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LIMITING DISTRIBUTION OF SUMS OF NONNEGATIVE STATIONARY RANDOM VARIABLES

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Summary

Let $\{X_{n,j}, -\infty<j<\infty\}$, $n\geq 1$, be a sequence of stationary sequences on some probability space, with nonnegative random variables. Under appropriate mixing conditions, it is shown that $S_n=X_{n,1}+\cdots+X_{n,n}$ has a limiting distribution of a general infinitely divisible form. The result is applied to sequences of functions $\{f_n(x)\}$ defined on a stationary sequence $\{X_j\}$, where $X_{n,j}=f_n(X_j)$. The results are illustrated by applications to Gaussian processes, Markov processes and some autoregressive processes of a general type.

1. Introduction and summary

The purpose of this paper is to present a limit theorem on the convergence of the distribution of the sum in an array of nonnegative random variables. The array is stationary in this sense: if the array is represented as $\{X_{n,j}, 1\leq j\leq n<\infty\}$, then, for each n, the random variables $X_{n,j}$, $j=1,\cdots,n$, form a stationary (though finite) sequence. Such a theorem was given earlier in [2], and was applied in the study of the sojourn times of stationary processes in "rare" sets such as the interval (u, ∞) for large u. The hypothesis of that theorem included certain kinds of mixing conditions which provide enough independence to give results similar to those for independent summands. The novelty of this work is the introduction of new mixing conditions having more flexibility in certain applications.

The previous mixing conditions were verified in applications to the sojourns and extremes of stationary Gaussian processes [3], and certain

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Markov processes [4]. The current theorem applies to similar problems for other classes of stationary processes including Gaussian processes not covered in the previous cases, general Markov processes, and autoregressive processes of a general nature. Among the new results is a limit theorem for the maximum in a stationary Gaussian sequence.

The class of limiting distributions obtained here is more general than the one in [2] because the latter were required to have finite first moments. The present class of limiting distributions is not restricted in this way. In the last section we show how the mixing conditions introduced here can be generalized to obtain limiting distributions which are mixtures of the original limit distributions.

The hypothesis of Theorem 1.1 below is formally stronger than the corresponding hypothesis in [2] in that the family $\{X_{n,j},\ j=1,\cdots n\}$ is required to be embeddable in an infinite stationary sequence $\{X_{n,j},\ j=0,\pm 1,\cdots\}$ for every n. This additional requirement is not essential but it makes the statement of the mixing condition simpler. In our applications, the embeddability requirement is actually satisfied.

The basic result, Theorem 1.1, is stated below, and the proof is given in Section 2. Let $\{X_{n,j},\ j=0,\pm 1,\cdots\}$ be a sequence (in n) of stationary sequences with nonnegative random variables. Let $\mathcal{F}_n(h)$ be the sigma-field generated by $X_{n,j}$, $j\leqq h$. We will suppress the subscript n, and write $\mathcal{F}_n(h)$ simply as $\mathcal{F}(h)$.

THEOREM 1.1. *Assume the following two conditions:*
i) *There exists a nonincreasing function $H(x)$, $x>0$, such that*

$$\int_0^\infty \min(x, 1)dH(x) > -\infty\ , \tag{1.1}$$

and

$$n\int_0^x \min(y, 1)d\,\mathrm{P}(X_{n,0}>y) \rightarrow \int_0^x \min(y, 1)dH(y)\ , \tag{1.2}$$

in the sense of complete convergence on $x>0$.
ii) *There exists a sequence $\{\gamma_n\}$ of positive integers such that for every $c>0$,*

$$n\sum_{1\leqq j\leqq \gamma_n} \mathrm{P}(X_{n,0}>c,\ X_{n,j}>c)\rightarrow 0\ , \tag{1.3}$$

and, for every $q>0$,

$$\operatorname{p}\lim_{n\rightarrow\infty} \frac{\mathrm{P}(X_{n,0}>x\,|\,\mathcal{F}(-q\gamma_n))}{\mathrm{P}(X_{n,0}>x)}=1\ . \tag{1.4}$$

Then $X_{n,1}+\cdots+X_{n,n}$ has, for $n\rightarrow\infty$, a limiting distribution with the Laplace-Stieltjes transform

(1.5) $$\Omega(s)=\exp\left[\int_0^\infty (1-e^{-sx})dH(x)\right] .$$

Remark 1. Condition (1.2) obviously implies

(1.6) $$\lim_{n\to\infty} n\,\mathrm{P}(X_{n,0}>x)=H(x)$$

in the sense of complete convergence on $x>c$, for any $c>0$.

Remark 2. Assumption i) is a condition only on the marginal distribution of the sequence. Assumption ii) is always satisfied in the case of independent $X_{n,j}$, $-\infty<j<\infty$, because the ratio in (1.4) is always equal to 1, and we can take γ_n as any sequence such that $\gamma_n/n \to 0$. Indeed, the left hand member of (1.3) is then equal to $n\,\mathrm{P}^2(X_{n,0}, 0>c)\gamma_n$, which, by (1.6), converges to 0. The theorem is, in fact, true if the $\{X_{n,j}\}$ are independent and assumption i) holds. For the logarithm of the Laplace-Stieltjes transform of $X_{n,1}+\cdots+X_{n,n}$ is $n\log \mathrm{E}(\exp(-sX_{n,0}))\sim -n\,\mathrm{E}(1-\exp[-sX_{n,0}])$, and the latter, by (1.2) converges to $\log\Omega(s)$.

A case of particular interest is where $X_{n,j}$ assumes only the values 0 and 1. Here the function $H(x)$ is constant except for a jump of magnitude $\lambda>0$ at $x=1$.

COROLLARY 1.1. *If $X_{n,j}$ assumes only the values* 0 *and* 1, *and*

(1.7) $$\lim_{n\to\infty} n\,\mathrm{P}(X_{n,0}=1)=\lambda ;$$

and if there is a sequence $\{\gamma_n\}$ such that

(1.8) $$n\sum_{1\le j\le\gamma_n} \mathrm{P}(X_{n,0}=1,\ X_{n,j}=1)\to 0 ,$$

and, for every $q>0$,

(1.9) $$\mathrm{p}\lim_{n\to\infty}\frac{\mathrm{P}(X_{n,0}=1\,|\,\mathcal{F}(-q\gamma_n))}{\mathrm{P}(X_{n,0}=1)}=1 ,$$

then $X_{n,1}+\cdots+X_{n,n}$ has a limiting Poisson distribution with mean λ.

We note that (1.9) is formally similar to certain conditions implicit in the work of Serfling [17] on the Poisson approximation for sums of dependent random variables. Indeed, Corollary 1.1 can, with a certain amount of labor, be deduced with the help of the estimates in [17]. However, I have not explicitly used these results because the Corollary is a direct consequence of the more general Theorem 1.1.

Related work on limit theorems for sums and extremes of dependent random variables has recently been done by Davis [6] and Davis and Resnick [7].

