')$ to find the value of M_b. As we shall see in Sections 4 and 5, $\partial \boldsymbol{F}(\Theta')$ can be easily determined for many families. In general, however, it can not be determined so easily, since it depends on $\mathcal{P}$ and C. To proceed evaluation of M_b, we prepare a specification of the structure of $\partial \boldsymbol{F}(\Theta')$.

THEOREM 3.3. *Let the following condition be satisfied for a positive integer* $p\ (\leqq m)$:

$(\mathrm{F.1})_p^*$ *For any set of pairs* (u_j, u_j'), $1 \leqq j \leqq p$, *with* $0 < u_j \leqq u_j' < u_{j+1} < 1$ *and* $(i_1, \cdots, i_p) \in \mathcal{I}(p)$, $\boldsymbol{F}\left[\overline{\bigcap_{j=1}^{p} F_{x_{i_j}}^{-1}([u_j, u_j'])}\right] \subset \boldsymbol{F}(\Theta')$.

Then:

(3.1) *For any* $(z_1, \cdots, z_m) \in \partial \boldsymbol{F}(\Theta')$ *the number of distinct* z_j*'s* $(j = 1, \cdots, m)$ *values such that* $0 < z_j < 1$ *is at most equal to* $p - 1$.

PROOF. Let $z'=(z_1,\cdots,z_m)\in\partial\boldsymbol{F}(\Theta')$. Then $0\leqq z_1\leqq\cdots\leqq z_m\leqq 1$. We prove the following fact:

(3.2) There is no $(j_1,\cdots,j_p)\in\mathcal{J}(p)$ such that $0<z_{j_1}<\cdots<z_{j_p}<1$.

Suppose that there is $(j_1,\cdots,j_p)\in\mathcal{J}(p)$ such that $0<z_{j_1}<\cdots<z_{j_p}<1$. Since $z'\in\partial\boldsymbol{F}(\Theta')$, there is a sequence $\{\theta'_n\}$ in Θ' such that $\lim_n \boldsymbol{F}(\theta'_n)=z'$. Take $\delta>0$ so that $2\delta<\min(z_{j_1}, 1-z_{j_p}, \min_{2\leqq k\leqq p}(z_{j_k}-z_{j_{k-1}}))$. Then $\theta'_n\in\bigcap_{k=1}^{p} F_{x_{j_k}}^{-1}\cdot[z_{j_k}-\delta, z_{j_k}+\delta])$ for infinitely many n and hence $z'\in\boldsymbol{F}(\Theta')$ by $(\mathrm{F}.1)_p^*$. This contradicts $z'\in\partial\boldsymbol{F}(\Theta')$. It is easily verified that (3.2) implies (3.1). This completes the proof.

Remark 3.2. (i) If $\boldsymbol{F}(\theta')$ is continuous on Θ', then the condition $(\mathrm{F}.1)_p$ implies the condition $(\mathrm{F}.1)_p^*$.
(ii) Theorems 3.1 and 3.3 remain valid if we assume that Θ' is an arbitrary nonempty set.

We shall give a sufficient condition for which $M_b=-\infty$ in terms of the q_{ij}'s in Section 2. We say that the pair $(\mathcal{P}, C)$ is p-regular if the statement (3.1) and the relation $\partial\boldsymbol{Z}(\Theta)\subset\mathcal{B}\cup(\boldsymbol{F}(\Theta')\times\{1\})$ hold. For each $z'=(z_1,\cdots,z_m)\in\mathcal{R}^m$, define

$$N(z')=\sum_{1\leqq i<j\leqq m; z_i=z_j} q_{ij}+\sum_{1\leqq j\leqq m; z_j=0} q_{0j}+\sum_{1\leqq i\leqq m; z_i=1} q_{i\,m+1}\,.$$

THEOREM 3.4. *Assume that* $q_{..}\neq 0$ *and the pair* $(\mathcal{P}, C)$ *is* p*-regular for a positive integer* $p\,(\leqq m)$. *If* $\inf\{N(z');\ z'\in\partial\boldsymbol{F}(\Theta')\}>0$, *then* $M_b=-\infty$.

PROOF. Let $z=(z_1,\cdots,z_{m+1})\in\mathcal{B}$ and put $z'=(z_1,\cdots,z_m)$. Then $z'\in\partial\boldsymbol{F}(\Theta')$. From $N(z')>0$ it follows that $\tilde{L}(z)=-\infty$. Hence $M_b=\sup\{\tilde{L}(z);\ z\in\mathcal{B}\}=-\infty$.

THEOREM 3.5. *Let* $m=0$. *If* $\partial\boldsymbol{Z}(\Theta)\subset\{1\}$ *and* $q_{..}\neq 0$, *or if* $\partial\boldsymbol{Z}(\Theta)\subset\{0\}$ *and* $q'\neq 0$, *then* $M_b=-\infty$.

PROOF. From (2.3) we have $\tilde{L}(z)=q_{..}\log(1-z)+q'\log z$. Our assertion follows from this expression.

4. Practical criteria (part I)

In this section we shall discuss a simple case where a stronger condition than the 1-regularity holds and which covers typical one-parameter families (see Examples 4.1–4.4). It is interesting to note that the number of parameters is not essential as shown by Examples 4.5–4.7. Throughout this section we assume that $m\geqq 1$ and the following:

(4.1) $\partial \boldsymbol{Z}(\Theta) \subset \mathcal{B} \cup (\boldsymbol{F}(\Theta') \times \{1\})$.

(4.2) $\partial \boldsymbol{F}(\Theta') \neq \phi$ and $\partial \boldsymbol{F}(\Theta') \subset \{\boldsymbol{a}_0, \cdots, \boldsymbol{a}_m\}$,

where $\boldsymbol{a}_i = (\overbrace{0, \cdots, 0}^{i}, \overbrace{1, \cdots, 1}^{m-i})$.

Note that $\boldsymbol{0} = (\overbrace{0, \cdots, 0}^{m}) = \boldsymbol{a}_m$ and $\boldsymbol{1} = (\overbrace{1, \cdots, 1}^{m}) = \boldsymbol{a}_0$. A general sufficient condition for which the relation (4.2) holds will be published elsewhere.

