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# ROBUST SLIPPAGE TESTS

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## Summary

The robust slippage testing problems of $k+1$ approximately known simple hypotheses are formulated as the slippage testing problems of $k+1$ composite hypotheses. It is shown that if there is a representative $k+1$-tuple (called a least favorable slippage tuple) of simple hypotheses, then maximin tests are given by the slippage analogues of the Neyman-Pearson tests for this tuple. The $k$-sample case is treated concerning this subject. In the general situations that there does not exist any least favorable slippage tuple, a method for constructing tests is proposed and applied to the case that composite hypotheses are described in terms of certain capacities ($\varepsilon$-contamination, total variation). The variants of the derived tests are also suggested.


## 1. Introduction

In this paper, the robust slippage testing problems of an approximately known simple hypothesis against $k$ approximately known simple hypotheses are formulated as the slippage testing problems of a composite hypothesis against $k$ composite hypotheses. This type of problems was treated in a less general form from a different point of view by Yao and Kudô [8], and Yeh and Kudô [9] (they considered the slippage problems of testing a composite hypothesis against $k$ simple hypotheses without the stand point of robustness and discussed the least favorable distributions). The present paper, however, inherits the spirit of Huber ([2], [3]), Huber and Strassen [5], and Rieder [7]. They formulated the robust testing problems between two approximately known simple hypotheses as the minimax testing problems between two composite hypotheses, and showed that if the composite hypotheses are described in terms of some types of neighborhoods (such as $\varepsilon$-contamination, total variation and alternating capacities of order 2), there exists a least favorable pair of simple hypotheses between two composite hypotheses and the Neyman-Pearson tests for the least favorable pair constitute a minimal essentially complete class of minimax tests. Their results are indispensable for our discussions.

In Section 2, the robust slippage testing problems are formulated and it is shown that if there is a representative $k+1$-tuple of simple hypotheses (called a least favorable slippage tuple), then maximin tests are given by the slippage analogues of the Neyman-Pearson tests for this tuple (Proposition 1). However, there does not exist any least favorable slippage tuple except for some special cases. This fact makes the problem more complicated and more difficult. A general method to construct tests is proposed in Section 3. Tne essence of this method lies in extending the sample space, constructing tests on each partitioned subset of the extended sample space and then unifying these tests. The derived tests, which are defined on the extended sample space, satisfy the size condition (Proposition 3) and their powers are evaluated (Proposition 4). The tests can also be expressed on the original sample space (Theorem 1).

Section 4 is concerned with the case that composite hypotheses are described in terms of the special capacities (a natural generalization of $\varepsilon$-contamination and total variation neighborhoods) introduced by Rieder [7], and presents an explicit substance to the general framework in Section 3. It is seen that the tests derived from our method are partially truncated versions of the slippage analogues of the Neyman-Pearson tests for the $k+1$ approximately known simple hypotheses. On the basis of this fact, the (completely truncated) variants of the above tests are suggested as the recommendable tests with excellent robustness properties.

The final Section 5 treats $k$-sample robust slippage testing problems. In this case, $k+1$ composite slippage hypotheses are formed by two specified composite hypotheses. It is shown that if there exists a least favorable pair between two specified composite hypotheses, then such a pair produces a least favorable slippage tuple (Theorem 3) and hence maximin tests. Some examples are exhibited to clarify our image of the maximin tests.

## 2. The formulation of the problem

Let $\mathcal{X}$ be a sample space, $\mathcal{B}$ a $\sigma$-field of subsets of $\mathcal{X}$ and $\mathcal{M}$ the set of all probability measures on $(\mathcal{X}, \mathcal{B})$. We assume that there is a group $G=\{g\}$ of transformations on $\mathcal{X}$ isomorphic to $T$ which is the symmetric group of all permutations on the set $\{1, 2, \cdots, k\}$ or its transitive subgroup. Denote the permutation corresponding to $g$ by $\tau_g$, $\tau_g(1, 2, \cdots, k)=(\tau_g 1, \tau_g 2, \cdots, \tau_g k)$ and define $\tau_g 0=0$ for all $g \in G$. Suppose that we are given $k+1$ distinct probability measures $\tilde{P}_0, \tilde{P}_1, \cdots, \tilde{P}_k$ $(\in \mathcal{M})$