The following two propositions will be used in some of the proofs:

PROPOSITION 1.1. Let $\{Y_n\}$ and $\{Y\}$ be nonnegative random variables such that

$$\mathrm{E}\, Y_n = \mathrm{E}\, Y \qquad \text{for } n \geqq 1 ; \ \mathrm{p}\lim_{n\to\infty} Y_n = Y ;$$

then,

$$\lim_{n\to\infty} \mathrm{E}\,|Y_n - Y| = 0 .$$

PROOF. The proof is implicit in that of Scheffe [14] and Lehmann ([14], p. 351). The latter proved the result under the hypothesis of almost everywhere convergence. Here I note that the proof is valid, through the dominated convergence theorem, for convergence in probability of Y_n to Y.

PROPOSITION 1.2. Let ξ be a random variable such that $\mathrm{E}\,|\xi| < \infty$. If $\mathcal{F}$ and $\mathcal{G}$ are sigma-fields such that $\mathcal{F} \subset \mathcal{G}$, then $\mathrm{E}\,|\mathrm{E}\,(\xi\,|\,\mathcal{F})| \leqq \mathrm{E}\,|\mathrm{E}\,(\xi\,|\,\mathcal{G})|$.

This follows after writing $\mathrm{E}\,(\xi\,|\,\mathcal{F})$ as $\mathrm{E}\,[\mathrm{E}\,(\xi\,|\,\mathcal{G})\,|\,\mathcal{F}]$.

As consequences we have:

LEMMA 1.1. *The relation* (1.4) *holds if and only if*

$$\lim_{n\to\infty} \mathrm{E} \left| \frac{\mathrm{P}\,(X_{n,0} > x\,|\,\mathcal{F}(-q\gamma_n))}{\mathrm{P}\,(X_{n,0} > x)} - 1 \right| = 0 . \tag{1.10}$$

PROOF. Clearly (1.10) implies (1.4). The converse follows from Proposition 1.1: we identify Y_n as $\mathrm{P}\,(X_{n,0} > x\,|\,\mathcal{F}(-q\gamma_n))/\mathrm{P}\,(X_{n,0} > x)$ and Y as 1.

LEMMA 1.2. *Condition* (1.4) *or* (1.10) *remains sufficient for Theorem* 1.1 *if the sigma-fields* $\{\mathcal{F}(-q\gamma_n)\}$ *are replaced by sigma-fields* $\{\mathcal{G}_n\}$ *with* $\mathcal{F}(-q\gamma_n) \subset \mathcal{G}_n$.

PROOF. By Proposition 1.2, the expectation in (1.10) cannot but increase if $\mathcal{F}(-q\gamma_n)$ is replaced by $\mathcal{G}_n$. As a consequence of Lemma 1.1 we infer that, under (1.2), the condition (1.4) is equivalent to

$$\lim_{n\to\infty} \mathrm{E}\,|n\,\mathrm{P}\,(X_{n,0} > x\,|\,\mathcal{F}(-q\gamma_n)) - H(x)| = 0 , \tag{1.11}$$

or to

$$\lim_{n\to\infty} n\,\mathrm{E}\,|\mathrm{P}\,(X_{n,0} > x\,|\,\mathcal{F}(-q\gamma_n)) - \mathrm{P}\,(X_{n,0} > x)| = 0 , \tag{1.12}$$

at all continuity points of H.

2. Proof of Theorem 1.1

LEMMA 2.1. *Under the conditions* (1.1) *and* (1.2), *it suffices, for the proof of Theorem* 1.1, *to consider only the case where there is a number* c, $0<c<1$, *such that for all* $n\geqq 1$,

$$P(X_{n,0}>1/c)=P(0<X_{n,0}\leqq c)=0 . \tag{2.1}$$

PROOF. For arbitrary $c>0$, let $X'_{n,j}$ be the random variable $X_{n,j}$ truncated in the standard way at $1/c$. Let A_n be the event $\{X_{n,j}>1/c$ for some j, $1\leqq j\leqq n\}$. For any $\varepsilon>0$, there exists $c>0$ such that for $n\geqq 1$, $P(A_n)\leqq nP(X_{n,0}>1/c)\leqq\varepsilon$; the latter is a consequence of (1.2). Therefore, for such c, the distributions of $\sum_{j=1}^{n} X_{n,j}$ and $\sum_{j=1}^{n} X'_{n,j}$ differ at no point by more than ε; therefore, we may as well assume that $X_{n,j}\leqq 1/c$.

Conditions (1.1) and (1.2) also imply

$$\limsup_{\varepsilon\to 0}\ \sup_{n\geqq 1} -n\int_0^{\varepsilon} x\,d\,P(X_{n,0}>x)=0 . \tag{2.2}$$

Write $X_{n,j}$ as

$$X_{n,j}I_{[X_{n,j}<c]}+X_{n,j}I_{[X_{n,j}\geqq c]}=X'_{n,j}+X''_{n,j} ,$$

where $I_{[\dots]}$ is the indicator function. Then $X_{n,j}\geqq X''_{n,j}$, and, by (2.2),

$$\begin{aligned}\limsup_{c\to 0}\ \sup_{n\geqq 1} E\left[\sum_{j=1}^{n} X_{n,j}-\sum_{j=1}^{n} X''_{n,j}\right] \\ =\limsup_{c\to 0}\ \sup_{n\geqq 1} E\sum_{j=1}^{n} X'_{n,j}=\limsup_{c\to 0}\ \sup_{n\geqq 1} -n\int_0^{c} x\,d\,P(X_{n,0}>x)=0 .\end{aligned} \tag{2.3}$$

Thus the limiting distributions of $\sum X_{n,j}$ and $\sum X''_{n,j}$ can be made arbitrarily close by choosing c sufficiently small. Since $X''_{n,j}$ has the property described by (2.1), the proof is complete. (Note that the moments in (2.3) exist because in the first part of the proof we deduced the permissibility of assuming $X_{n,j}\leqq 1/c$.)

The proof of Theorem 1.1 is based on the familiar construction of blocks of nearly independent subsums. We use the notation of [2]. Let p and q be arbitrary positive numbers with $p+q=1$, and where q is meant to be close to 0. For any integer $k\geqq 1$ decompose the index set $(1,\cdots,n)$, for large n, into $2k$ consecutive subsets $A_1, B_1,\cdots, A_k, B_k$, where each A_j has $[pn/k]$ members, and each B_j has approximately $[qn/k]$ members.

For an arbitrary A-set, we write

$$\Sigma_A X=\sum_{j\in A} X_{n,j} ,$$

and, analogously, $\Sigma_B X$ for the B-sets. As shown in [2], the contribution to the sum from the terms whose indices are in B-sets can be made arbitrarily small by choosing q sufficiently small:

$$\mathrm{E}\sum_{h=1}^{k}\Sigma_{B_h}X \approx qn\,\mathrm{E}\,X_{n,0} \to -q\int_0^{1/c} x dH(x)\,;$$

the latter follows from (1.2) and Lemma 2.1.

