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EXTENSIONS OF WILKS' INTEGRAL EQUATIONS AND DISTRIBUTIONS OF TEST STATISTICS

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Summary

Wilks [26] introduced two integral equations in connection with distribution problems in statistics. He called them Type A and Type B equations. Tretter and Walster ([22], [24]) solved the Type B equation and obtained the null and non-null distributions of the likelihood ratio criterion for testing linear hypotheses in the multinormal case. In this article we present several types of solutions of these equations along with new equations called Types C, D, E and F with their solutions. These include the integral equations satisfied by the density of a random variable which is (a) product of independent real gamma variates; (b) products of independent real beta variates; (c) ratio of products of independent beta and gamma variates; (d) arbitrary powers of products of gamma and beta variates; (e) arbitrary powers of products and ratios of beta and gamma variates, and more general cases.

1. Introduction

In order to give a technique of deriving the distribution of a statistic when its hth moment is available Wilks [26] introduced two integral equations called Type A and Type B equations. Consider the equation

$$(1.1)\qquad \int_0^\infty x^h f(x)dx = B^h \prod_{j=1}^p \{\Gamma(\alpha_j+h)/\Gamma(\alpha_j)\} \qquad \text{(Type A)}$$

where h and α_j's are real and positive, B and $f(x)$ are free of h. This he called Type A equation. Type B is given by (1.2).

$$(1.2)\qquad \int_0^B x^h f(x)dx = CB^h \prod_{j=1}^p \{\Gamma(b_j+h)/\Gamma(c_j+h)\} \qquad \text{(Type B)}$$

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where $C=\prod_{j=1}^{p}\{\Gamma(c_j)/\Gamma(b_j)\}$ and B and $f(x)$ are free of h. The b_j's and c_j's are two sets of positive numbers such that there exists at least one way of pairing them so that each b_i is less than the corresponding c_i. Without loss of generality we will assume that $b_i<c_i$, $i=1, \cdots, p$. Type A equation can result from the following situation. Let $X=X_1\cdots X_p$ where $X_1, \cdots, X_p$ are independent real gamma variates with the densities

$$f_i(x_i)=\{d_i^{a_i}\Gamma(a_i)\}^{-1}x_i^{a_i-1}e^{-x_i/d_i}\,, \quad x_i>0\,,\quad a_i>0\,,\quad d_i>0\,,\quad i=1,\cdots,p \tag{1.3}$$

and $f_i(x_i)=0$ elsewhere. If $f(x)$ is the density of X then the hth moment of X is given by (1.1) with $B=d_1d_2\cdots d_p$. Thus X is structurally the product of p independent real gamma variates. In (1.2) change X to X/B then B will disappear from the right side. Hence for convenience we will assume $B=1$ in the following discussion. Let $X=Y_1\cdots Y_p$ where $Y_1, \cdots, Y_p$ are independent real beta variables with the density functions

$$g_i(y_i)=\{\Gamma(\alpha_i+\beta_i)/\Gamma(\alpha_i)\Gamma(\beta_i)\}\,x_i^{\alpha_i-1}(1-x_i)^{\beta_i-1}\,, \quad 0<x_i<1\,,\quad \alpha_i>0\,,\quad \beta_i>0\,,\quad i=1,\cdots,p \tag{1.4}$$

and $g_i(y_i)=0$ elsewhere. Then the hth moment of X with $\alpha_i=b_i$, $\alpha_i+\beta_i=c_i$, $i=1, \cdots, p$ gives (1.2) for $B=1$. Thus $f(x)$ in (1.2) can be looked upon as the density of X where X is structurally a product of p independent real beta variables. X in (1.1) is associated with the determinant of the sample covariance matrix when the sample comes from a multivariate normal distribution. X in (1.2) is associated with several test statistics for testing hypotheses on the parameters of one or more multivariate normal populations. It is also connected with problems in geometric probability, see for example Mathai [8]. Since Type B is relatively more important than Type A as far as statistical applications are concerned we will look into the solution of Type B first. Throughout this article an empty product is interpreted as unity and the corresponding empty sum as zero. All the parameters are assumed to be real unless otherwise specified.

2. Solutions of Type B integral equation

Wilks [26] gave a multiple integral representation for the solution of Type B equation. This is obtained by working out the density of a product of p independent real beta variables by the method of transformation of variables. His solution for $B=1$ is the following.

$$f(x)=Kx^{b_p-1}(1-x)^{\gamma_p-\beta_p-1}\int_0^1\cdots\int_0^1\left\{\prod_{i=1}^{p-1} v_i^{c_i-b_i-1}(1-v_i)^{\gamma_{p-i}-\beta_{p-i}-1}\xi_i^{b_i-c_{i+1}}dv_i\right\} \tag{2.1}$$

where

$$\xi_i=\{v_1+v_2(1-v_1)+\cdots+v_i(1-v_1)(1-v_2)\cdots(1-v_{i-1})\}(1-x)\,, \tag{2.2}$$

$$\gamma_i=\sum_{j=0}^{i-1} c_{p-j}\,,\quad \beta_i=\sum_{j=0}^{i-1} b_{p-j}\,,\quad K=\prod_{i=1}^{p}\Gamma(c_i)/\{\Gamma(b_i)\Gamma(c_i-b_i)\}\,.$$

By induction he showed that $0<\xi_i<1$ for $i=1,\cdots,p-1$. By successive integration of (2.1) Wilks [25], [26] derived the exact densities in particular cases for the likelihood ratio statistics for testing general linear hypothesis and independence of subvectors in multinormal populations. By using Wilks' multiple integral representation Wald and Brookner [23] derived the null density for the problem of testing independence.