such that

$$\tilde{P}_i g^{-1} = \tilde{P}_{\tau_g i} \qquad i=0,\cdots,k, \text{ all } g \in G, \tag{2.1}$$

where $Pg^{-1}$ ($P \in \mathcal{M}$) denotes the probability measure induced from $P$ by $g$, and consider the problem of testing $\tilde{P}_0$ against $\tilde{P}_i$ $i=1,\cdots,k$ based on a random element $X$ taking values in $\mathcal{X}$. We assume, however, that the distribution $\mathcal{L}(X)$ of $X$ is only known to lie near one of $\tilde{P}_0$, $\tilde{P}_1,\cdots,\tilde{P}_k$. Let $\mathcal{P}_0, \mathcal{P}_1,\cdots,\mathcal{P}_k$ be $k+1$ disjoint neighborhoods of $\tilde{P}_0, \tilde{P}_1, \cdots, \tilde{P}_k$, respectively, such that

$$\mathcal{P}_i g^{-1} = \mathcal{P}_{\tau_g i} \qquad i=0,\cdots,k, \text{ all } g \in G, \tag{A.1}$$

where $\mathcal{P}_i g^{-1} = \{P_i g^{-1} \mid P_i \in \mathcal{P}_i\}$. Then we are interested in testing

$$H_0: \mathcal{L}(X) \in \mathcal{P}_0 \text{ against } H_i: \mathcal{L}(X) \in \mathcal{P}_i \qquad i=1,\cdots,k. \tag{RSTP}$$

We call (RSTP) a robust slippage testing problem. This is a robust version of the slippage problem of testing $H_0$: $\mathcal{L}(X)=\tilde{P}_0$ against $H_i$: $\mathcal{L}(X)=\tilde{P}_i$ $i=1,\cdots,k$, which was formalized by Hall and Kudô [1].

A slippage test based on $X$ is denoted by $\varphi(x)=(\varphi_0(x), \varphi_1(x),\cdots,\varphi_k(x))$ where $\varphi_i(x)$ means the conditional probability that $\varphi$ takes $H_i$ given $X=x$. A test $\varphi$ is called of size $\alpha$ if it satisfies

$$\inf_{P_0 \in \mathcal{P}_0} \mathrm{E}_{P_0}[\varphi_0(X)] \geqq 1-\alpha, \qquad 0<\alpha<1, \tag{2.2}$$

where $\mathrm{E}_P$ denotes the expectation under $P$. Let $\Phi_\alpha$ be the class of all size $\alpha$ tests. A test $\hat{\varphi}$ ($\in \Phi_\alpha$) is called maximin if it satisfies

$$\inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\hat{\varphi}_i(X)] = \sup_{\varphi \in \Phi_\alpha} \inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i(X)] \tag{2.3}$$

where $(P_1,\cdots,P_k)$ ranges over $\mathcal{P}_1 \times \cdots \times \mathcal{P}_k$.

In this set up, we might have a preference for maximin tests. The concept of the least favorable distribution in slippage testing problems as well as in ordinary testing problems plays an important role for the purpose of finding maximin tests.

A $k+1$-tuple $(P_0, P_1,\cdots,P_k)$, $P_i \in \mathcal{P}_i$, of probability measures satisfying (2.1) with $\tilde{P}_i$ replaced by $P_i$ is called a slippage tuple (with respect to $G$). For any slippage tuple $Q=(P_0, P_1,\cdots,P_k)$, consider the following tests $\varphi^Q=(\varphi_0^Q, \varphi_1^Q,\cdots,\varphi_k^Q)$:

$$\begin{aligned} \varphi_0^Q(x) &= 1,\ \xi(x),\ 0 \qquad \text{if } \max_{1\le j\le k} \pi_j(x) <, =, > \lambda \\ \varphi_i^Q(x) &= 0,\ \eta_i(x) \qquad \text{if } \pi_i(x) <, = \max_{1\le j\le k} \pi_j(x),\ i=1,\cdots,k \end{aligned} \tag{2.4}$$

where $\pi_i(x)$ is a version of the Radon-Nikodym derivative $dP_i/dP_0$,

$$\frac{dP_i}{dP_0}=\left\{\frac{q_i}{q_0}\middle| q_j \in \frac{dP_j}{d(P_0+P_i)},\ q_j \geqq 0,\ j=0, i,\ q_0+q_i>0\right\}\ ,$$

$\xi(x)$ and $\eta_i(x)$ are arbitrary, subject to the condition that $\varphi^Q$ is a test, and $\lambda$ is a constant.