Now we choose the number k of A-sets to depend on n: Letting $\{\gamma_n\}$ be the sequence in (1.3) and (1.4), we assume

$$k = k_n \sim n/\gamma_n\,, \tag{2.4}$$

and so $k_n \to \infty$.

Put $W_A = Indicator\ of\ \{X_{n,j} > 0\ for\ at\ most\ one\ j \in A\}$. Since, by (2.1), $X_{n,j} > 0$ implies $X_{n,j} > c$, and since each A-set has approximately $p\gamma_n$ elements, it follows from (1.3) and (2.4) that

$$\begin{aligned}\mathrm{E}\left(\sum_{h=1}^{k_n}(1-W_{A_h})\right) &\leqq \sum_{h=1}^{k_n}\sum_{i\neq j;\, i,j\in A_h}\mathrm{P}\,(X_{n,i}>c,\, X_{n,j}>c)\\ &\leqq n\sum_{1\leqq j\leqq \gamma_n}\mathrm{P}\,(X_{n,0}>c,\, X_{n,j}>c)\to 0\,.\end{aligned} \tag{2.5}$$

For $n \geqq 1$, let $\{Z_{n,j}\}$ be a stationary array with the same marginal distributions as $\{X_{n,j}\}$ but where the Z's are mutually independent. We use the same notation for the subsums of the Z's as for the X's, namely, $\Sigma_A Z$ and $\Sigma_B Z$. By Remark 2 following Theorem 1.1, the latter holds for $\{Z_{n,j}\}$.

We apply the comparison method of Lévy and Loève (see [15], p. 41). Let the array $\{X_{n,j}\}$ be defined on some probability space, and let the associated array $\{Z_{n,j}\}$ be defined on the same space in such a way that the two arrays are independently distributed. As shown above, the contribution of the B-index sets may be ignored in calculating the limiting distribution of the sum. Hence, we will prove the theorem by comparing the distribution of $\sum_{h=1}^{k}\Sigma_{A_h}X$ to that of $\sum_{h=1}^{k}\Sigma_{A_h}Z$, and then noting that the limit of the latter distribution (for $k=k_n$ and $n\to\infty$) has already been found.

Let $\mathrm{E}_h(\cdot)$ represent the conditional expectation operator relative to the sigma field generated by $\mathcal{F}(j)$, $j \in A_h$, $h=1,\cdots,k$, and $\mathrm{E}_0(\cdot)$ the operator relative to $\mathcal{F}(0)$. According to the Lévy-Loève method, the limiting distributions of the X-sums and Z-sums are identical if

$$\lim_{n\to\infty}\sum_{h=1}^{k_n}\mathrm{E}\,|\mathrm{E}_{h-1}[\exp(-s\Sigma_{A_h}X)] - \mathrm{E}\,[\exp(-s\Sigma_{A_h}Z)]| = 0\,. \tag{2.6}$$

We show that in the evaluation of (2.6) the expression $exp\,(-s\Sigma_A X)$ may be replaced by

$$\Sigma_A(e^{-sX}-1) \tag{2.7}$$

and $\exp(-s\Sigma_A Z)$ by $\Sigma_A(e^{-sZ}-1)$. First we note the elementary identity,

$$W_A \exp(-s\Sigma_A X)=W_A[1+\Sigma_A(e^{-sX}-1)] .$$

The latter implies

$$\begin{aligned}\exp(-s\Sigma_A X)=[1+\Sigma_A(e^{-sX}-1)]-(1-W_A)\Sigma_A(e^{-sX}-1)\\ +(1-W_A)[\exp(-s\Sigma_A X)-1] .\end{aligned} \tag{2.8}$$

Put $A=A_h$, and apply the conditional expectation operator E_{h-1} to each member of (2.8); then $\mathrm{E}_{h-1}[\exp(-s\Sigma_{A_h}X)]$ is the sum of three terms:

$$\mathrm{E}_{h-1}[1+\Sigma_{A_h}(e^{-sX}-1)] \tag{2.9}$$

$$\mathrm{E}_{h-1}(1-W_{A_h})\Sigma_{A_h}(1-e^{-sX}) \tag{2.10}$$

$$\mathrm{E}_{h-1}(1-W_{A_h})[\exp(-s\Sigma_{A_h}X)-1] . \tag{2.11}$$

We will take the absolute values of the terms (2.10) and (2.11), sum over $h=1,\cdots,k_n$, take the unconditional expectation, and then pass to the limit over n. For (2.10) we have,

$$\sum_{h=1}^{k_n} \mathrm{E}\,|\mathrm{E}_{h-1}(1-W_{A_h})\Sigma_{A_h}(1-e^{-sX})|\leqq\sum_{h=1}^{k_n} \mathrm{E}\,[(1-W_{A_h})\Sigma_{A_h}I_{[X>c]}] . \tag{2.12}$$

It follows from the definition of W_A that

$$(1-W_{A_h})\Sigma_{A_h}I_{[X>c]}\leqq\sum_{i\neq j;\, i,j\in A_h} I_{[X_{n,i}>c,X_{n,j}>c]} ,$$

hence, by (2.4) and stationarity, the second member of (2.12) is at most equal to

$$n\sum_{1\leqq j\leqq r_n} \mathrm{P}\,(X_{n,0}>c,\, X_{n,j}>c) ,$$

which, by the hypothesis (1.3), converges to 0. A similar analysis holds for (2.11):

$$\sum_{h=1}^{k_n} \mathrm{E}\,|\mathrm{E}_{h-1}(1-W_{A_h})(\exp[-s\Sigma_{A_h}X]-1)|\leqq\mathrm{E}\sum_{h=1}^{k_n}(1-W_{A_h}) ,$$

which, by (2.5), converges to 0.

It follows from (2.8), and from the foregoing estimates of the sums of the expectations of (2.10) and (2.11) that

$$\sum_{h=1}^{k_n} \mathrm{E}\,|\mathrm{E}_{h-1}\{\exp(-s\Sigma_{A_h}X)-1-\Sigma_{A_h}(e^{-sX}-1)\}|$$

has the limit 0; and a similar relation holds for the Z-array. Therefore, as we intended to show, (2.6) is equivalent to

$$\lim_{n\to\infty}\sum_{h=1}^{k_n} \mathrm{E}\,|\Sigma_{A_h}[\mathrm{E}_{h-1}(e^{-sX}-1)-\mathrm{E}\,(e^{-sZ}-1)]|=0\ ,$$

which is implied by

$$\lim_{n\to\infty}\sum_{h=1}^{k_n} \Sigma_{A_h}\,\mathrm{E}\,|\mathrm{E}_{h-1}(e^{-sX}-1)-\mathrm{E}\,(e^{-sZ}-1)|=0\ . \tag{2.13}$$