We prove

THEOREM 4.1. *An MLE exists if the following condition is satisfied*:

(4.3) $\boldsymbol{a}_h \notin \partial \boldsymbol{F}(\Theta')$ *or* $\sum_{j=1}^{h} q_{\cdot j} + \sum_{i=h+1}^{m} q_{i\cdot} \neq 0$, $h = 0, \cdots, m$,

where $q_{\cdot j} = \sum_{i=0}^{j-1} q_{ij}$ *and* $q_{i\cdot} = \sum_{j=i+1}^{m+1} q_{ij}$.

PROOF. From (4.1) and (4.2), we see that the pair $(\mathcal{P}, \mathcal{C})$ is 1-regular. Since $N(\boldsymbol{a}_h) = \sum_{j=1}^{h} q_{\cdot j} + \sum_{i=h+1}^{m} q_{i\cdot}$, $0 \leqq h \leqq m$, (4.3) yields that $\inf \{N(\boldsymbol{z}'); \boldsymbol{z}' \in \partial \boldsymbol{F}(\Theta')\} > 0$. Our assertion follows from Corollary 2.1 and Theorem 3.4.

The following theorem shows that under some restrictions, the condition (4.3) is necessary for the existence of an MLE.

THEOREM 4.2. *Let* $q_{\cdot} = 0$, $M \neq -\infty$ *and* $\boldsymbol{F}(\Theta') \subset (0, 1)^m$. *Assume that* $\sup \{q_{\cdot\cdot} \log (1-\eta) + q' \log \eta;\ \ \eta \in \mathcal{H}(\theta')\} \leqq \sup \{q_{\cdot\cdot} \log (1-\eta) + q' \log \eta;\ \ \eta \in \boldsymbol{h}(\boldsymbol{a})\}$ *for all* $\theta' \in \Theta'$ *and for all* $\boldsymbol{a} \in \partial \boldsymbol{F}(\Theta')$. *Then an MLE exists if and only if the condition* (4.3) *is satisfied.*

PROOF. Consider the case $\boldsymbol{1} \in \partial \boldsymbol{F}(\Theta')$ and $\sum_{i=1}^{m} q_{i\cdot} = 0$. Then $q_{\cdot\cdot} = \sum_{j=1}^{m+1} q_{0j}$. From $m \geqq 1$ and $q_{\cdot} = 0$, $\sum_{j=1}^{m} q_{0j} \neq 0$. This and $\boldsymbol{F}(\Theta') \subset (0, 1)^m$ yield that $\sum_{j=1}^{m} q_{0j} \log z_j < 0$ for all $(z_1, \cdots, z_m) \in \boldsymbol{F}(\Theta')$. Take an arbitrary $\boldsymbol{z}_0 = (z_1, \cdots, z_{m+1}) \in \boldsymbol{Z}(\Theta)$ such that $\tilde{L}(\boldsymbol{z}_0) > -\infty$ and $\boldsymbol{z}_0 = \boldsymbol{Z}(\theta)$ with $\theta = (\theta', \eta)$. Since $z_{m+1} = \eta \in \mathcal{H}(\theta')$,

$$\begin{aligned} \tilde{L}(\boldsymbol{z}_0) &= \sum_{j=1}^{m} q_{0j} \log z_j + q_{\cdot\cdot} \log (1-\eta) + q' \log \eta \\ &< q_{\cdot\cdot} \log (1-\eta) + q' \log \eta \\ &\leqq \sup \{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z} \in \{1\} \times \boldsymbol{h}(\boldsymbol{1})\} \leqq M_b \ . \end{aligned}$$

In view of this and Theorem 2.1, an MLE does not exist. Similarly we can prove, for the remaining cases, that an MLE does not exist.

We say that the g.c. data sample $\mathcal{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_q\}$ of size (q', q) is a grouped data sample of size q if

(4.4) $q'=0$ and each C_i, $1\leqq i\leqq q$, is an element of the set $\{[y_0, y_1), \cdots, [y_r, y_{r+1})\}$, where r is a positive integer, $y_0=T'<y_1<\cdots<y_r<y_{r+1}=T$ and r and the y_i's are independent of q and θ.

The existence of an MLE from the grouped data sample C has been studied by many statisticians. When $\mathcal{P}$ is the family of one-parameter exponential distributions or the family of normal distributions with location parameter, Kulldorff ([7]; Theorems 2.1 and 8.1) showed that an MLE exists if and only if $q_{01}<q$ and $q_{r\,r+1}<q$. Carter et al. ([3]; Theorem 5.1) obtained the same result when $\mathcal{P}$ is the family of Poisson distributions. They proved their results by solving the likelihood equation whose form depends on the family $\mathcal{P}$. Since $\partial\boldsymbol{F}(\Theta')=\{0, 1\}$ for each of the above families, their results immediately follow from Theorem 4.2 (see Examples 4.1–4.3).

We shall give seven examples of $\mathcal{P}$ for which the conditions (4.1), (4.2) and $(\mathrm{F}.1)_1$ (see Theorem 3.3) are satisfied. The proofs for the structure of $\partial\boldsymbol{F}(\Theta')$ in the examples are not given here, since they can be easily carried out (see, for example, Nakamura and Kariya [10]). If no confusion can arise, we write $\mathcal{P}=\{P_\theta([-\infty, x));\ \theta\in\Theta\}$ instead of $\mathcal{P}=\{P_\theta;\ \theta\in\Theta\}$. In the examples, we assume that $F(x)$ is a distribution function (d.f.) on $\mathcal{R}$ such that $F(x)$ is continuous on $\mathcal{R}$ and is strictly increasing on the set $F^{-1}((0, 1))=\{x\in\mathcal{R};\ 0<F(x)<1\}$, and assume that $\mathcal{H}(\theta')=\mathcal{H}$ for all $\theta'\in\Theta'$, where $\mathcal{H}=[0, 1)$ or $\{0\}$.

Example 4.1. Location parameter. Let $\mathcal{P}=\{(1-\eta)F(x-\theta');\ (\theta', \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=(-\infty, \infty)$ and $F^{-1}((0, 1))=\mathcal{R}$. In this case, $\partial\boldsymbol{F}(\Theta')=\{0, 1\}$.

Example 4.2. Scale parameter. Let $\mathcal{P}=\{(1-\eta)F(\theta' x);\ (\theta', \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=(0, \infty)$ and $F^{-1}((0, 1))=(0, \infty)$. In this case, $\partial\boldsymbol{F}(\Theta')=\{0, 1\}$.