We notice that $\varphi^Q$ are the most powerful tests for the testing problem of $P_0$ against $P_1, P_2, \cdots, P_k$ (see Theorem 1 of Hall and Kudô [1]). A slippage tuple $\hat{Q}=(\hat{P}_0, \hat{P}_1, \cdots, \hat{P}_k)$ is called the least favorable for $(\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_k)$ if any $\varphi^{\hat{Q}}$ of (2.4) satisfies that for all $P_i \in \mathcal{P}_i$ $i=0, \cdots k$

$$\mathrm{E}_{\hat{P}_0}[\varphi_0^{\hat{Q}}(X)] \leqq \mathrm{E}_{P_0}[\varphi_0^{\hat{Q}}(X)] \tag{2.5}$$

$$\sum_{i=1}^{k} \mathrm{E}_{\hat{P}_i}[\varphi_i^{\hat{Q}}(X)] \leqq \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i^{\hat{Q}}(X)]\ . \tag{2.6}$$

PROPOSITION 1. If $\hat{Q}=(\hat{P}_0, \hat{P}_1, \cdots, \hat{P}_k)$ is a least favorable slippage tuple for $(\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_k)$, then for any $\alpha \in (0, 1)$ $\varphi^{\hat{Q}}$ are maximin tests for the problem (RSTP).

PROOF. We first note that $\varphi^{\hat{Q}} \in \Phi_\alpha$ follows from (2.5) and $\mathrm{E}_{\hat{P}_0}[\varphi_0^{\hat{Q}}(X)] =1-\alpha$. Let $\varphi \in \Phi_\alpha$. Then

$$\begin{aligned}&\inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i^{\hat{Q}}(X)] \\ &\quad \geqq \sum_{i=1}^{k} \mathrm{E}_{\hat{P}_i}[\varphi_i^{\hat{Q}}(X)] \geqq \sum_{i=1}^{k} \mathrm{E}_{\hat{P}_i}[\varphi_i(X)] \geqq \inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i(X)]\ .\end{aligned}$$

Hence

$$\inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i^{\hat{Q}}(X)] \geqq \sup_{\varphi\in\Phi_\alpha} \inf_{(P_1,\cdots,P_k)} \sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i(X)]\ . \qquad \text{Q.E.D.}$$

According to this proposition, for the purpose of obtaining maximin tests it is sufficient to find a least favorable slippage tuple. The proposition will be used in Section 5.

Though digressing from main subjects we wish to present some results here.

PROPOSITION 2. If $\varphi$ is invariant under $G$ (i.e. $\varphi_i(x)=\varphi_{\tau_g i}(gx)$), then

(1) $\inf_{P_i \in \mathcal{P}_i} \mathrm{E}_{P_i}[\varphi_i(X)] = \inf_{P_j \in \mathcal{P}_j} \mathrm{E}_{P_j}[\varphi_j(X)]$

(2) $\sup_{P_i \in \mathcal{P}_i} \mathrm{E}_{P_i}[\varphi_0(X)] = \sup_{P_j \in \mathcal{P}_j} \mathrm{E}_{P_j}[\varphi_0(X)]$

(3) $\sup_{P_0 \in \mathcal{P}_0} \mathrm{E}_{P_0}[\varphi_i(X)] = \sup_{P_0 \in \mathcal{P}_0} \mathrm{E}_{P_0}[\varphi_j(X)]$.

Furthermore, if $T$ is 2-ply transitive, then

(4) $\sup_{P_i \in \mathcal{P}_i} \mathrm{E}_{P_i}[\varphi_j(X)] = \sup_{P_{i'} \in \mathcal{P}_{i'}} \mathrm{E}_{P_{i'}}[\varphi_{j'}(X)]$

for $i, i', j, j'=1, \cdots, k,\ i\neq j,\ i'\neq j'$.