Let us estimate a typical term of the sum above. Let $\mathrm{P}_h(\cdot)$ be the conditional probability measure corresponding to the conditional expectation $\mathrm{E}_h(\cdot)$. If $j\in A_h$, and $X=X_{n,j}$, then

$$\begin{aligned} &\mathrm{E}\,|\mathrm{E}_{h-1}(e^{-sX}-1)-\mathrm{E}\,(e^{-sZ}-1)| \\ &\quad =\mathrm{E}\left|\int_0^\infty (e^{-sx}-1)d[\mathrm{P}_{h-1}(X_{n,j}\leqq x)-\mathrm{P}\,(X_{n,j}\leqq x)]\right| . \end{aligned} \tag{2.14}$$

By integration by parts, the second member above is equal to

$$\begin{aligned} &s\,\mathrm{E}\left|\int_0^\infty e^{-sx}[\mathrm{P}_{h-1}(X_{n,j}>x)-\mathrm{P}\,(X_{n,j}>x)]dx\right| \\ &\quad \leqq s\int_0^\infty e^{-sx}\,\mathrm{E}\,|\mathrm{P}_{h-1}(X_{n,j}>x)-\mathrm{P}\,(X_{n,j}>x)|dx\ . \end{aligned} \tag{2.15}$$

According to Lemma 2.1, the relation (2.13) follows from the estimate (2.15) of (2.14) if we can show that

$$\lim_{n\to\infty}\sum_{h=1}^{k_n} \Sigma_{A_h}\,\mathrm{E}\,|\mathrm{P}_{h-1}(X_{n,j}>x)-\mathrm{P}\,(X_{n,j}>x)|=0\ , \tag{2.16}$$

for all $x>0$ in the continuity set of H. Since each of the B-blocks of indices contains about $q\gamma_n$ terms, the index j of the random variable $X_{n,j}$ in (2.16) differs from the indices of the random variable in the conditioning set by at least $q\gamma_n$. Thus, by stationarity, the sum (2.16) is at most equal to

$$n\sup_{q'\geqq q}\mathrm{E}\,|\mathrm{P}\,(X_{n,0}>x\,|\,\mathcal{F}(-q'\gamma_n))-\mathrm{P}\,(X_{n,0}>x)|\ .$$

By Proposition 1.2, the latter is at most equal to

$$n\,\mathrm{E}\,|\mathrm{P}\,(X_{n,0}>x\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}\,(X_{n,0}>x)|\ ,$$

which, by (1.12), converges to 0 for $n\to\infty$. Hence, (2.16) has the limit 0, and so the proof of Theorem 1.1 is complete.

3. Application to functions defined on a stationary sequence

Stationary arrays arise naturally as follows. Let $\{X_j\}$ be a stationary sequence with values in some measure space $(\boldsymbol{X}, \mathcal{M})$, and let $f_n(x)$, $x\in\boldsymbol{X}$, $n\geqq 1$, be a sequence of nonnegative measurable functions. Then

$$X_{n,j}=f_n(X_j) \tag{3.1}$$

defines a stationary array, and $\mathcal{F}(h)$ may be defined as the sigma field generated by X_j, $j\leqq h$.

Suppose that the marginal distribution of X_0 and the sequence $\{f_n\}$ are such that the conditions (1.1) and (1.2) of assumption i) of Theorem 1.1 are valid. Our interest is now in the mixing assumption ii). The Poisson limit theorem for dependent random variables has traditionally included two kinds of mixing assumptions, local and global. The former limits the number of random variables in a relatively small time set that can significantly contribute to the sum. The latter specifies the rate at which the parts of the process become independent as their index sets become separated in time. The Poisson theorem has been very useful, particularly in extreme value theory, because the global mixing conditions have been relatively easy to verify. For example, the conditions are generally weaker than the global mixing conditions used to prove the Central Limit Theorem. See [17]. However, the local mixing condition of the Poisson theorem is generally not used in the Central Limit Theorem, so that the requirements of the two theorems are not strictly comparble.

The conditions (1.3) and (1.4) place upper and lower bounds, respectively, on the growth of the sequence $\{\gamma_n\}$: If (1.3) holds, then it holds for any $\{\gamma'_n\}$ with $\gamma'_n\leqq\gamma_n$ for all large n; and, if (1.4) holds, then it holds for any $\{\gamma'_n\}$ with $\gamma_n\leqq\gamma'_n$ for all large n. The strength of the local mixing is described by the growth of the most rapidly growing sequence $\{\gamma_n\}$ for which (1.3) holds. The strength of the global mixing is, similarly, described by the growth of the least rapidly growing sequence for which (1.4) holds.

The local and global mixing conditions can often be established by a suitable selection of $\{\gamma_n\}$.

According to (1.12) an equivalent condition for our global mixing condition (1.4) is

$$n\,\mathrm{E}\,|\mathrm{P}\,(f_n(X_0)>x\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}\,(f_n(X_0)>x)|\to 0\,;$$

the latter is implied by

$$n\sup_C \mathrm{E}\,|\mathrm{P}\,(X_0\in C\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}\,(X_0\in C)|\to 0\,, \tag{3.2}$$

where C is an arbitrary measurable subset of the state space. Put

$$\alpha(m)=\sup_C \mathrm{E}\,|\mathrm{P}\,(X_0\in C\,|\,\mathcal{F}(-m))-\mathrm{P}\,(X_0\in C)|\,;$$

then (3.2) is equivalent to

$$n\alpha(q\gamma_n)\to 0\,. \tag{3.3}$$

If $\{\alpha(m)\}$ is known for a given process, then, in verifying (1.3), $\{\gamma_n\}$ can be chosen as small as desired subject only to the minimal growth rate required by (3.3).

As an example, consider a stationary Markov chain $\{X_j\}$. Under Doeblin's hypothesis [8], p. 217, we have $\alpha(m)=O(\rho^m)$ for some $0<\rho<1$. According to (3.3), any sequence $\{\gamma_n\}$ such that $\gamma_n/\log n\to\infty$ is sufficient for global mixing. If, for such a sequence $\{\gamma_n\}$, the relation (1.3) holds, then the conditions of assumption ii) of Theorem 1.1 are satisfied. In the case of the Markov chain, the relation (1.3) is reducible to a condition on the transition probability function because, by the Markov property,

$$\mathrm{P}(f_n(X_j)>x, f_n(X_0)>x)=\int_{\{f_n(X_0)>x\}} \mathrm{P}(f_n(X_j)>x\,|\,X_{j-1})dP .$$

Leadbetter [11] recently introduced a method for relaxing the local mixing condition for the Poisson limit theorem.