Example 4.3. Power series distribution. Let $\mathcal{P}=\{(1-\eta)F(x, \theta');\ (\theta', \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=(0, \alpha)$ $(0<\alpha\leqq\infty)$, $F(x, \theta')=\sum_{0\leqq j<x} a_j\theta'^j/f(\theta')$ and $f(\theta')=\sum_{i=0}^{\infty} a_i\theta'^i$ with $a_i\geqq 0$ and $\sum_{i=1}^{\infty} a_i>0$. Here we assume that at least two of the a_i's are positive, $\lim_{\theta'\to\alpha} f(\theta')=\infty$ and $f(\theta')$ is finite on Θ'. In this case, $\partial\boldsymbol{F}(\Theta')=\{0, 1\}$.

Example 4.4. Gamma distribution with shape parameter. Let $\mathcal{P}=\{(1-\eta)F(x, \theta');\ (\theta', \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=(0, \infty)$ and $F(x, \theta')$ is the gamma d.f., that is, $F(x, \theta')=0$ if $x\leqq 0$ and

$$F(x, \theta')=\int_0^x \Gamma(\theta')^{-1}v^{\theta'-1}\exp(-v)dv \qquad \text{if } x>0\,.$$

Here $\Gamma(\theta')$ is the gamma function. In this case, $\partial \boldsymbol{F}(\Theta')=\{\mathbf{0}, \mathbf{1}\}$.

The parameter space Θ' does not need to be an interval of $\mathcal{R}$ as shown by the following examples.

Example 4.5. Hypergeometric distribution. Let $\mathcal{P}=\{(1-\eta)F(x, \theta');\ (\theta', \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=\{(h, k);\ h$ and k are integers such that $0<h<k,\ r<k$ and $h<h_0\}$ with fixed positive integers r and h_0, and $F(x, \theta')$ $(\theta'=(h,k)\in\Theta')$ is defined by $F(x, \theta')=\sum_{a\leq j<x}\binom{k-h}{r-j}\binom{h}{j}\Big/\binom{k}{r}$ if $x\leq \min(r, h)$, where $a=\max(0, r-k+h)$, and by $F(x, \theta')=1$ if $x>\min(r, h)$. In this case, $\partial\boldsymbol{F}(\Theta')=\{\mathbf{1}\}$.

Example 4.6. Location and scale parameters. Let $\mathcal{P}=\{(1-\eta)F((x-\mu)/\sigma);\ (\mu, \sigma, \eta)\in\Theta'\times\mathcal{H}\}$, where $\Theta'=\{\mu_1,\cdots, \mu_{m+1}\}\times(0, \alpha]$ $(0<\alpha<\infty)$, the μ_i's are real numbers such that $\mu_1<x_1<\cdots<\mu_m<x_m<\mu_{m+1}$ and $F^{-1}((0, 1))=\mathcal{R}$. In this case, $\partial\boldsymbol{F}(\Theta')=\{\boldsymbol{a}_0,\cdots, \boldsymbol{a}_m\}$.

Example 4.7. Polynomial distribution. Let $\mathcal{P}=\left\{(1-\eta)F\left(\sum_{i=1}^{r}(\alpha_i x)^i\right);\ (\alpha_1,\cdots, \alpha_r, \eta)\in\Theta'\times\mathcal{H}\right\}$, where $\Theta'=\left\{(\alpha_1,\cdots, \alpha_r)\in[0, \infty)^r;\ \sum_{i=1}^{r}\alpha_i\neq 0\right\}$ and $F^{-1}((0, 1))=(0, \infty)$. In this case, $\partial\boldsymbol{F}(\Theta')=\{\mathbf{0}, \mathbf{1}\}$.

Remark 4.1. The p.c.b. analysis gives a version of Cramer's theorem (cf. [6]; p. 37) which states that the probability of the existence of an MLE from the grouped data sample C tends to unity as $q\to\infty$. Kulldorff ([6]; p. 37) proved, by utilizing the likelihood equation, this theorem under some regularity conditions. We can prove this theorem under the conditions (4.1)–(4.3) (no likelihood equation is needed). This is illustrated by

Example 4.8. Let $\mathcal{P}$ be the family in Example 4.1 with $\mathcal{H}=\{0\}$ and C be a grouped data sample of size q. By the strong law of large numbers, we see that $\lim_{q\to\infty}\Pr\left(\bigcap_{i=0}^{r}\{q_{i\,i+1}\neq 0\}\right)=1$. Note that $\bigcap_{i=0}^{r}\{q_{i\,i+1}\neq 0\}\neq\phi$ implies $m=r$. Thus Cramér's theorem follows from Theorem 4.1.

5. Practical criteria (part II)

The condition (4.2) is not satisfied for typical two-parameter families. In this section we shall discuss, as did in Section 4, a case where a stronger condition than the 2-regularity holds. Typical two-parameter families satisfy this strong condition (see Examples 5.1–5.4). The number of parameters is, again, not essential as shown by Examples 5.5 and 5.6. Throughout this section we assume that $m \geqq 2$, the condition (H.1) and the following:

$$(5.1) \quad \partial \boldsymbol{F}(\Theta') \neq \phi \text{ and } \partial \boldsymbol{F}(\Theta') \subset \{\boldsymbol{a}_0, \cdots, \boldsymbol{a}_m\} \cup \mathcal{A} \cup \left(\bigcup_{i=1}^{m} \mathcal{A}_i\right), \text{ where}$$

$$\mathcal{A} = \{z\mathbf{1};\ 0<z<1\} \text{ and } \mathcal{A}_i = \{(\overbrace{0, \cdots, 0}^{i-1}, z, \overbrace{1, \cdots, 1}^{m-i});\ 0<z<1\}.$$

A general sufficient condition for which the relation (5.1) holds will be published elsewhere. From (H.1), (5.1) and Theorem 3.1, it follows that the pair $(\mathcal{P}, C)$ is 2-regular.