Nair [20] showed that $f(x)$ of (1.2) satisfied the differential equation

$$x\prod_{j=1}^{p}\left(x\frac{d}{dx}-c_j+2\right)f(x)=B\prod_{j=1}^{p}\left(x\frac{d}{dx}-b_j+1\right)f(x)\,. \tag{2.3}$$

He gave a general method of solving such a differential equation with the help of the method of Frobenius and calculus of residues. We will show in Section 3 that the differential equation (2.3) is a particular case of a differential equation coming from a more general moment structure and the general solutions of such differential equations are also available. In a series of articles Davis, see for example Davis [4], worked out the explicit unique solutions of (2.3) for particular cases and thus obtained the exact distributions of several test statistics in particular cases. Mathai [13] used a general expansion of a G-function and derived an explicit general solution for (1.2) for $B=1$ in the form of the following series which is suitable for computational purposes.

$$f(x)=C\Sigma_j A_j x^{B_j}(-\log x)^{C_j}\,,\qquad 0<x<1 \tag{2.4}$$

and $f(x)=0$ elsewhere, where C, A_j, B_j and C_j do not depend on x. Further, C_j is a non-negative integer such that $0\leqq C_j\leqq p-1$. Explicit forms of A_j, B_j and C_j will be given in Theorem 2 later on. This author has noticed that (2.4) is the most suitable form for computational purposes compared to other series forms which will be discussed in the next paragraphs. Computability of (2.4) is illustrated by computing the exact percentage points for several test statistics, see for example Mathai and Katiyar [14].

Following the suggestion in Wilks [26], Tretter and Walster [22] expanded $\xi_i^{k_i}$ for $i=1,\cdots,p-1$ and obtained the density for Wilks' Λ in the form of a multiple sum. They noted the recursive relationship

$\xi_i = v_i + (1-v_i)\xi_{i-1}$, $i=1, \cdots, p-1$. When k_i is a positive integer $\xi_i^{k_i}$ results in a finite sum when expanded. Thus successive expansions lead to a finite sum. Then successive integrations give the final solution to (2.1). But note that $k_i = b_i - c_{i+1} < 0$ in Wilks' representation and hence $\xi_i^{k_i}$ for $i=1, \cdots, p-1$ give multiple infinite series. They obtained a series of the form

$$f(u) = Cu^a \Sigma_j \sigma_j (1-u)^j \tag{2.5}$$

where C, a and σ_j do not contain u and σ_j's involve multiple sums and products. By using (2.5) they derived the exact null density of Wilks' Λ for testing general linear hypothesis. Later Walster and Tretter [24] extended their results to the non-null case as well as to other test criteria. By using the same procedure Gupta [7] worked out the exact null density of the likelihood ratio criterion for testing sphericity. When p becomes large σ_j's become unmanageably complicated in this multiple sum representation which then becomes unsuitable for computational purposes. This is a drawback in any multiple series representation. If a multiple sum representation is looked for then we will give the following new representation which is more compact and valid for general values of the parameters.

LEMMA 1. *If $Y_1, \cdots, Y_p$ are independent real beta variates with the densities given in* (1.4) *and if $U = Y_1 \cdots Y_p$ then the density of U, denoted by $g(u)$, is given by*

$$g(u) = C_p u^{\alpha_p - 1}(1-u)^{\delta_p - 1} \times \int_0^1 \cdots \int_0^1 \left\{ \prod_{i=1}^{p-1} t_i^{\delta_i - 1}(1-t_i)^{\beta_{i+1}-1}(1-t_{p-1}t_{p-2}\cdots t_{p-i}(1-u))^{-\gamma_{p-i}} dt_i \right\}, \quad 0<u<1 \tag{2.6}$$

and $g(u)=0$ elsewhere, where,

$$C_p = \prod_{i=1}^{p} \{\Gamma(\alpha_i + \beta_i)/\Gamma(\alpha_i)\Gamma(\beta_i)\}, \quad \delta_i = \beta_1 + \cdots + \beta_i, \quad \gamma_i = \alpha_{i+1} + \beta_{i+1} - \alpha_i . \tag{2.7}$$

PROOF. Consider two independent real beta variables Y_1 and Y_2 having densities given in (1.4). By transformation of variables it is trivial to note that the density of $U_1 = Y_1 Y_2$ is given by

$$h_1(u_1) = C_2 u_1^{\alpha_2 - 1} \int_{u_1}^1 t^{\alpha_1 - \alpha_2 - \beta_2}(1-t)^{\beta_1 - 1}(t-u_1)^{\beta_2 - 1} dt$$

where C_2 is defined in (2.7). Make the substitution $1-t_1 = (t-u_1)/(1-u_1)$. Then,

$$h_1(u_1) = C_2 u_1^{\alpha_2 - 1}(1-u_1)^{\beta_1 + \beta_2 - 1} \int_0^1 t_1^{\beta_1 - 1}(1-t_1)^{\beta_2 - 1}(1-t_1(1-u_1))^{-(\alpha_2 + \beta_2 - \alpha_1)} dt_1 .$$

Now by induction and making the above substitution each time the result follows.

The form (2.6) is more convenient compared to Wilks' representation (2.1) with ξ_i defined in (2.2). It is readily seen that each factor in (2.6) is expansible since $0<t_{p-1}t_{p-2}\cdots t_{p-i}(1-u)<1$ for $i=1,\cdots,p-1$. In the persuit of obtaining new results in Special Functions by using statistical techniques and getting a multiple integral representation for a G-function Mathai and Saxena [19] arrived at the representation in (2.6).

THEOREM 1. *A solution of the Type B integral equation given in* (1.2) *with* $B=1$, $b_i=\alpha_i$, $c_i=\alpha_i+\beta_i$, $i=1,\cdots,p$ *is given by*

$$(2.8)\quad g(u)=C_p\left\{\prod_{j=2}^{p}\Gamma(\beta_j)\right\}u^{\gamma_p-1}(1-u)^{\delta_p-1}\sum_{r=0}^{\infty}\Sigma_R \times\prod_{i=1}^{p-1}\{(\gamma_i)_{r_i}\Gamma(\delta_i+R_i)(1-u)^r/(\Gamma(\delta_{i+1}+R_i)r_i!)\}\,,\qquad 0<u<1$$

and $g(u)=0$ *elsewhere, where* C_p, δ_j*'s are defined in* (2.7), $R_j=r_1+\cdots+r_j$, $R=(r_1,\cdots,r_{p-1})$ *is the partitioning of the integer* r *such that* $r_i\geqq 0$, $i=1,\cdots,p-1$, $r_1+\cdots+r_{p-1}=r$.