4. Application to a particular class of functions defined on a stationary sequence in R^d

Let $\{X_j\}$ be a stationary sequence taking values in R^d, $d\geqq 1$. Let G be a fixed measurable subset of R^d, $\chi_G(x)$ the indicator of G, and

$$f_n(x)=\chi_G(n^{1/d}x) . \tag{4.1}$$

As in (3.1) define the array $\{X_{n,j}\}$ and the sigma-fields $\mathcal{F}(h)$. Let $\mathrm{mes}(G)$ be d-dimensional Lebesgue measure, and suppose $0<\mathrm{mes}(G)<\infty$. We prove the following limit theorem for $X_{n,1}+\cdots+X_{n,n}$:

THEOREM 4.1. *Assume that*

$$\int_{R^d} |\mathrm{E}\, e^{iu'X_0}|du<\infty ; \tag{4.2}$$

$$\int_{R^d} \mathrm{E}\,|\mathrm{E}\,(e^{iu'X_0}|\mathcal{F}(-1))|du<\infty ; \tag{4.3}$$

and

(4.4) $\mathcal{F}(-\infty)$ *contains only sets of probability* 0 *and* 1 .

Then $f_n(X_1)+\cdots+f_n(X_n)$ *has a limiting Poisson distribution with mean*

$$\lambda=\frac{\mathrm{mes}(G)}{(2\pi)^d}\int_{R^d} \mathrm{E}\, e^{iu'X_0}du . \tag{4.5}$$

(*Here* $u'X$ *is the usual scalar product.*)

Remark. Under (4.2), the integral in (4.5) is proportional to the density of X_0 at the origin. Thus, it is nonnegative. If it is equal to 0, then $\lambda=0$ and the Poisson distribution is degenerate.

PROOF. We apply Corollary 1.1. (4.1) implies that $X_{n,0}=1$ if and only if $n^{1/d}X_0 \in G$. Hence, by (4.2) and the inversion formula for the characteristic function,

$$
(4.6) \qquad n\,\mathrm{P}(X_{n,0}=1)=(2\pi)^{-d}\int_{R^d}\int_G e^{-i(u'y)n^{-1/d}}\,\mathrm{E}\,e^{iu'X_0}dydu \\
\to \frac{\operatorname{mes}(G)}{(2\pi)^d}\int_{R^d}\mathrm{E}\,e^{iu'X_0}du\ ;
$$

therefore, (1.7) holds.

According to Proposition 1.2, the condition (4.3) implies that

$$
\int_{R^d}\mathrm{E}\,|\mathrm{E}\,[e^{iu'X_0}|\mathcal{F}(-h)]|du<\infty
$$

for all $h\geqq 1$. The sequence $\mathrm{E}\{\exp(iu'X_0)|\mathcal{F}(-h)\}$, $h\geqq 1$, is a reversed martingale, and so, by (4.4) and the martingale convergence theorem, $\mathrm{E}|\mathrm{E}\{\exp(iu'X_0)|\mathcal{F}(-h)\}-\mathrm{E}\{\exp(iu'X_0)\}|\to 0$, for every u. Furthermore, by Proposition 1.2, the convergence is monotonic, so that

$$
(4.7) \qquad \lim_{h\to\infty}\int_{R^d}\mathrm{E}\,|\mathrm{E}\,(e^{iu'X_0}|\mathcal{F}(-h))-\mathrm{E}\,(e^{iu'X_0})|du=0\ .
$$

By the same calculation as in (4.6) it follows that

$$
n\,\mathrm{E}\,|\mathrm{P}\,(X_{n,0}=1\,|\,\mathcal{F}(-h))-\mathrm{P}\,(X_{n,0}=1)| \\
=(2\pi)^{-d}\,\mathrm{E}\left|\int_{R^d}\left\{\int_G e^{-i(u'y)n^{-1/d}}dy\right\}\{\mathrm{E}\,(e^{iu'X_0}|\mathcal{F}(-h))-\mathrm{E}\,e^{iu'X_0}\}du\right| \\
\leqq \frac{\operatorname{mes}(G)}{(2\pi)^d}\int_{R^d}\mathrm{E}\,|\mathrm{E}\,(e^{iu'X_0}|\mathcal{F}(-h))-\mathrm{E}\,e^{iu'X_0}|du\ .
$$

Thus it follows from (4.7) that

$$
\lim_{h\to\infty} n\,\mathrm{E}\,|\mathrm{P}\,(X_{n,0}=1\,|\,\mathcal{F}(-h))-\mathrm{P}\,(X_{n,0}=1)|=0\ ,
$$

uniformly in n. In particular, for any sequence $\{\gamma_n\}$ such that $\gamma_n\to\infty$ and for any $q>0$,

$$
(4.8) \qquad \lim_{n\to\infty} n\,\mathrm{E}\,|\mathrm{P}\,(X_{n,0}=1\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}\,(X_{n,0}=1)|=0\ .
$$

The relation (4.6) and Lemma 1.1 now imply that (4.8) is equivalent to the condition (1.9).

In order to complete the proof of the theorem, it suffices to show that there exists a sequence $\{\gamma_n\}$ such that $\gamma_n\to\infty$ and (1.8) holds. The expression in (1.8) takes the particular form,

$$n \sum_{1 \leq j \leq r_n} \mathrm{P}\,(n^{1/d}X_0 \in G,\ n^{1/d}X_j \in G)\ , \tag{4.9}$$

which, by the inversion formula, is expressible as

$$(2\pi)^{-d}\, n \sum_{1 \leq j \leq r_n} \mathrm{E}\left\{\chi_G(n^{1/d}X_{-j}) \int_{R^d} \left(\int_G e^{-iu'y}dy\right) \mathrm{E}\,[e^{iu'X_0 n^{1/d}} \,|\, \mathcal{F}(-j)]du\right\}$$

which is, by reasoning similar to that in (4.6), at most equal to

$$\frac{\text{mes}\,(G)}{(2\pi)^d} \sum_{1 \leq j \leq r_n} \mathrm{E}\left\{\chi_G(n^{1/d}X_{-j}) \int_{R^d} |\mathrm{E}\,(e^{iu'X_0} \,|\, \mathcal{F}(-j))|du\right\}\ . \tag{4.10}$$

The term of fixed index $j \geqq 1$ in (4.10) converges to 0 for $n \to \infty$; this follows from the convergence of $\chi_G(n^{1/d}X_{-j})$ in the mean to 0, and from the domination of the expectation of the integral in (4.10) in accordance with (4.3).

Let $A_{n,k}$ be the partial sum of the terms in (4.10),

$$A_{n,k} = \sum_{j=1}^{k} \mathrm{E}\left\{\chi_G(n^{1/d}X_{-j}) \int_{R^d} |\mathrm{E}\,(e^{iu'X_0} \,|\, \mathcal{F}(-j))|du\right\}\ ;$$

then $A_{n,k} \to 0$ for $n \to \infty$, for each k. It can be shown by an elementary argument that there exists a sequence of positive integers $\gamma_n \to \infty$ such that $A_{n,\gamma_n} \to 0$. This sequence is used in (4.10) to get the limit 0, so that (4.9) has the same limit.