With the aid of Corollary 2.1 and Theorem 3.4 we have

THEOREM 5.1. *Let $q_{\cdot\cdot} \neq 0$. Then an MLE exists if the following conditions are satisfied*:

$$(5.2) \quad \partial \boldsymbol{F}(\Theta') \cap \mathcal{A} = \phi \quad or \quad \sum_{1 \leqq i<j \leqq m} q_{ij} \neq 0\ .$$

$$(5.3) \quad \partial \boldsymbol{F}(\Theta') \cap \mathcal{A}_h = \phi \quad or \quad \sum_{j=1}^{h-1} q_{\cdot j} + \sum_{i=h+1}^{m} q_{i\cdot} \neq 0\ , \qquad h=1, \cdots, m\ .$$

$$(5.4) \quad \partial \boldsymbol{F}(\Theta') \cap \{\boldsymbol{a}_h\} = \phi \quad or \quad \sum_{j=1}^{h} q_{\cdot j} + \sum_{i=h+1}^{m} q_{i\cdot} \neq 0\ , \qquad h=0, \cdots, m\ .$$

We shall give three examples of $\mathcal{F} = \{F(x, \theta');\ \theta' \in \Theta'\}$ for which (5.1) is satisfied. Details of the computation are not given here, since they can be easily carried out (see, for example, Nakamura and Kariya [10]).

Example 5.1. Uniform distribution with two parameters. Let $\mathcal{F} = \{F(x, \theta');\ \theta' \in \Theta'\}$, where $\Theta' = \mathcal{R} \times (0, \infty)$ and $F(x, \theta')$ $(\theta' = (\alpha, \beta) \in \Theta')$ is defined by $F(x, \theta')=0$ if $x<\alpha-\beta$, by $F(x, \theta')=(2\beta)^{-1}(x-\alpha+\beta)$ if $\alpha-\beta \leqq x \leqq \alpha+\beta$ and by $F(x, \theta')=1$ if $x>\alpha+\beta$. In this case, $\partial \boldsymbol{F}(\Theta') = \mathcal{A}$.

Example 5.2. Beta distribution with two parameters. Let $\mathcal{F} = \{F(x, \theta');\ \theta' \in \Theta'\}$, where $\Theta' = (0, \infty) \times (0, \infty)$ and $F(x, \theta')$ $(\theta' = (\alpha, \beta) \in \Theta')$ is defined by $F(x, \theta')=0$ if $x \leqq 0$, by $F(x, \theta')=1$ if $x \geqq 1$ and by

$$F(x, \theta') = \int_0^x B(\alpha, \beta)^{-1} v^{\alpha-1}(1-v)^{\beta-1} dv \qquad \text{if } 0<x<1\ .$$

Here $B(\alpha, \beta)$ is the beta function. In this case, $\partial \boldsymbol{F}(\Theta') = \{\boldsymbol{a}_0 \cdots, \boldsymbol{a}_m\} \cup \mathcal{A} \cup \left(\bigcup_{i=1}^{m} \mathcal{A}_i\right)$.

Example 5.3. Gamma distribution with three parameters. Let $\mathcal{F} = \{F(x, \theta');\ \theta' \in \Theta'\}$, where $\Theta' = (0, \infty) \times (0, \infty) \times [\lambda_1, \lambda_2]$, $-\infty < \lambda_1 \leqq \lambda_2 < x_1$ and $F(x, \theta')$ $(\theta' = (\alpha, \beta, \lambda) \in \Theta')$ is defined by $F(x, \theta')=0$ if $x \leqq \lambda$ and by

$$F(x, \theta')=\int_{\lambda}^{x}(\Gamma(\alpha)\beta^{\alpha})^{-1}(v-\lambda)^{\alpha-1}\exp(-(v-\lambda)/\beta)dv \qquad \text{if } x>\lambda .$$

In this case, $\partial \boldsymbol{F}(\boldsymbol{\Theta}')$ is the same as in Example 5.2.

To find a necessary and sufficient condition for the existence of an MLE, we consider the following condition:

(P.1) For each h, $1\leqq h\leqq m$, with $\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap \mathcal{A}_h\neq\phi$ and for each $\theta'\in\boldsymbol{\Theta}'$, there exist $u\in(0, 1)$, a positive number t_0 and a mapping $\rho(t)$ from $(0, t_0)$ into $\boldsymbol{\Theta}'$ such that:

(5.5) $\theta'\in\{\rho(t);\ t\in(0, t_0)\}\subset F_{x_h}^{-1}([u, u])$.

(5.6) For every sequence $\{t_n\}$ in $(0, t_0)$ with limit 0, the sequence $\{\boldsymbol{F}(\rho(t_n))\}$ has no cluster point in $\boldsymbol{F}(\boldsymbol{\Theta}')$.

(5.7) For every pair (i, j), $1\leqq i<h<j\leqq m$, $F(x_i, \rho(t))$ (resp. $F(x_j, \rho(t))$) is strictly increasing (resp. strictly decreasing) on $(0, t_0)$.

THEOREM 5.2. *Let the condition* (P.1) *be satisfied, let* $\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\left(\bigcup_{h=1}^{m}\mathcal{A}_h\right)\neq\phi$ *and let* $\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\mathcal{A}_h\neq\phi$ *for each* h, $1\leqq h\leqq m$, *with* $\boldsymbol{a}_h\in\partial \boldsymbol{F}(\boldsymbol{\Theta}')$. *Suppose that* $M\neq-\infty$, $q.=0$ *and* $\sum_{1\leqq i<j\leqq m} q_{ij}\neq 0$. *Then an MLE exists if and only if for every* h, $1\leqq h\leqq m$, *with* $\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\mathcal{A}_h\neq\phi$,