PROOF. Since

$$(1-\delta(1-u))^{-\gamma}=\sum_{r=0}^{\infty}(\gamma)_r\delta^r(1-u)^r/r!\,,\qquad \text{for } |\delta|<1,\ 0<u<1$$

where $(\gamma)_r=\gamma(\gamma+1)\cdots(\gamma+r-1)$, by expanding and integrating out $t_1,\cdots,t_{p-1}$ in (2.6) the result follows. All the steps are valid in this case.

In the case of Wilks' Λ in the real null case the parameters are (see for example Mathai [12]) $\alpha_j=(N-j)/2$, $\beta_j=q/2$, $j=1,\cdots,p$, $\gamma_j=(q-1)/2$, $j=1,\cdots,p-1$, $N>p$ and N is the sample size. Thus (2.8) gives a multiple series representation. But in Mathai [12] it is shown that in the case of Wilks' Λ one can get the density as a finite sum for all cases, where p and q are not both odd, by using calculus of residues. Also it is shown in Mathai and Rathie [16] that generalized partial fraction technique is applicable in this case. The general non-null case for Wilks' Λ is given in Mathai [9] and Walster and Tretter [24]. The main advantage of the representation in (2.8) is that it is a general representation for all parameters in (1.4) and the distribution function is available in terms of incomplete beta functions by term by term integration and the steps are all valid in this case.

THEOREM 2. *Consider the integral equation*

$$(2.9)\qquad \int_0^1 x^h f(x)dx=C_p\prod_{i=1}^{p}\{\Gamma(\alpha_i+h)/\Gamma(\alpha_i+\beta_i+h)\}=C_p\Delta(h)$$

where C_p *is given in* (2.7), α_i, β_i, $i=1, \cdots, p$ *are complex numbers such that* $R(h)>-\min R(\alpha_j)$, $j=1, \cdots, p$, $\sum_{i=1}^{p} R(\beta_i)+1/2>0$ *where* $R(\cdot)$ *denotes the real part of* $(\cdot)$, $f(x)$ *is assumed to be real and defined and single valued almost everywhere for* $0<x<1$ *such that* $f(x)\geqq 0$ *and absolutely integrable over the range* (0, 1). *Then* $f(x)$ *for* $0<x<1$ *is given by a series of the form*

$$f(x)=C_p \sum_{i=1}^{\infty} \mu_i x^{f_i}(-\log x)^{g_i} , \qquad 0<x<1 \tag{2.10}$$

and $f(x)=0$ *elsewhere, where* μ_i, f_i *and* g_i *do not depend on* x *and* $0\leqq g_i\leqq p-1$.

PROOF. From the conditions stated above $f(x)$ is available as the inverse Mellin transform of the right side of (2.9) with h replaced by $h-1$. That is,

$$f(x)=C_p x^{-1}(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Delta(h)x^{-h}dh , \qquad i=(-1)^{1/2}$$

where $c>-\min R(\alpha_j)$, $j=1, \cdots, p$. From the theory of G-functions $f(x)$ is defined for $0<x<1$ and it is available as the sum of the residues at the poles of $\Delta(h)$. Let $h=-a_i$, $i=1, \cdots$ be the poles of $\Delta(h)$ with orders b_i, $i=1, \cdots$. Evidently $1\leqq b_i\leqq p$ since there are only p gammas in the numerator of $\Delta(h)$. The residue for a pole of order b at $h=-a$ is given by

$$\begin{aligned} \sigma &= \{1/(b-1)!\} \lim_{h\to -a} \frac{\partial^{b-1}}{\partial h^{b-1}} \{(h+a)^b \Delta(h) x^{-h}\} \\ &= \{x^a/(b-1)!\} \sum_{r=0}^{b-1} \binom{b-1}{r} (-\log x)^{b-1-r} c_r \end{aligned} \tag{2.11}$$

where

$$c_r = \lim_{h\to -a} \frac{\partial^r}{\partial h^r} \{(h+a)^b \Delta(h)\} . \tag{2.12}$$

Summing up the residues by using (2.11) one has

$$f(x)=C_p \sum_{i=1}^{\infty} \{x^{a_i-1}/(b_i-1)!\} \sum_{r=0}^{b_i-1} \binom{b_i-1}{r} \sigma_{ir}(-\log x)^{b_i-1-r} , \qquad 0<x<1$$

where σ_{ir} is c_r of (2.12) with a and b replaced by a_i and b_i respectively. Since $0\leqq b_i-1-r\leqq p-1$ for all values of b_i's and r the result follows.

3. Type C and Type D integral equations and their solutions

Consider the equation (3.1) which we will call Type C equation.

$$\int_0^\infty x^h f(x)dx = Ca^h \prod_{j=1}^{m} \Gamma(\alpha_j+h) \prod_{j=1}^{m'} \Gamma(\beta_j-h) \qquad \text{(Type C)} \tag{3.1}$$

where h is such that the gamma products exist and $C^{-1}=\prod_{j=1}^{m}\Gamma(\alpha_j)\prod_{j=1}^{m'}\Gamma(\beta_j)$. A sufficient condition for the existence is that $m\geqq 0$, $m'\geqq 0$, $m+m'\geqq 1$, a real and positive, $-\min R(\alpha_j)<R(h)<\min R(\beta_k)$, $R(\alpha_j)>0$, $R(\beta_k)>0$, $j=1,\cdots,m$, $k=1,\cdots,m'$, where $R(\cdot)$ denotes the real part of $(\cdot)$. $f(x)$ is assumed to be real and defined and single valued almost everywhere for $x\geqq 0$ and absolutely integrable over the range $(0,\infty)$. For simplicity we will assume that the parameters are all positive and that $f(x)$ is a density function. Since by changing x to x/a one can remove a from the right side we will assume $a=1$ without loss of generality. The hth moment of,

$$X=X_1\cdots X_m/(Y_1\cdots Y_{m'}) \tag{3.2}$$

can give (3.1) where $X_1,\cdots,X_m$ and $Y_1,\cdots,Y_{m'}$ are mutually independent real gamma random variables defined in (1.3). A typical example of (3.1) is the ratio of the determinants of two independent sample covariance matrices when the samples come from multinormal populations. When $m=m'=p$ such a ratio of the Wilks' concept of generalized variances has applications in analysis of variance and other problems in statistics.