Example 4.1. Let Y_j, $-\infty < j < \infty$, be a sequence of i.i.d. random variables, and define

$$X_j = \sum_{h=0}^{\infty} c_h Y_{j-h}\ , \qquad j = 0, \pm 1, \cdots\ , \tag{4.11}$$

for a real sequence (c_h). Put $\phi(u) = \mathrm{E}\,[\exp\,(iuY_0)]$; then the series (4.11) converges almost surely if and only if the infinite product

$$\pi(u) = \prod_{j=0}^{\infty} \phi(uc_j) \tag{4.12}$$

converges for all u. In this case (4.11) defines a stationary sequence $\{X_j\}$. A sufficient condition for the hypothesis of Theorem 4.1 is

$$\int_{-\infty}^{\infty} |\phi(u)|du < \infty\ . \tag{4.13}$$

Since $|\phi(u)| \leqq 1$ for all u, (4.13) implies the integrability of $|\pi(u)|$, which is the modulus of the characteristic function of X_0; thus (4.2) holds. Since X_j, with $j \leqq -h$, are measurable with respect to the sigma-field generated by Y_j, $j \leqq -h$, and since the Y_j are mutually independent, Proposition 1.2 implies

$$\int_{-\infty}^{\infty} \mathrm{E}\,|\mathrm{E}\,(e^{iuX_0}|\mathcal{F}(-1))|\,du$$
$$\leqq \int_{-\infty}^{\infty} \mathrm{E}\,|\mathrm{E}\,(e^{iuX_0}|Y_j,\, j\leqq -1)|\,du$$
$$= \int_{-\infty}^{\infty} \mathrm{E}\,|\mathrm{E}\, e^{iuY_0c_0}|\,du = \int_{-\infty}^{\infty} |\phi(c_0u)|\,du < \infty\ ;$$

thus (4.3) holds. Since the tail field of the X's is contained in that of the Y's, and the latter have a trivial tail field, the same is true of the X's so that (4.4) also holds. This example obviously generalizes to random vectors in R^d.

Example 4.2. Let $\{X_j\}$ be a stationary sequence in R^d satisfying the conditions in the hypothesis of Theorem 4.1; and let $\|X_j\|$ be the usual Euclidian norm. Theorem 4.1 implies

$$\lim_{n\to\infty} \mathrm{P}\,(n^{1/d} \min_{1\leq j\leq n} \|X_j\| > x) = \exp\,(-cx^d)\ , \tag{4.14}$$

for all $x>0$, for a specified constant $c>0$. According to the remark following Theorem 4.1, we will assume that the integral in (4.5) is not equal to 0, so that the Poisson distribution is not degenerate. The result (4.14) is apparently new even in the case of independent X's.

For the proof of (4.14), let G be the ball with center at the origin, and of radius $x>0$; then, as is well known,

$$\mathrm{mes}\,(G) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}\, x^d\ . \tag{4.15}$$

If $f_n(x)$ is defined as in (4.1), then the event $n^{1/d}\min\,(\|X_1\|,\cdots,\|X_n\|)>x$ is identical with the event $\sum_{j=1}^{n} f_n(X_j)=0$. By Theorem 4.1, the sum has a limiting Poisson distribution, so that the probability of the latter event converges to $e^{-\lambda}$, where λ is determined by (4.5) with the particular form (4.15).

We note that a version of the Poisson limit theorem for the visits of a stationary Gaussian sequence to a shrinking neighborhood of the origin was given by Zahle [18].

Theorem 4.1 can be extended to the case where χ_G in (4.1) is replaced by an arbitrary nonnegative measurable function.

THEOREM 4.2. *Let $f(x)$, $x \in R^d$, be a nonnegative measurable function such that*

$$\mathrm{mes}\,\{x : f(x) > y\} < \infty\ , \qquad \textit{for } y>0\ , \tag{4.16}$$

and

$$\int_{R^d} \min(1, f(x))dx < \infty . \tag{4.17}$$

Assume (4.2), (4.3) *and* (4.4), *and define*

$$H(y) = \frac{\text{mes}\{x: f(x) > y\}}{(2\pi)^d} \int_{R^d} \mathrm{E}\, e^{iu'X_0} du , \tag{4.18}$$

and

$$S_n = \sum_{j=1}^{n} f(n^{1/d} X_j) . \tag{4.19}$$

Then S_n *has, for* $n \to \infty$, *a limiting distribution with the Laplace-Stieltjes transform* (1.5). *The distribution is degenerate if and only if* f *is constant or the integral in* (4.18) *is equal to* 0.

PROOF. The proof in the case of independent $\{X_j\}$ is a routine computation involving the probability $\mathrm{P}(f(n^{1/d}X_0) > y) = \mathrm{P}(n^{1/d}X_0 \in f^{-1}(y, \infty))$. The verification of the conditions of assumption ii) of Theorem 1.1 in the general case is similar to that of Theorem 4.1. The details are omitted.

5. Application to the limiting distribution of the maximum in a stationary sequence

Let $\{X_j, j=0, \pm 1, \cdots\}$ be a real stationary sequence, and define $Z_n = \max(X_1, \cdots, X_n)$, for $n \geqq 1$. We will apply Corollary 1.1 to the distribution of Z_n for $n \to \infty$. Suppose that the marginal distribution $F(x)$ of X_0 is in the domain of attraction of an extreme value limiting distribution $G(x)$: If $Y_1, \cdots, Y_n$ are independent random variables with the common distribution function F, then there are sequences $\{a_n\}$ and $\{b_n\}$ with $a_n > 0$ such that $\mathrm{P}(\max(Y_1, \cdots, Y_n) \leqq a_n x + b_n) \to G(x)$, for all x. The condition is equivalent to

$$\lim_{n \to \infty} n(1 - F(u_n(x))) = -\log G(x) , \tag{5.1}$$

where

$$u_n(x) = a_n x + b_n . \tag{5.2}$$

Here we give conditions under which Z_n has the same limiting distribution as $\max(Y_1, \cdots, Y_n)$. By the standard argument we identify $X_{n,j}$ as the indicator of the event $X_j > a_n x + b_n$, and apply Corollary 1.1 to obtaine $e^{-\lambda}$ as the limiting distribution of the maximum.

THEOREM 5.1. *Let the distribution function* F *of* X_0 *satisfy* (5.1). *If there is a sequence* $\{\gamma_n\}$ *such that*

$$n \sum_{1 \leq j \leq r_n} \mathrm{P}(X_0 > u_n(x),\, X_j > u_n(x)) \to 0 \tag{5.3}$$

for every x, and

$$\mathrm{p}\lim_{n\to\infty} \frac{\mathrm{P}(X_0 > u_n(x) \mid \mathcal{F}(-q r_n))}{\mathrm{P}(X_0 > u_n(x))} = 1 \tag{5.4}$$

for every x and $q>0$, then $\mathrm{P}(\max(X_1, \cdots, X_n) \leq a_n x + b_n) \to G(x)$, *for all x.*

This follows from Corollary 1.1 by (5.1) and the identification $\lambda = -\log G(x)$.