$$\sum_{j=1}^{h-1} q_{\cdot j}+\sum_{i=h+1}^{m} q_{i\cdot}\neq 0 .$$

PROOF. The "if" part of the theorem follows from Theorem 5.1. In order to prove the "only if" part of the theorem, assume that $\sum_{j=1}^{h-1} q_{\cdot j}+\sum_{i=h+1}^{m} q_{i\cdot}=0$ for some h, $1\leqq h\leqq m$, with $\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\mathcal{A}_h\neq\phi$. Choose an arbitrary point $\boldsymbol{z}_0\in\boldsymbol{Z}(\boldsymbol{\Theta})$ with $\tilde{L}(\boldsymbol{z}_0)>-\infty$, and let $\theta_0=(\theta_0', \eta)\in\boldsymbol{\Theta}$ with $\boldsymbol{Z}(\theta_0)=\boldsymbol{z}_0$. Because of (P.1), there exist $u\in(0, 1)$, a positive number t_0 and a mapping $\rho(t)$ from $(0, t_0)$ into $\boldsymbol{\Theta}'$ which satisfy the conditions (5.5) -(5.7) with θ' replaced by θ_0'. Put $\boldsymbol{z}(t)=(\boldsymbol{F}(\rho(t)), \eta)$. We show $\lim_{t\to 0}\boldsymbol{z}(t)\in(\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\mathcal{A}_h)\times\overline{\mathcal{H}(\theta_0')}$. In view of (5.6) and (5.7), $\lim_{t\to 0}\boldsymbol{F}(\rho(t))=(z_1', \cdots, z_m')\in\partial \boldsymbol{F}(\boldsymbol{\Theta}')$. Since the pair $(\mathcal{P}, C)$ is 2-regular, it follows from (5.7) that $z_{h-1}'=0$ and $z_{h+1}'=1$. Hence $\lim_{t\to 0}\boldsymbol{z}(t)\in(\partial \boldsymbol{F}(\boldsymbol{\Theta}')\cap\mathcal{A}_h)\times\overline{\mathcal{H}(\theta_0')}$. We finally show that $\tilde{L}(\boldsymbol{z}(t))$ is strictly decreasing on $(0, t_0)$. Since $q_{\cdot j}=0$ for all $j\leqq h-1$ and $q_{i\cdot}=0$ for all $i\geqq h+1$,

$$\tilde{L}(\boldsymbol{z}(t))=\sum_{1\leqq i<j\leqq m;\, i\leqq h\leqq j} q_{ij}\log(F(x_j, \rho(t))-F(x_i, \rho(t)))$$
$$+\sum_{j=h+1}^{m} q_{0j}\log F(x_j, \rho(t))$$

$$+\sum_{i=1}^{h-1} q_{i\,m+1} \log(1-F(x_i, \rho(t)))$$
$$+q_{..} \log(1-\eta)+q' \log \eta \ .$$

This, together with (5.7), yields that $\tilde{L}(z(t))$ is strictly decreasing on $(0, t_0)$. Because of (5.5), $\tilde{L}(z_0)<\lim_{t\to 0} \tilde{L}(z(t))=\tilde{L}(\lim_{t\to 0} z(t))\leqq M_b$. The proof is completed because of Theorem 2.1.

Remark 5.1. Let (P.1)′ denote the condition (P.1) with (5.6) replaced by

(5.6)′ For any sequence $\{t_n\}$ in $(0, t_0)$ with limit 0, the sequence $\{\rho(t_n)\}$ has no cluster point in Θ'.

It can be seen that the conditions $(F.1)_2$, $(F.2)_2$ and (P.1)′ imply the condition (P.1).

We shall give three examples of $\mathcal{F}=\{F(x, \theta'); \theta' \in \Theta'\}$ for which the condition (P.1)′ is satisfied. It can be easily shown that $\partial F(\Theta')=\{\boldsymbol{a}_0, \cdots, \boldsymbol{a}_m\} \cup \mathcal{A} \cup \left(\bigcup_{i=1}^{m} \mathcal{A}_i\right)$ in these examples.

Example 5.4. Location and scale parameters. Let $\mathcal{F}=\{F((x-\mu)/\sigma); (\mu, \sigma) \in \Theta'\}$, where $\Theta'=\mathcal{R}\times(0, \infty)$ and $F(x)$ is a continuously differentiable d.f. on $\mathcal{R}$ with the positive density function. Take $\rho(t)=(x_h-(x_h-\mu)\cdot\sigma^{-1}t, t)$ $(t>0)$ for each h, $1\leqq h\leqq m$, and for each $(\mu, \sigma) \in \Theta'$.

Example 5.5. Scale, power and location parameters. Let $\mathcal{F}=\{F(x, \theta'); \theta' \in \Theta'\}$, where $\Theta'=(0, \infty)\times(0, \infty)\times[\lambda_1, \lambda_2]$, $-\infty<\lambda_1\leqq\lambda_2<x_1$ and $F(x, \theta')$ $(\theta'=(\alpha, \beta, \lambda) \in \Theta')$ is defined by $F(x, \theta')=0$ if $x\leqq\lambda$ and $F(x, \theta')=F(\alpha\cdot(x-\lambda)^{1/\beta})$ if $x>\lambda$. Here $F(x)$ is a continuously differentiable d.f. on $(0, \infty)$ with the positive density function. Take $\rho(t)=(\alpha(x_h-\lambda)^{1/\beta-1/t}, t, \lambda)$ $(t>0)$ for each h, $1\leqq h\leqq m$, and for each $(\alpha, \beta, \lambda) \in \Theta'$

Example 5.6. Trinomial distribution. Let $\mathcal{F}=\{F(x, \theta'); \theta' \in \Theta'\}$, where $\Theta'=\{(\alpha_1, \alpha_2, \alpha_3) \in (0, 1)^3; \alpha_1+\alpha_2+\alpha_3=1\}$ and $F(x, \theta')$ $(\theta'=(\alpha_1, \alpha_2, \alpha_3) \in \Theta')$ is defined by $F(x, \theta')=0$ if $x\leqq y_1$, by $F(x, \theta')=1$ if $x>y_3$ and by $F(x, \theta')=\alpha_1+\cdots+\alpha_i$ if $y_i<x\leqq y_{i+1}$; $i=1, 2$, where the y_i's are numbers such that $-\infty<y_1<y_2<y_3<\infty$. In this case, $m=2$. Take $\rho(t)=(\alpha_1, 1-\alpha_1-t, t)$ $(t<1-\alpha_1)$ for each $(\alpha_1, \alpha_2, \alpha_3) \in \Theta'$ in case $h=1$ and $\rho(t)=(t, \alpha_1+\alpha_2-t, \alpha_3)$ $(t<\alpha_1+\alpha_2)$ for each $(\alpha_1, \alpha_2, \alpha_3) \in \Theta'$ in case $h=2$.