PROOF. As all the proofs are similar, we give only the proof of (4). Because of 2-ply transitivity of $T$, there exists $g$ such that $\tau_g i = i'$ and $\tau_g j = j'$ for any $i \neq j$, $i' \neq j'$. Let $P_i \in \mathcal{P}_i$. Then, since $\varphi$ is invariant, we have

$$\mathrm{E}_{P_i}[\varphi_j(X)] = \mathrm{E}_{P_i}[\varphi_{\tau_g j}(gX)] = \mathrm{E}_{P_{\tau_g j}}[\varphi_{\tau_g j}(X)] = \mathrm{E}_{P_{i'}}[\varphi_{j'}(X)] .$$

Noting that by (A.1) the correspondence $P_i \to P_{i'}$ from $\mathcal{P}_i$ to $\mathcal{P}_{i'}$ is bijective (i.e., one-one, onto), (4) follows immediately. Q.E.D.

## 3. A general method for constructing tests

Let $\mathcal{X}_i$ $i=1,\cdots,k$ be $k$ measurable subsets of $\mathcal{X}$ such that $g\mathcal{X}_i = \mathcal{X}_{\tau_g i}$ for $i=1,\cdots,k$, all $g \in G$, and $\mathcal{X} = \bigcup_{i=1}^{k} \mathcal{X}_i$. Here we do not assume that $\mathcal{X}_i$ are disjoint. In other words, $\{\mathcal{X}_i\}$ is a covering of $\mathcal{X}$, but may not be a partition of $\mathcal{X}$. If $\{\mathcal{X}_i\}$ is not a partition of $\mathcal{X}$, we shall extend the sample space $\mathcal{X}$ to a new sample space $\mathcal{X}^*$ and construct a partition $\{\mathcal{X}_i^*\}$ of $\mathcal{X}^*$. On the other hand, if $\{\mathcal{X}_i\}$ is a partition of $\mathcal{X}$, or if $\{\mathcal{X}_i\}$ satisfies $P(\mathcal{X}_j \cap \mathcal{X}_l)=0$ for $j \neq l$ and all $P \in \bigcup_{i=0}^{k} \mathcal{P}_i$, then it is unnecessary to extend $\mathcal{X}$ to $\mathcal{X}^*$. Therefore, in what follows, we can eliminate asterisk * from all notations.

Let $\mathcal{S} = \{S\}$ be the family of all nonempty subsets of $\{1, 2, \cdots, k\}$ and let $S(x) = \{i \mid x \in \mathcal{X}_i\}$. For any $S \in \mathcal{S}$ define $\mathcal{X}_S = \{x \mid S(x) = S\}$. Then $\{\mathcal{X}_S\}$ is a partition of $\mathcal{X}$. We denote by $|S|$ the number of elements in a set $S$. Let $x$ be any element of $\mathcal{X}$ and suppose $S(x) = \{i_1, i_2, \cdots, i_{|S(x)|}\}$. Then $x$ is contained in $|S(x)|$ sets $\mathcal{X}_{i_1}, \mathcal{X}_{i_2}, \cdots, \mathcal{X}_{i_{|S(x)|}}$. We consider $|S(x)|$ copies $x_{i_1}, x_{i_2}, \cdots, x_{i_{|S(x)|}}$ of $x$, and by putting asterisk * on them we regard $x_{i_1}^*, x_{i_2}^*, \cdots, x_{i_{|S(x)|}}^*$ as $|S(x)|$ distinct points. Let $\mathcal{X}_i^* = \{x_i^* \mid x \in \mathcal{X}_i\}$ and $\mathcal{X}^* = \bigcup_{i=1}^{k} \mathcal{X}_i^*$. It is clear that $\{\mathcal{X}_i^*\}$ is a partition of the extended sample space $\mathcal{X}^*$.