A solution of (3.1) can be found by identifying it with a G-function of the type $G^{m,m'}_{m',m}(x)$. Various existence conditions for $f(x)$ of (3.1) are available from the existence conditions for the G-function, see for example Mathai and Saxena [18]. A computable series representation of the type

$$f(x)=\Sigma_j a_j x^{b_j}(\log x)^{c_j} \tag{3.3}$$

where a_j, b_j and c_j do not depend on x and $0\leqq c_j\leqq m-1$ is available from Mathai [10] along with applications in statistics. Since the case $m=m'=p$ is closely accociated with the ratio of two independent generalized variances we will give here another new representation for the case $m=m'=p$ and $a=1$ in (3.1).

LEMMA 2. *Let $m=m'=p$, $a=1$, α_j, β_j, $j=1,\cdots,p$ be real positive numbers in Type C equation given in* (3.1). *Then the density of U, denoted by $g(u)$, is determined as follows where $U=(1+X)^{-1}$.*

$$(3.4)\quad g(u)=C_p u^{\beta_1-1}(1-u)^{\alpha_1-1}\int_0^1\cdots\int_0^1\left\{\prod_{i=1}^{p-1} t_i^{\beta_1+\alpha_{i+1}-1}(1-t_i)^{\alpha_1+\beta_{i+1}-1}\right\}$$
$$\times(ut_1\cdots t_{p-1}+(1-u)(1-t_1)\cdots(1-t_{p-1}))^{-(\alpha_1+\beta_1)}dt_1\cdots dt_{p-1}\,,$$
$$0<u<1\,,$$

where C_p is defined in (2.7).

PROOF. We will rewrite X of (3.2) in the form $X=Z_1\cdots Z_p$ where $Z_i=X_i/Y_i$ and it is trivial to note that Z_i has a beta type-2 density given by

$$K_i(z_i)=\{\Gamma(\alpha_i+\beta_i)/\Gamma(\alpha_i)\Gamma(\beta_i)\}z_i^{\alpha_i-1}(1+z_i)^{-(\alpha_i+\beta_i)}\,,$$

$0<z_i<\infty$, $\alpha_i>0$, $\beta_i>0$ and $K_i(z_i)=0$ elsewhere, $i=1,\cdots,p$. Thus $f(x)$ of (3.1) is available as the density of the product of the independent beta type-2 random variables $Z_1,\cdots,Z_p$. Let $U_1=Z_1Z_2$. By transformation of variables the density of U_1, denoted by $P_1(u_1)$, is given by

$$P_1(u_1)=C_2u_1^{\alpha_1-1}\int_0^\infty t^{\alpha_2-\alpha_1-1}(1+t)^{-(\alpha_2+\beta_2)}(1+u_1/t)^{-(\alpha_1+\beta_1)}dt\,.$$

Change t to w_1^{-1} to get

$$P_1(u_1)=C_2u_1^{\alpha_1-1}\int_0^\infty w_1^{\alpha_1+\beta_2-1}(1+w_1)^{-(\alpha_2+\beta_2)}(1+u_1w_1)^{-(\alpha_1+\beta_1)}dw_1\,.$$

By induction it is easily seen that the density of X is given by

$$P(x)=C_px^{\alpha_1-1}\int_0^\infty\cdots\int_0^\infty\left\{\prod_{i=1}^{p-1} w_i^{\alpha_1+\beta_{i+1}-1}(1+w_i)^{-(\alpha_{i+1}+\beta_{i+1})}\right\}$$
$$\times(1+xw_1\cdots w_{p-1})^{-(\alpha_1+\beta_1)}dw_1\cdots dw_{p-1}\,,\qquad 0<x<\infty$$

where C_p is defined in (2.7). Put $u=(1+x)^{-1}$, $t_i=(1+w_i)^{-1}$, $i=1,\cdots,p-1$ so that $0<u<1$, $0<t_i<1$, $i=1,\cdots,p-1$ then the density of U reduces to (3.4).

THEOREM 3. *A solution of Type C equation of* (3.1) *with $a=1$, $m=m'=p$, α_j, β_j, $j=1,\cdots,p$ real positive numbers, is,*

$$(3.5)\quad f(x)=C_px^{\beta_1-1}(1+x)^{-(\alpha_1+\beta_1)}$$
$$\times\sum_{r=0}^\infty\{(\alpha_1+\beta_1)_r/r!\}\sum_{r_1=0}^{r}\binom{r}{r_1}(-1)^{r_1}\sum_{r_2=0}^{r_1}\binom{r_1}{r_2}x^{r_2}(1+x)^{-r_1}$$
$$\times\prod_{i=1}^{p-1}\{\Gamma(\beta_1+\alpha_{i+1}+r_2)\Gamma(\alpha_1+\beta_{i+1}+r_1-r_2)/$$
$$\Gamma(\alpha_1+\beta_1+\alpha_{i+1}+\beta_{i+1}+r_1\}\,,\qquad 0<x<\infty$$

and $f(x)=0$ elsewhere, where C_p is given in (2.7).