6. Practical criteria (part III)

In case $\partial F(\Theta')\cap\mathcal{A}\neq\phi$ and $\sum_{1\leqq i<j\leqq m} q_{ij}=0$, Theorem 5.1 does not assure the existence of an MLE even if the conditions (5.3) and (5.4) are

satisfied. This is shown by

Example 6.1. Let $\mathcal{F}$ be the family in Example 5.4, $m=2$, $q'=q.=0$ and $\mathcal{H}(\theta')=\{0\}$ for all $\theta'\in\Theta'$. Choose $\{q_{ij}\}$ so that $q_{01}\neq 0$, $q_{02}\neq 0$, $q_{23}\neq 0$ and $q_{ij}=0$ for any other pair (i, j). Let $z_0=(z_1, z_2, z_3)\in \boldsymbol{Z}(\Theta)$. From (2.3), $\tilde{L}(z_0)=q_{01}\log z_1+q_{02}\log z_2+q_{23}\log(1-z_2)$. Hence $\tilde{L}(z_0)<q_{0\cdot}\log z_2+q_{\cdot 3}\cdot\log(1-z_2)\leqq\sup\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in(\partial\boldsymbol{F}(\Theta')\cap\mathcal{A})\times\{0\}\}\leqq M_b$. We see, from Theorem 2.1, that an MLE does not exist.

In this section we shall give some practical criteria for the existence of an MLE in such a case. The equality $\sum_{1\leqq i<j\leqq m} q_{ij}=0$ holds when our observation is restricted two possible types (e.g. an individual is living at time t, or was dead before time t). The sample C in this case is called a binary response data sample. Throughout this section we assume that $m\geqq 2$, $\partial\boldsymbol{F}(\Theta')\cap\mathcal{A}\neq\phi$, $q_{\cdot\cdot}\neq 0$, $\sum_{1\leqq i<j\leqq m} q_{ij}=0$ and the conditions (H.1) and (5.1)-(5.4) except for (5.2) are satisfied. In this case, $\mathcal{H}(\theta')$ is independent of θ' and $M_b=\sup\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in(\partial\boldsymbol{F}(\Theta')\cap\mathcal{A})\times\overline{\mathcal{H}}\}$, where $\mathcal{H}$ denotes the set $\mathcal{H}(\theta')$ $(\theta'\in\Theta')$. For simplicity we put $\tilde{L}(z, \eta)=\tilde{L}(\boldsymbol{z})$ $(\boldsymbol{z}=(z,\cdots, z, \eta)\in(\partial F(\Theta')\cap\mathcal{A})\times\overline{\mathcal{H}})$.

To find a sufficient condition for the existence of an MLE, put $\mathcal{A}^*=\{z;\ z\boldsymbol{1}\in\partial\boldsymbol{F}(\Theta')\cap\mathcal{A}\}$ and consider the following condition:

(P.2) For each $z\in\mathcal{A}^*$, there exist a positive number t_0, a mapping $\rho(t)$ from $(0, t_0)$ into Θ' and a positive function $w(t)$ defined on $(0, t_0)$ such that:

(6.1) For each i, $1\leqq i\leqq m$, $F(x_i, \rho(t))\to z$ as $t\to 0$.

(6.2) For each i, $1\leqq i\leqq m$, $F(x_i, \rho(t))$ is differentiable on $(0, t_0)$, and $W(x_i; z)=\lim_{t\to 0} w(t)dF(x_i, \rho(t))/dt$ exists and is finite.

Hereafter the symbol $\sum$ denotes the summation from $i=1$ to m.

LEMMA 6.1. *Assume that the condition* (P.2) *is satisfied. Let* $(\hat{z}, \hat{\eta})\in\mathcal{A}^*\times\mathcal{H}$ *such that* $\tilde{L}(\hat{z}, \hat{\eta})=\max\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in(\partial\boldsymbol{F}(\Theta')\cap\mathcal{A})\times\overline{\mathcal{H}}\}$. *Then an MLE exists if*

$$(6.3)\qquad (1-\hat{z})(1-\hat{z}+\hat{z}\hat{\eta})\sum q_{0i}W_i > \hat{z}(1-\hat{z}+\hat{z}\hat{\eta})\sum q_{i\,m+1}W_i+\hat{z}(1-\hat{z})(1-\hat{\eta})\sum q_i W_i\,,$$

where $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$.

PROOF. Because of (P.2), there exist a positive number t_0, a mapping $\rho(t)$ from $(0, t_0)$ into Θ' and a positive function $w(t)$ defined on $(0, t_0)$ such that (6.1) and (6.2) with z replaced by $\hat{z}$ are satisfied. Put $\boldsymbol{z}(t)=$

$(\boldsymbol{F}(\rho(t)), \hat{\eta})$, $\tilde{L}'(t)=d\tilde{L}(\boldsymbol{z}(t))/dt$ and $F_i'(t)=dF(x_i, \rho(t))/dt$, $1\leqq i\leqq m$. It is easy to see that $\lim_{t\to 0} \boldsymbol{z}(t)=(\hat{z},\cdots,\hat{z},\hat{\eta})\in(\partial \boldsymbol{F}(\Theta')\cap\mathcal{A})\times\mathcal{H}$ and

$$\tilde{L}'(t)=\sum q_{0i}F_i'(t)F(x_i,\rho(t))^{-1}-\sum q_{i\,m+1}F_i'(t)(1-F(x_i,\rho(t))^{-1}$$
$$-(1-\hat{\eta})\sum q_iF_i'(t)(1-F(x_i,\rho(t))+F(x_i,\rho(t))\hat{\eta})^{-1}.$$

Multiplying the above equality by $w(t)$ and letting $t\to 0$, we have

$$\lim_{t\to 0} w(t)\tilde{L}'(t)=\hat{z}^{-1}\sum q_{0i}W_i-(1-\hat{z})^{-1}\sum q_{i\,m+1}W_i$$
$$-(1-\hat{z}+\hat{z}\hat{\eta})^{-1}(1-\hat{\eta})\sum q_iW_i$$
$$=[\hat{z}(1-\hat{z})(1-\hat{z}+\hat{z}\hat{\eta})]^{-1}[(1-\hat{z})(1-\hat{z}+\hat{z}\hat{\eta})Q_1$$
$$+\hat{z}\hat{\eta}Q_2-\hat{z}(1-\hat{z}+\hat{z}\hat{\eta})(Q_2+Q_3)],$$

where $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$, $Q_1=\sum q_{0i}W_i$, $Q_2=\sum q_iW_i$ and $Q_3=\sum q_{i\,m+1}\cdot W_i$. From the inequality (6.3), it follows that $\tilde{L}'(t)>0$ for sufficiently small t. Thus there exists $\boldsymbol{z}\in\boldsymbol{Z}(\Theta)$ such that $\tilde{L}(\boldsymbol{z})>\tilde{L}(\hat{z},\hat{\eta})=M_b$, and our assertion follows from Theorem 2.1.