We will apply Theorem 5.1 to a stationary Gaussian sequence. If $F(x)$ is the standard normal distribution function, then, as is well known, it is in the domain of attraction of $G(x) = \exp(-e^{-x})$ with

$$a_n = (2\log n)^{-1/2} \qquad b_n = a_n^{-1} - \frac{1}{2} a_n (\log\log n + \log 4\pi) \tag{5.5}$$

in (5.2). Suppose that $\mathcal{F}(-\infty)$ is trivial; then, by the classical Wold decomposition, the sequence $\{X_j\}$ has the representation (see [8], p. 576)

$$X_j = \sum_{h=0}^{\infty} c_h Y_{j-h}\,, \tag{5.6}$$

where $\{Y_i\}$ is a sequence of independent, standard normal variables, and

$$\sum_{h=0}^{\infty} c_h^2 = 1\,. \tag{5.7}$$

THEOREM 5.2. *If $\{X_j\}$ is of the form* (5.6), *and*

$$\lim_{n\to\infty} (\log n) \sum_{h=n}^{\infty} c_h^2 = 0\,, \tag{5.8}$$

then $(Z_n - b_n)/a_n$, with a_n and b_n in (5.5), *has the same limiting distribution* $\exp(-e^{-x})$.

PROOF. Put $r_n = \mathrm{cov}(X_0, X_n)$; then (5.6) implies

$$r_n = \sum_{h=0}^{\infty} c_h c_{h+n}\,, \tag{5.9}$$

and (5.7) implies $r_n \to 0$ for $n \to \infty$. The latter implies the existence of δ, $0<\delta<1$, such that [1]

$$\delta = \sup_{n\geq 1} |r_n|\,. \tag{5.10}$$

Then, for any c such that

$$0 < c < \frac{1-\delta}{1+\delta}\,, \tag{5.11}$$

we define $\{\gamma_n\}$ as an integer valued sequence such that

$$n^c \leqq \gamma_n < (n+1)^c . \tag{5.12}$$

We will show that $\{\gamma_n\}$ satisfies the requirements of (5.3) and (5.4). It follows from (5.5) that

$$u_n^2(x) = 2 \log n - \log \log n + O(1) \tag{5.13}$$

for $n \to \infty$. The formula [5], p. 27 implies

$$\mathrm{P}(X_0 > u_n, X_j > u_n) \leqq \mathrm{P}^2(X_0 > u_n) + (2\pi)^{-1} \int_0^{\delta} (1-y^2)^{-1/2} \exp\left(-\frac{u_n^2}{1+y}\right) dy ,$$

which, by (5.13), and the well known asymptotic formula for the tail of the standard normal distribution,

$$1 - \Phi(u) \sim \Phi'(u)/u , \tag{5.14}$$

implies

$$\sup_{j \geqq 1} \mathrm{P}(X_0 > u_n, X_j > u_n) = O(n^{-2/(1+\delta)}) .$$

The latter relation and (5.11) and (5.12) imply that the left hand member of (5.3) tends to 0, so that (5.3) is confirmed.

Now we verify (5.4). The conditional distribution of X_0, given X_j, $j \leqq -m$ (which is identical with conditioning by Y_j, $j \leqq -m$) is normal with mean

$$\hat{X}_m = \sum_{i=m}^{\infty} c_i Y_{-i}$$

and variance

$$\sigma_m^2 = \sum_{i=0}^{m-1} c_i^2 .$$

Therefore, the ratio in (5.4) is equal to

$$\frac{1 - \Phi((u_n - \hat{X}_m)/\sigma_m)}{1 - \Phi(u_n)} , \tag{5.15}$$

where $m \sim q\gamma_n$. (5.14) implies that the ratio (5.15) is asymptotically equal to

$$\exp\left\{-\frac{1}{2}\left[\frac{(u_n - \hat{X}_m)^2}{\sigma_m^2} - u_n^2\right]\right\} .$$

To verify (5.4), it suffices to show that the exponent converges in probability to 0. The latter may be written as

$$-\frac{1}{2}\left(\frac{u_n-\hat{X}_m}{\sigma_m}-u_n\right)\left(\frac{u_n-\hat{X}_m}{\sigma_m}\right)+u_n\right) ,$$

which is asymptotically in probability equal to

$$-u_n^2(1-\sigma_m)-\frac{1}{2}u_n\hat{X}_m . \tag{5.16}$$

By (5.8) and (5.13), we have

$$\lim_{n\to\infty} u_n^2(1-\sigma_n)=\lim_{n\to\infty} (2\log n)\frac{1-\sigma_n^2}{1+\sigma_n}=0 . \tag{5.17}$$

Since $\hat{X}_m$ and $X_0-\hat{X}_m$ are independent, $\operatorname{var}\hat{X}_m=\operatorname{var}X_0-\operatorname{var}(X_0-\hat{X}_m)=1-\sigma_m^2$. Thus, by (5.13)

$$\operatorname{var}(u_n\hat{X}_m)\sim(2\log n)(1-\sigma_m^2) ,$$

which, by the assumption $m\sim qn^c$, is asymptotically equal to $(2/c)(\log m)\cdot(1-\sigma_m^2)$, which, by (5.17), converges to 0. From this and from (5.17) it follows that the expression (5.16) converges in probability to 0. This completes the verification of (5.4). The proof of the theorem is complete.

The limit theorem for the maximum of a stationary Gaussian process was shown by the author [1] to hold if either $r_n\log n\to 0$ or $\sum r_n^2<\infty$. This result was generalized by Leadbetter, Lindgren and Rootzen [12], and also involves the rate of decay of r_n. The current condition (5.8) is novel because the relation between the asymptotic properties of the sequence $\{r_n\}$ and $\{1-\sigma_n^2\}$ at the rate of decay $(\log n)^{-1}$ is apparently too complicated to be usefully recorded. The only simple relation between the two sequences is $r_n^2\leqq 1-\sigma_n^2$, so that (5.8) implies only the condition $r_n^2\log n\to 0$, which is stronger than the condition $r_n\log n\to 0$.

Finster [9] recently derived the limiting distribution of the maximum for certain classes of non-Gaussian stationary sequences which have the representation (5.6). He also used the Lévy-Loève comparison method.

6. Extension to nonergodic arrays with mixed limiting distributions

In this section we will assume, as we already have in the applications in Sections 3–5, that the stationary sequences $\{X_{n,j},\ j=0,\pm 1,\cdots\}$ are really defined on the same space, and that the sigma-fields generated by $\{X_{n,j},\ j\leqq h\}$ are identical for $n\geqq 1$, for each h. The mixing condition (1.4) requires that the tail field have no role in the limiting distribution. Indeed, by Proposition 1.2, it implies that

$$\operatorname{p}\lim_{n\to\infty} \frac{\mathrm{P}(X_{n,0}>x\,|\,\mathcal{F}(-\infty))}{\mathrm{P}(X_{n,0}>x)}=1 .$$

Let $\mathcal{I}$ be the invariant sigma-field of the process. Since $\mathcal{I}\subset\mathcal{F}(-\infty)$, it also has no role in the limiting distribution.