PROOF. Since $u+(1-u)=1$ and $0<t_1\cdots t_{p-1}<1$, $0<(1-t_1)\cdots(1-t_{p-1})$

<1 we have $0<\Phi<1$ where $\Phi=ut_1\cdots t_{p-1}+(1-u)(1-t_1)\cdots(1-t_{p-1})$. Hence

$$\begin{aligned}\Phi^{-(\alpha_1+\beta_1)}&=\{1-(1-\Phi)\}^{-(\alpha_1+\beta_1)}\\&=\sum_{r=0}^{\infty}\{(\alpha_1+\beta_1)_r/r!\}(1-\Phi)^r\\&=\sum_{r=0}^{\infty}\{(\alpha_1+\beta_1)_r/r!\}\sum_{r_1=0}^{r}\binom{r}{r_1}(-1)^{r_1}\sum_{r_2=0}^{r_1}\binom{r_1}{r_2}(ut_1\cdots t_{p-1})^{r_2}\\&\quad\times\{(1-u)(1-t_1)\cdots(1-t_{p-1})\}^{r_1-r_2}\end{aligned}$$

where for example, $(a)_r=a(a+1)\cdots(a+r-1)$, $\binom{m}{n}=m!/\{n!(m-n)!\}$, $0!=1$. The factors containing t_i in (3.4) give

$$\begin{aligned}&\int_0^1 t_i^{\beta_1+\alpha_{i+1}+r_2-1}(1-t_i)^{\alpha_1+\beta_{i+1}+r_1-r_2-1}dt_i\\&\quad=\Gamma(\beta_1+\alpha_{i+1}+r_2)\Gamma(\alpha_1+\beta_{i+1}+r_1-r_2)/\Gamma(\alpha_1+\beta_1+\alpha_{i+1}+\beta_{i+1}+r_1)\,.\end{aligned}$$

Changing U back to X the result follows.

For special values of the parameters α_j's, β_j's, m and m' of (3.1) one can write $f(x)$ in terms of elementary functions such as Bessel, Whittaker, and Struve functions. Several such reduction formulae are available from Mathai and Saxena [18].

For the Type A equation given in (1.1) one can obtain a multiple integral representation by using the procedure discussed above by treating X/B as a product of independent real gamma variables with parameters $\alpha_1,\cdots,\alpha_p$. One such representation is given in Wilks [26]. But it is easy to note that $f(x)$ of (1.1) is nothing but a G-function of the type $G_{0,p}^{p,0}(x)$. A general series representation of the type (3.3) is available from Mathai [11]. Wilks [26] remarked that he was unable to get explicit forms of $f(x)$ for various particular values of p. This is due to the fact that for general parameter values $G_{0,p}^{p,0}(x)$ does not reduce to elementary functions except for $p=2$ which leads to a Bessel function. For special values of the parameters several reduction formulae are available, see for example Mathai [11]. One such case is when $\alpha_i=\alpha+(i-1)/2$, $i=1,\cdots,p$. Combining the gammas by using the multiplication formula

$$\text{(3.6)}\qquad \Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{j=1}^{m}(z+(j-1)/m)\,,\qquad m=1,2,\cdots$$

one gets from (1.1),

$$\int_0^\infty\{p(x/B)^{1/p}\}^s f(x)dx=\Gamma(p\alpha+s)/\Gamma(p\alpha)\,,\qquad s=ph\,.$$

Hence $p(X/B)^{1/p}$ has a gamma density with the parameter $p\alpha$. In the case of Wilks' generalized variance in the multinormal case the parameters a_j's differ by 1/2. Hence by combining the gammas by using (3.6) for $m=2$ the number of gammas on the right side of (1.1) can be reduced. Bagai [1] obtained explicit expressions for the density of the generalized variance in the multinormal case in terms of multiple integrals and multiple series for $p=2$ to 10.

Consider a more general case of integral equation for $f(x)$ where $f(x)$ will be assumed to be real, single valued and non-negative almost everywhere for $x\geqq 0$ and absolutely integrable over the range $(0, \infty)$, which we will call Type D equation.

$$(3.7)\quad \int_0^\infty x^{h-1}f(x)dx = Ca^{h-1}\left\{\prod_{j=1}^{m}\Gamma(b_j+h)\prod_{j=1}^{n}\Gamma(1-a_j-h)\right\}\Big/ \left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-h)\prod_{j=n+1}^{q}\Gamma(a_j+h)\right\} \qquad \text{(Type D)}$$

where C is a constant such that $\int_0^\infty f(u)du=1$, the points $h=-v-b_j$, $j=1, \cdots, m$, $v=0, 1, \cdots$ and $h=1-a_j+\lambda$, $j=1, \cdots, n$, $\lambda=0. 1, \cdots$ are separated. a_j, $j=1, \cdots, p$ and b_j, $j=1, \cdots, q$ are some complex numbers. h is such that $-\min R(b_j)<R(h)<1-\max R(a_k)$, $j=1, \cdots, m$, $k=1, \cdots, n$. $m+n-p/2-q/2>0$ or $m+n-p/2-q/2=0$ and $0<|x|<1$ in which case it is assumed that $f(x)=0$ for $|x|\geqq 1$.

Since by changing x to x/a one can get rid of a^{h-1} we will take $a=1$ in the following discussions. A particular case of the gamma product is the hth moment of V where $V=(X_1\cdots X_r)/(Y_1\cdots Y_s)$ where $X_1, \cdots, X_r, Y_1, \cdots, Y_s$ are mutually independent, some of which are gamma variates and others are beta variates. But this does not exhaust all possible cases in (3.7). From the gamma product in (3.7) it is easily seen that $f(x)$ is a general G-function of the type $G_{p,q}^{m,n}(x)$. Existence conditions for $f(x)$ and a series representation of the type (3.3) are available from Mathai and Saxena [18]. Also it should be remarked that the integral equations of Types A, B and C are special cases of Type D. For particular values of the parameters one can write the general G-function in terms of elementary functions. Several such cases are available in the literature, see for example Mathai and Saxena [18]. In all such cases $f(x)$ of (3.7) can be written in terms of elementary functions.