THEOREM 6.1. *Let the condition* (P.2) *be satisfied. Assume that* $\sum q_{0i}\sum(q_{i\,m+1}+q_i)\neq 0$, $q'=0$, $\mathcal{A}^*=(0,1)$ *and* $\mathcal{H}=\{0\}$ *or* $[0,1)$. *Then an MLE exists if*

$$\sum q_{0i}W_i/\sum q_{0i}>\sum(q_{i\,m+1}+q_i)W_i/\sum(q_{i\,m+1}+q_i), \tag{6.4}$$

where $\hat{z}=\sum q_{0i}/\sum(q_{0i}+q_{i\,m+1}+q_i)$ *and* $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$.

PROOF. Since the proof for the case $\mathcal{H}=\{0\}$ is similar to that for the case $\mathcal{H}=[0,1)$, we prove the theorem for the latter case. We show $\tilde{L}(\hat{z},0)=\max\{\tilde{L}(z,\eta);\ (z,\eta)\in[0,1]\times[0,1]\}$. It is easy to see that

$$\partial\tilde{L}(z,\eta)/\partial\eta=[(1-\eta)(1-z+z\eta)]^{-1}[(q_{..}+q_{.})z(1-\eta)-q_{..}]. \tag{6.5}$$

Put $z'=q_{..}/(q_{..}+q_{.})$ and define $V(z)$ by $V(z)=0$ if $0\leqq z\leqq z'$ and by $V(z)=(z-z')/z$ if $z'<z\leqq 1$. By (6.5), $\tilde{L}(z,V(z))=\max\{\tilde{L}(z,\eta);\ \eta\in[0,1]\}$ and

$$\tilde{L}(z,V(z))=\begin{cases} a\log z+(b+q_.)\log(1-z) & \text{if } 0\leqq z\leqq z', \\ q_{..}\log z'+q_.\log(1-z')+b\log(1-z) & \\ \qquad -(b+q_{0\,m+1})\log z & \text{if } z'<z\leqq 1, \end{cases}$$

where $a=\sum q_{0i}$ and $b=\sum q_{i\,m+1}$. Note that $q_{..}=a+b+q_{0\,m+1}$. Since $\hat{z}\leqq z'$, $\tilde{L}(\hat{z},0)=\max\{\tilde{L}(z,V(z));\ 0\leqq z\leqq z'\}$. On the other hand, $b\log(1-z)-(b+q_{0\,m+1})\log z$ is decreasing on $(0,1)$. Hence $\max\{\tilde{L}(z,V(z));\ z'\leqq z\leqq 1\}=\tilde{L}(z',V(z'))\leqq\tilde{L}(\hat{z},0)$. Thus $\tilde{L}(\hat{z},0)=\max\{\tilde{L}(z,\eta);\ (z,\eta)\in[0,1]\times[0,1]\}$. Put $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$, and $\hat{\eta}=0$. From (6.4), the left-hand side of (6.3)$=(b+q.)^2(a+b+q.)^{-2}\sum q_{0i}W_i>a(b+q.)(a+b+q.)^{-2}\sum (q_{i\,m+1}+q_i)W_i=$ the right-hand side of (6.3). Now our assertion follows from Lemma 6.1.

This theorem gives an answer to the problem raised by Moran ([8]; p. 5): Does an MLE from a binary response data sample exist when $\mathcal{P}$ is the family of two-parameter normal distributions?

Similaly we have

THEOREM 6.2. *Let the condition* (P.2) *be satisfied. Assume that* $\sum q_{0i}\sum q_{i\,m+1}\neq 0$, $q.=0$, $\mathcal{A}^*=(0, 1)$ *and* $\mathcal{H}=\{0\}$ *or* $[0, 1)$. *Then an MLE exists if the inequality* (6.4) *with* $q.=0$ *holds.*

Finally we consider the case where $\mathcal{A}^*$ consists of only one point. We easily obtain

LEMMA 6.2. *Assume that* $q\neq q'$, $\mathcal{A}^*=\{\hat{z}\}$ *and* $\mathcal{H}=[0, 1)$. *Let* $\hat{\eta}$ *be the greater solution of the quadratic equation*: $(1-\hat{z}+\hat{z}\eta)(q\eta-q')=q.\eta$. *Then* $\hat{\eta}\in\mathcal{H}$ *and* $\tilde{L}(\hat{z}, \hat{\eta})=\max\{\tilde{L}(\boldsymbol{z});\ \boldsymbol{z}\in(\partial\boldsymbol{F}(\Theta')\cap\mathcal{A})\times\bar{\mathcal{H}}\}$.

By Lemmas 6.1 and 6.2 and by the same argument as in the proof of Theorem 6.1, we have

THEOREM 6.3. *Let the condition* (P.2) *be satisfied. Assume that* $\sum q_{0i}\sum q_{i\,m+1}\neq 0$, $\mathcal{A}^*=\{\hat{z}\}$ *and* $\mathcal{H}=[0, 1)$. *Put* $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$. *Then the following assertions hold*:

(i) *In case* $q.=0$, *an MLE exists if*

$$(1-\hat{z})\sum q_{0i}W_i>\hat{z}\sum q_{i\,m+1}W_i\ . \tag{6.6}$$

(ii) *In case* $q.\neq 0$, $q'=0$ *and* $\hat{z}\leqq q../q$, *an MLE exists if*

$$(1-\hat{z})\sum q_{0i}W_i>\hat{z}\sum (q_{i\,m+1}+q_i)W_i\ . \tag{6.7}$$

(iii) *In case* $q.\neq 0$, $q'=0$ *and* $\hat{z}>q../q$, *an MLE exists if*

$$q_{..}^{-1}[(1-\hat{z})\sum q_{0i}W_i-\hat{z}\sum q_{i\,m+1}W_i]>q_{.}^{-1}(1-\hat{z})\sum q_iW_i\ . \tag{6.8}$$

THEOREM 6.4. *Let the condition* (P.2) *be satisfied. Assume that* $\sum q_{0i}\sum q_{i\,m+1}\neq 0$, $\mathcal{A}^*=\{\hat{z}\}$ *and* $\mathcal{H}=\{0\}$. *Put* $W_i=W(x_i;\hat{z})$, $1\leqq i\leqq m$. *Then an MLE exists if the inequality* (6.7) *holds.*

We give an example of $\mathcal{F}=\{F(x, \theta');\ \theta'\in\Theta'\}$ for which $\mathcal{A}^*$ consists of only one point.