Loève [15], p. 44, showed that the comparison method, which we used in the proof of Theorem 1.1, can be extended to the case of sequences where there is conditional mixing, given a sub-sigma-field. In this case, the conditional distributions have limits which depend on the sub-sigma-field, and the unconditional distribution is obtained from these by integration. The mixing condition (1.4) has a relatively uncomplicated extension to conditional mixing when the conditioning sigma-field is contained in $\mathcal{F}(-\infty)$.

THEOREM 6.1. *Let* $\{X_{n,j},\ -\infty<j<\infty\}$, $n\geqq 1$, *be stationary sequences on a probability space with (common) invariant sigma-field* $\mathcal{I}$. *Assume the condition* i) *of Theorem* 1.1 *and also its conditional form given* $\mathcal{I}$, *namely, there exists a random* $\mathcal{I}$*-measurable monotone function* $H_\omega(x)$ *such that*

$$\int_0^\infty \min(x,1)dH_\omega(x)<\infty \tag{6.1}$$

almost surely, and

$$n\int_0^x \min(y,1)d\,\mathrm{P}(X_{n,0}>y\,|\,\mathcal{I})\to\int_0^x \min(y,1)dH_\omega(y) \tag{6.2}$$

in the sense of complete convergence, almost surely. Condition ii) *is modified*: (1.3) *is retained, but* (1.4) *is replaced by*

$$\lim_{n\to\infty} n\,\mathrm{E}\,|\mathrm{P}(X_{n,0}>x\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}(X_{n,0}>x\,|\,\mathcal{I})|=0 . \tag{6.3}$$

Then $X_{n,1}+\cdots+X_{n,n}$ *has a limiting distribution with the Laplace-Stieltjes transform*

$$\mathrm{E}\left\{\exp\left[\int_0^\infty (1-e^{-sx})dH_\omega(x)\right]\right\} . \tag{6.4}$$

Remark. By (1.12), condition (6.3) is equivalent to (1.4) when $\mathcal{I}$ is trivial.

PROOF. We adapt the proof of Theorem 1.1.

First of all, Lemma 2.1 remains valid because it depends only on assumption i) of Theorem 1.1, which is also assumed here. The index set is, as before, decomposed into blocks such that each A-block has approximately $p\gamma_n$ elements. The relation (2.5) remains valid.

The Z-array is defined as follows: For each n, the random variables $Z_{n,j}$, $j=0, \pm 1, \cdots$, are conditionally independent, given $\mathcal{J}$, with the common distribution identical with the conditional distribution of $X_{n,0}$, given $\mathcal{J}$. The Z's are also assumed to be conditionally independent of the X's.

The theorem is indeed valid for the Z-array. As in Remark 2 following Theorem 1.1, the logarithm of the conditional Laplace-Stieltjes transform of $Z_{n,1}+\cdots+Z_{n,n}$ is $n \log \mathrm{E}(\exp(-sZ_{n,0})|\mathcal{J})$. Under our hypothesis, this converges to $\int_0^\infty (1-e^{-sx})dH_\omega(x)$. Hence, the unconditional limiting Laplace-Stieltjes transform is equal to (6.4).

We prove the theorem for the X-array by comparing it to the conditioned Z-array. As in the proof of Theorem 1.1, we define the conditional expectation operators E_h relative to $\mathcal{F}(j)$, $j \in A_h$, $h=1, \cdots, k_n$. Now we also define the conditional expectation operator E_ω relative to $\mathcal{J}$.

Next we apply the Lèvy-Loéve method in conditional form. Put

$$\xi_h=\exp(-s\Sigma_{A_h}X) \qquad \text{and} \qquad \eta=\exp(-s\Sigma_{A_h}Z) ,$$

and estimate

$$\left|\mathrm{E}\prod_{h=1}^{k_n}\xi_h-\mathrm{E}\prod_{h=1}^{k_n}\eta_h\right| . \tag{6.5}$$

The latter is dominated by

$$\mathrm{E}\,|\mathrm{E}_\omega \prod_h \xi_h-\mathrm{E}_\omega \prod_h \eta_h| ,$$

which, by the conditional independence of (η_h) and their $\mathcal{J}$-measurability, and by the inclusion $\mathcal{J}\subset \bigcap_j \mathcal{F}(j)$, is equal to

$$\mathrm{E}\,|\mathrm{E}_\omega [\prod_h \mathrm{E}_{h-1}(\xi_h)-\prod_h \mathrm{E}_\omega(\eta_h)]| .$$

The latter is dominated by

$$\mathrm{E}\,|\prod_h \mathrm{E}_{h-1}(\xi_h)-\prod_h \mathrm{E}_\omega(\eta_h)| ,$$

which is at most equal to

$$\sum_h \mathrm{E}\,|\mathrm{E}_{h-1}(\xi_h)-\mathrm{E}_\omega(\eta_h)| , \tag{6.6}$$

because $|\prod \alpha_i-\prod \beta_i|\leqq\sum|\alpha_i-\beta_i|$ for $|\alpha_i|\leqq 1$ and $|\beta_i|\leqq 1$. The sum in (6.6) is, by definition, equal to

$$\sum_{h=1}^{k_n}\mathrm{E}\,|\mathrm{E}_{h-1}[\exp(-s\Sigma_{A_h}X)]-\mathrm{E}_\omega[\exp(-s\Sigma_{A_h}Z)]| . \tag{6.7}$$

By exactly the same argument as that leading from (2.6) to (2.13), it follows that the convergence to 0 of the sum (6.7) is implied by that of

(6.8) $$\sum_{h=1}^{k_n} \Sigma_{A_h} \mathrm{E}|\mathrm{E}_{h-1}(e^{-sX}-1)-\mathrm{E}_\omega(e^{-sZ}-1)| .$$

According to the part of the proof of Theorem 1.1 which follows (2.13), the convergence to 0 of (6.8) is implied by that of

$$n\,\mathrm{E}|\mathrm{P}(X_{n,0}>x\,|\,\mathcal{F}(-q\gamma_n))-\mathrm{P}(X_{n,0}>x\,|\,\mathcal{J})| ,$$

and the latter convergence is assumed in (6.3). This shows that the expression (6.5) converges to 0, and this completes the proof.

As an application, we remark that Theorem 4.1 has a direct extension to the present context. If condition (4.4) is replaced by the condition

(6.9) $$\mathcal{J}=\mathcal{F}(-\infty) ,$$

then the conclusion of the theorem remains valid for the conditional limiting distribution of the sum when the constant λ in (4.5) is replaced by the random variable

$$\lambda_\omega=\frac{\mathrm{mes}\,(G)}{(2\pi)^d}\int_{R^d}\mathrm{E}\,(e^{iu'X_0}\,|\,\mathcal{J})du .$$

Thus the unconditional limiting distribution is a mixture of Poisson distributions. The proof of this extension is a direct adaptation of the one in Section 4: The conditional characteristic function is used in the place of the unconditional one.

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