$f(x)$ in (3.7) also satisfies a homogeneous linear differential equation of the type

$$(3.8)\qquad \left\{(-1)^{p-m-n}x\prod_{j=1}^{p}(\theta-a_j+1)-\prod_{j=1}^{q}(\theta-b_j)\right\}f(x)=0$$

where θ is the differential operator $\theta=x\dfrac{d}{dx}$. From (3.8) one can easily write down the differential equations satisfied by $f(x)$ of Types A, B and C integral equations. For Type B Nair [20] derived the differential equation directly from the moment expression. It is easy to verify that Nair's differential equation is a particular case of (3.8). Also particular cases of (3.8) are worked out by Davis in connection with various test statistics, see for example Davis [4]. Hence one way of solving for $f(x)$ is to find the unique solutions of (3.8). But this procedure, in general, is complicated, see for example Nair [20] and it is more so when $n\neq 0$ and $q\neq m$.

In some cases the hth moment of the density $f(x)$ may not have the structure given in Types A to D but transformable to one of these types. As an example consider the hth null moment of U associated with the likelihood ratio criterion for testing sphericity in the multinormal case. Here

$$\text{(3.9)}\qquad \mathrm{E}\,(U^h)=\{p^{ph}\Gamma(np/2)/\Gamma(ph+np/2)\}\times\prod_{j=1}^{p}\{\Gamma(h+(n+1-j)/2)/\Gamma((n+1-j)/2)\}\,.$$

By expanding $\Gamma(ph+np/2)$ with the help of the multiplication formula (3.6) one can easily see that (3.9) reduces to a Type B equation.

4. Type E and Type F integral equations

In all the cases discussed in Sections 1 to 3 the integral equations have gamma products on the right side with h having the coefficient ± 1. In this section we will consider two cases, which are applicable to statistical problems, where the coefficients of h are different from ± 1. Consider the following equation which will be called Type E equation.

$$\text{(4.1)}\qquad \int_0^\infty x^h f(x)dx=Ca^h\prod_{i=1}^{m}\Gamma(b_j+\beta_j h)\Big/\prod_{i=1}^{m'}\Gamma(a_j+\alpha_j h)\qquad\text{(Type E)}$$

where h is such that the gammas exist, α_j, $j=1,\cdots,m'$, β_j, $j=1,\cdots,m$ and a are positive real numbers and a_j's and b_j's in general could be complex numbers but we restrict them to be real numbers for the time being. $f(x)$ is assumed to be real, non-negative and single valued almost everywhere for $x\geqq 0$ and absolutely integrable over the range $(0,\infty)$ and C is a normalizing constant such that $\int_0^\infty f(x)dx=1$. Let

$$\text{(4.2)}\qquad \mu'=\sum_{j=1}^{m}\beta_j-\sum_{j=1}^{m'}\alpha_j \quad\text{and}\quad \beta'=\prod_{j=1}^{m'}\alpha_j^{\alpha_j}\prod_{j=1}^{m}\beta_j^{-\beta_j}\,.$$

Then it is easy to show from the theory of H-functions, see Mathai and Saxena [17], that $f(x)$ exists for all x if $\mu'>0$ and for $0<x<a/\beta'$ if $\mu'=0$. Thus when $\mu'=0$ we define $f(x)$ in (4.1) to be $f(x)\geqq 0$ for $0<x<a/\beta'$ and $f(x)=0$ elsewhere. A special case of (4.1), the case when $\mu'=0$ is the moment expression considered by Box [2]. Hence we will call this as Type E or Box's integral equation. Some simplifications in Box's procedures may be seen from Gleser and Olkin [5]. A special case of the moment structure in (4.1) can also be obtained by the hth moment of the random variable $V=X_1\cdots X_r$ where $X_1, \cdots, X_r$ are mutually independent, some of which are powers of real beta random variables and the others are powers of real gamma or generalized gamma variables. Many of the likelihood ratio test criteria, in the null cases, for testing hypotheses such as sphericity, equality of covariance matrices, equality of populations, H_{vc}, H_{mvc} (in Wilks' notation), in multinormal populations fall in the category of Type E. The exact null densities of many of these test statistics are given in Mathai and Saxena [18] which give $f(x)$ of (4.1) for various particular values of the parameters. A series representation in terms of gamma densities is available from Box [2] for the case $\mu'=0$ when α_j's and β_j's can be made arbitrarily large whereas $(a_j-\alpha_j)$, $j=1, \cdots, m'$, $(b_j-\beta_j)$, $j=1, \cdots, m$ are bounded. The general solution for (4.1) is available as a particular case of the computable representation of a general H-function given in Mathai and Saxena [17]. This is a series of the type (3.3). When α_j's and β_j's are rational numbers, that is numbers of the type m'_j/n'_j where m'_j and n'_j are positive integers, there exists a number N such that $N\beta_j=m_j$, $j=1, \cdots, m$ and $N\alpha_j=n_j$, $j=1, \cdots, m'$ are all positive integers. Replace h by Nh in (4.1). This is equivalent to considering the hth moment of U where $U=X^N$. That is,

$$\text{(4.3)} \qquad \mathrm{E}\,(U^h)=C(a^N)^h \prod_{j=1}^{m} \Gamma(b_j+m_jh)\Big/\prod_{j=1}^{m'} \Gamma(a_j+n_jh)\,.$$

Now expand all the gammas in (4.3) by using the multiplication formula (3.6). Then the density of $U=X^N$ satisfies either Type B equation or a special case of Type D equation of which the general solutions are already discussed. The likelihood ratio statistics discussed earlier belong to (4.3) which reduce to Type B when the gammas are expanded. Densities of random r-contents associated with problems in geometric probabilities which are discussed in Mathai [8] and in the references therein also belong to the type (4.3) which reduce to Type B. Thus only when at least one of the α_j's or β_j's in (4.1) is irrational (4.1) can not be reduced to the Type D equation.

It is pointed out in (3.8) that $f(x)$ in Types A to D satisfies a homogeneous linear differential equation. But so far nobody has worked out a differential equation for $f(x)$ in (4.1) when (4.1) can not be reduced to a Type D equation. Hence the method of differential equation fails in (4.1) for the general case.