Example 6.2. Scale parameter. Let $\mathcal{F}=\{F((x-\mu_0)/\theta');\ \theta'\in\Theta'\}$, where $\Theta'=(0, \infty)$, $\mu_0\in\mathcal{R}$ and $F(x)$ is a continuously differentiable d.f. on $\mathcal{R}$ such that $dF(x)/dx>0$ for all $x\in\mathcal{R}$. Put $F(0)=z$. It is easy to see that $\partial\boldsymbol{F}(\Theta')=\{\boldsymbol{1}, z\boldsymbol{1}\}$ if $\mu_0<x_1$; $\partial\boldsymbol{F}(\Theta')=\{\boldsymbol{a}_k, z\boldsymbol{1}\}$ if $x_k<\mu_0<x_{k+1}$, $1\leqq k\leqq$

$m-1$; $\partial F(\Theta')=\{0, z1\}$ if $x_m<\mu_0$; $\partial F(\Theta')=\{z1, (\overbrace{0,\cdots,0}^{k-1}, z, 1,\cdots,1)\}$ if $\mu_0=x_k$, $1\leqq k\leqq m$. In each case, $\mathcal{A}^*=\{z\}$.

Remark 6.1. In order to prove that all families in Examples 5.1–5.6 and 6.2 satisfy the condition (P.2), we may take $\rho(t)$ as follows (results in Examples 5.2 and 5.3 are given by the referee):

In Example 5.1, $\rho(t)=((1-2z)/t, 1/t)$, $t>0$; $z\in(0, 1)$. In this case, $W(x_i, z)=x_i$ and hence the inequality (6.4) becomes

$$(6.9)\qquad (\sum q_{i\,m+1}+q.)^{-1}\sum(q_{i\,m+1}+q_i)x_i<(\sum q_{0i})^{-1}\sum q_{0i}x_i\ .$$

In Example 5.2, $\rho(t)=((1-z)t, zt)$ $t>0$; $z\in(0, 1)$. In this case, $W(x_i; z)=\log(x_i/(1-x_i))$ and hence the inequality (6.4) becomes (6.9) with x_i replaced by $\log(x_i/(1-x_i))$.

In Example 5.3, $\rho(t)=(t, z^{-1/t}, \lambda)$, $t>0$; $z\in(0, 1)$, where λ is an arbitrary number with $\lambda\in[\lambda_1, \lambda_2]$. In this case, $W(x_i; z)=C+\log(x_i-\lambda)$ with Euler's constant C and hence the inequality (6.4) becomes (6.9) with x_i replaced by $\log(x_i-\lambda)$.

In Example 5.4, $\rho(t)=(x-F^{-1}(z)/t, 1/t)$, $t>0$; $z\in(0, 1)$, where x is an arbitrary number and $F^{-1}(z)$ is the inverse function of $F(x)$. In this case, $W(x_i; z)=x_i-x$ and hence the inequality (6.4) becomes (6.9).

In Example 5.5, $\rho(t)=((x-\lambda)^{-t}F^{-1}(z), 1/t, \lambda)$, $t>0$; $z\in(0, 1)$, where λ and x are arbitrary numbers with $\lambda\in[\lambda_1, \lambda_2]$ and $\lambda<x$. In this case, $W(x_i; z)=\log((x_i-\lambda)/(x-\lambda))$ and hence the inequality (6.4) becomes (6.9) with x_i replaced by $\log(x_i-\lambda)$.

In Example 5.6, $\rho(t)=(z, t, 1-z-t)$, $t<1-z$; $z\in(0, 1)$. In this case, $W(x_1; z)=0$ and $W(x_2; z)=1$, and hence the inequality (6.4) becomes

$$(\sum q_{i3}+q.)^{-1}(q_{23}+q_2)<(\sum q_{0i})^{-1}q_{02}\ .$$

In Example 6.2 with $\mu_0=x_k$ for some k, $1\leqq k\leqq m$, $\rho(t)=1/t$, $t>0$. Put $z=F(0)$. In this case, $W(x_i; z)=x_i-x_k$ and hence the inequality (6.6) becomes

$$x_k((1-z)\sum q_{0i}-z\sum q_{i\,m+1})<(1-z)\sum q_{0i}x_i-z\sum q_{i\,m+1}x_i\ .$$

Remark 6.2. With the aid of Remark 6.1, we see that the p.c.b. analysis gives a version of Cramer's theorem (see Remark 4.1). This is illustrated by

Example 6.3. Let $\mathcal{F}$ be the family in Example 5.4 and let $\mathcal{C}=\bigcup_{i=1}^{k}\mathcal{C}^{(i)}$ be a binary response data sample such that each $\mathcal{C}^{(i)}$ is a grouped data sample of size n_i defined by (4.4) with $r=1$ and $y_1=t_i$, where $-\infty<t_1<\cdots<t_k<\infty$. In this case, $q'=q.=0$ and $q_{ij}=0$ for each pair (i, j),

$1 \leqq i < j \leqq m$. If $\mathcal{S} = \bigcap_{i=1}^{k} \{q_{0i} \neq 0 \text{ and } q_{i\,m+1} \neq 0\} \neq \phi$, then $k=m$ and (5.4) is satisfied. By the strong law of large numbers, we see that $\Pr(\mathcal{S} \neq \phi) \to 1$ as $\min_{1 \leqq i \leqq k} n_i \to \infty$. Using Corollary 1 of Nakamura and Kariya [10], we can show that the probability that the inequality (6.9) holds tends to unity as $\min_{1 \leqq i \leqq k} n_i \to \infty$.

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