Now we will consider the most general case in this category and call it Type F integral equation.

$$(4.4)\qquad \int_0^\infty x^{h-1}f(x)dx=Ca^{h-1}\prod_{j=1}^{m}\Gamma(b_j+\beta_jh)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_jh)\Big/ \left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_jh)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_jh)\right\} \qquad \text{(Type F)}$$

where the gamma products exist and the points $h=-(b_j+v)/\beta_j$, $j=1, \cdots, m$, $v=0, 1, \cdots$ and $h=(1-a_j+\lambda)/\alpha_j$, $j=1, \cdots, n$, $\lambda=0, 1, \cdots$ are separated. $f(x)$ is assumed to be real single valued almost everywhere for $x\geqq 0$ and absolutely integrable over $(0, \infty)$. For simplicity we will assume that all the parameters are real, α_j, $j=1, \cdots, p$, β_j, $j=1, \cdots, q$ are positive numbers but in general a_j, $j=1, \cdots, p$ and b_j, $j=1, \cdots, q$ could be complex numbers. C is a normalizing constant such that $\int_0^\infty f(x)dx=1$. Let

$$(4.5)\qquad \mu=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j \quad\text{and}\quad \beta=\prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j}\,.$$

From the theory of H-functions it is easy to show that, see Mathai and Saxena [17], $f(x)$ is defined for all x when $\mu>0$ and for $0<x<a/\beta$ when $\mu=0$.

A special case of the moment structure in (4.4) could be generated by the hth moment of $W=X_1\cdots X_r/(Y_1\cdots Y_s)$ where the random variables $X_1, \cdots, X_r$, $Y_1, \cdots, Y_s$ are real and mutually independent, some of them are powers of beta variables and others are powers of gamma variates or generalized gamma variates. It is easy to note that the hth moment of W does not exhaust all possible cases in (4.4). If products and ratios of independent likelihood ratio test criteria, such as the ones for testing sphericity, equality of covariance matrices, eqaulity of populations, in the multinormal case, are considered then the hth moment of such a quantity also falls into Type F equation.

A general solution of Type F eqaution is available and a computable representation of $f(x)$ in (4.4), whenever $f(x)$ exists, is available from the computable representation of the H-function given in Mathai and Saxena [17]. For many special parameter values $f(x)$ is available in terms of elementary special functions which are obtained from the many special cases of the H-function.

When $\alpha_1=\cdots=\alpha_p=1=\beta_1=\cdots=\beta_q$ Type F equation reduces to Type D equation. Also when $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q$ are rational numbers Type F can be reduced to Type D as discussed earlier. The technique of differential equation fails because a differential equation for $f(x)$ of (4.4) is not available for general values of the parameters. Also it should be remarked that the general solutions of Types D and F need not give a non-negative $f(x)$ for all parameters. Hence restrictions on the parameters become necessary if $f(x)$ is to be kept non-negative for all x.

5. Type G equation and its solution

In all the equations considered so far the right sides contain gamma products containing the complex variable h. The only other factors on the right sides are of the form a^h where a is free of h. Here we will consider the case where a factor of the form $(h+a)^{k(h+a)}$ is present. The hth null moments of the likelihood ratio criteria for testing the hypotheses H_1: $\Sigma=\Sigma_0$, H_2: $\mu=\mu_0$, $\Sigma=\Sigma_0$ where μ_0 and Σ_0 are given quantities, μ and Σ are the parameters in a multinormal $N_p(\mu, \Sigma)$, are of this type. They are

$$(5.1) \qquad \mathrm{E}\,(\lambda_1^{h-1})=(2e/N)^{pN(h-1)/2}h^{-p(Nh-1)/2}\prod_{i=1}^{p}\{\Gamma(Nh-i)/2)/\Gamma(N-i)/2)\}$$

and

$$(5.2) \qquad \mathrm{E}\,(\lambda_2^{h-1})=(2e/N)^{pN(h-1)/2}h^{-pNh/2}\prod_{i=1}^{p}\{\Gamma(Nh-i)/2)/\Gamma((N-i)/2)\}$$

where N is the sample size. Hence we will consider the following integral equation and call it Type G.

$$(5.3) \qquad \int_0^\infty x^{h-1}f(x)dx=a^{h-1}h^{-b}h^{-kh}\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_jh)/\Gamma(a_j+\alpha_j)\}$$
$$\times\prod_{j=1}^{p'}\{\Gamma(a'_j+\alpha'_j)/\Gamma(a'_j+\alpha'_jh)\} \qquad \text{(Type G)}$$

where $f(x)$ is assumed to be real, non-negative and single valued almost everywhere for $x\geqq 0$ and absolutely integrable over $(0, \infty)$, $a>0, b, k$ do not contain h, $R(h)>-\min(R(a_j/\alpha_j), R(a'_r/\alpha'_r), 0)$, $j=1, \cdots, p$, $r=1, \cdots, p'$, α_j, $j=1, \cdots, p$ and α'_r, $r=1, \cdots, p'$ are real and positive numbers. $R(\cdot)$ denotes the real part of $(\cdot)$, $p\geqq 0$, $p'\geqq 0$, an empty product is interpreted as unity and an empty sum as zero. Generalizations of (5.3) along the lines of Type F equation can be given but there does not seem to be any test statistic in the current literature having the hth moment of such a general nature. Hence we will confine to the solution of Type G.

THEOREM 4. *A solution of the Type G equation given in* (5.3), *when* a_j, $j=1, \cdots, p$, a'_j, $j=1, \cdots, p'$ *are bounded*, α_j, $j=1, \cdots, p$, α'_j, $j=1$,

$\cdots, p'$ *can be made arbitrarily large,* $\sum_{j=1}^{p} a_j - \sum_{j=1}^{p'} \alpha'_j = k$, $b' = b + (p-p')/2 + \sum_{j=1}^{p'} a'_j - \sum_{j=1}^{p} a_j > 0$, *is given by*

(5.4) $$f(x) = C^{-1} \sum_{j=0}^{\infty} c_j \{\log C - \log x\}^{b'+j-1}, \qquad 0<x<C$$

and $f(x)=0$ *elsewhere, where* c_j*'s do not contain* x,

(5.5) $$C = a \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{p'} \alpha_j'^{-\alpha'_j} e^{-\rho}, \qquad \rho = \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{p'} \alpha'_j .$$

PROOF. When a_j is bounded and $\alpha_j \to \infty$ one has

(5.6) $$\Gamma(a_j+\alpha_j h) = (2\pi)^{1/2}(\alpha_j h)^{a_j+\alpha_j h-1/2} \times \exp\left\{-\alpha_j h + \sum_{r=1}^{\infty} (-1)^r B_{r+1}(a_j)/(r(r+1)h^r \alpha_j^r)\right\}$$

where $B_r(\alpha)$ are the Bernoulli polynomials defined by

$$xe^{\alpha x}/(e^x-1) = \sum_{r=0}^{\infty} x^r B_r(\alpha)/r! .$$

Expanding all the gammas in (5.3) by using (5.6) one gets

(5.7) $$\begin{aligned} \Delta(h) &= \prod_{j=1}^{p} \{\Gamma(a_j+\alpha_j h)/\Gamma(a_j+\alpha_j)\} \prod_{j=1}^{p'} \{\Gamma(a'_j+\alpha'_j)/\Gamma(a'_j+\alpha'_j h)\} \\ &= \left(\prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{p'} \alpha_j'^{-\alpha'_j} e^{-\rho}\right)^{h-1} h^{\rho h} h^{\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{p'} a'_j\right)-(p-p')/2} \exp\{\beta\}, \end{aligned}$$

where ρ is defined in (5.5) and

(5.8) $$\begin{aligned} &\beta = \sum_{r=1}^{\infty} \beta_r(-1+h^{-r}) \quad \text{and} \\ &\beta_r = (-1)^{r+1}\left\{\sum_{j=1}^{p} B_{r+1}(a_j)/\alpha_j^r - \sum_{j=1}^{p'} B_{r+1}(a'_j)/\alpha_j'^r\right\} \Big/ \{r(r+1)\} . \end{aligned}$$

Since $k=\rho$, substituting (5.7) in (5.3) one gets

$$\int_0^\infty x^{h-1} f(x)dx = C^{h-1} h^{-b'} e^{\beta}$$

where C is defined in (5.5), β in (5.8) and $b' = b + (p-p')/2 + \sum_{j=1}^{p'} a'_j - \sum_{j=1}^{p} a_j$. Expanding e^β and writing it as a power series in $(1/h)$ one has

$$e^\beta = \sum_{j=0}^{\infty} b_j h^{-j} .$$

Then

$$\int_0^\infty x^{h-1} f(x) dx = C^{h-1} \sum_{j=0}^{\infty} b_j h^{-(b'+j)} . \tag{5.9}$$

For computational purposes one many require the right side of (5.9) expressed in ascending powers of $(1/\alpha_j)$ and $(1/\alpha'_j)$. In this case we may write

$$\begin{aligned} e^\beta = 1+\beta+\beta^2/2!+\cdots = 1 &+ \{\beta_1(1/h)-\beta_1\} \\ &+ \{(\beta_2+\beta_1^2/2)(1/h^2)-\beta_1^2(1/h)+(-\beta_2+\beta_1^2/2)\} \\ &+\cdots . \end{aligned}$$

From the conditions stated in (5.3) $f(x)$ is available as the inverse Mellin transform of the right side of (5.9). Since $\alpha_j \to \infty$, $\alpha'_j \to \infty$ for all j it is easy to note that the right side of (5.9) goes to zero uniformly with respect to arg h as $|h| \to \infty$. Hence $f(x)$ is available by term by term inversion. The inverse of $h^{-(b'+j)}$ is $(-\log x)^{b'+j-1}/\Gamma(b'+j)$ for $0<x<1$ and zero elsewhere. Hence from (5.9) we have the density of $U=X/C$, denoted by $g(u)$, given by

$$g(u) = \sum_{j=0}^{\infty} c_j (-\log u)^{b'+j-1} , \qquad 0<u<1 \tag{5.10}$$

and $g(u)=0$ elsewhere, where $c_j = b_j/\Gamma(b'+j)$. Thus $f(x)$ is established as given in (5.4).

Note that if the right side of (5.9) contains the factor $(h+d)^{-(b'+j)}$ for $d>0$ instead of $h^{-(b'+j)}$ then in (5.10) there will be an additional factor u^d on the right side.

Identifying the parameters in (5.1) with the equation (5.3) we have, $k=Np/2$, $b=-p/2$, $a=(2e/N)^{Np/2}$, $\alpha_j=N/2$, $a_j=-j/2$, $j=1, \cdots, p$, $p'=0$. Hence $\sum \alpha_j = Np/2 = k$, $C=1$, $b'=p(p+1)/4$. The density of λ_1 is given by (5.4) with $C=1$, $b'=p(p+1)/4$. Comparing (5.2) and (5.3) one has $a=(2e/N)^{Np/2}$, $b=0$, $k=Np/2$, $\alpha_j=N/2$, $a_j=-j/2$, $j=1, \cdots, p$, $p'=0$, $b'=p/2+p(p+1)/4$, $C=1$. Then the density of λ_2 is given by (5.4) with $C=1$, $b'=p/2+p(p+1)/4$.

The hth non-null moments of a large number of test criteria associated with the multinormal populations have the structure of Type E or Type G multiplied by a hypergeometric function of one matrix argument, see for example Mathai [9], Pillai and Jouris [21], or a hypergeometric function of many matrix arguments, see for example Mathai and Rathie [15]. In a number of other cases the hth non-null moments are not available in the literature yet. In the cases where they are available it is seen that the hypergeometric functions are expansible in terms of zonal polynomials. For a discussion of zonal polynomials, see for example, Constantine [3], James [6]. Once these hypergeometric functions are expanded and the terms rearranged then the kernel containing h falls in the category of Type E or Type G equation, see for example Mathai [9]. By modifying the equations Types A to G the corresponding non-null cases can be covered. Since in a large number of cases the non-null moments are not yet available further discussion of integral equations to cover non-null cases in deleted.

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