Let $\psi$ be the mapping from $\mathcal{X}^*$ onto $\mathcal{X}$ which maps $x^*$ to its generator $x$, namely, $\psi(x_i^*) = x$ for all $i \in S(x)$. Let $\mathcal{B}_i^* = \psi^{-1}\mathcal{B} \cap \mathcal{X}_i^*$ and let $\mathcal{B}^*$ be the smallest $\sigma$-field containing $\bigcup_{i=1}^{k} \mathcal{B}_i^*$. Note that $\psi_i$, the restriction of $\psi$ to $\mathcal{X}_i^*$, is a one-one mapping from $\mathcal{X}_i^*$ onto $\mathcal{X}_i$ and $\mathcal{B}_i^* = \psi_i^{-1}(\mathcal{B} \cap \mathcal{X}_i)$. Thus we get a new extended measurable space $(\mathcal{X}^*, \mathcal{B}^*)$ and its measurable subspaces $(\mathcal{X}_i^*, \mathcal{B}_i^*)$. Let $\mathcal{M}^*$ be the set of all probability measures on $(\mathcal{X}^*, \mathcal{B}^*)$ and $\mathcal{M}_i^* = \{P^* \in \mathcal{M}^* \mid P^*(\mathcal{X}_i^*) = 1\}$ be all probability measures (concentrating) on $(\mathcal{X}_i^*, \mathcal{B}_i^*)$. Also, let us define transformations $g^*$ on $\mathcal{X}^*$ by $g^*(x_i^*) = (g\psi(x_i^*))_{\tau_g i}^*$. It is easily seen that $G^* = \{g^* \mid g \in G\}$ forms a transformation group on $\mathcal{X}^*$ with the operation $g_1^* \circ g_2^* = (g_1 \circ g_2)^*$ and isomorphic to $T$ by the correspondence $g^* \to$

$\tau_{g^*}$ $(=\tau_g)$.

Any probability measure $P$ on $(\mathcal{X}, \mathcal{B})$ is transferred to the probability measure $P^*$ on $(\mathcal{X}^*, \mathcal{B}^*)$ defined by

$$P^*(B^*)=\sum_{i=1}^{k}\sum_{S\in\mathcal{S}}\frac{1}{|S|}P(\psi(B^*\cap\mathcal{X}_i^*)\cap\mathcal{X}_S)\,,\qquad B^*\in\mathcal{B}^*\,. \tag{3.1}$$

We note here that $P^*$ is the probability measure with the probability $P(\mathcal{X}_S)/|S|$ on $\psi^{-1}(\mathcal{X}_S)\cap\mathcal{X}_i^*$ for all $i\in S$. Let $\mathcal{P}_i^*=\{P_i^*\,|\,P_i\in\mathcal{P}_i\}$ $i=0,\cdots,k$. Clearly, $\mathcal{P}_i^*$ $i=0,\cdots,k$ are disjoint, and noting $P_i^*g^{*-1}=(P_ig^{-1})^*$ it follows that

$$\mathcal{P}_i^*g^{*-1}=\mathcal{P}_{\tau_{g^*}i}^*\qquad i=0,\cdots,k,\ \text{all } g^*\in G^*\,. \tag{3.2}$$

Now our original testing problem (RSTP) is expressed as the problem of testing

(RSTP)* $H_0^*:\ \mathcal{L}(X^*)\in\mathcal{P}_0^*$ against $H_i^*:\ \mathcal{L}(X^*)\in\mathcal{P}_i^*$ $\quad i=1,\cdots,k$.

Let $P_{i|j}^*$ $(i=0,\cdots,k,\ j=1,\cdots,k)$ be the conditional probability measure of $P_i^*$ given $\mathcal{X}_j^*$, namely,

$$P_{i|j}^*(B^*)=\frac{1}{P_i^*(\mathcal{X}_j^*)}P_i^*(B^*\cap\mathcal{X}_j^*)\qquad B^*\in\mathcal{B}^*\,, \tag{3.3}$$

and let $\mathcal{P}_{i|j}^*=\{P_{i|j}^*\,|\,P_i^*\in\mathcal{P}_i^*\}$. For any $P^*\in\mathcal{M}^*$ we define

$$\bar{P}^*=\frac{1}{|G^*|}\sum_{g^*\in G^*}P^*g^{*-1}\,,$$

where $|G^*|$ means the number of elements of $G^*$.

Suppose that we have subsets $\mathcal{P}_{0i}^*$, $\mathcal{P}_{i0}^*$ $(i=1,\cdots,k)$ of $\mathcal{M}_i^*$ such that

$$\mathcal{P}_{0i}^*g^{*-1}=\mathcal{P}_{0\tau_{g^*}i}^*\qquad i=1,\cdots,k,\ \text{all } g^*\in G^*\,, \tag{A.2}$$

$$\mathcal{P}_{i0}^*g^{*-1}=\mathcal{P}_{\tau_{g^*}i0}^*\qquad i=1,\cdots,k,\ \text{all } g^*\in G^*\,, \tag{A.3}$$

$$\mathcal{P}_{0i}^*\cap\mathcal{P}_{i0}^*=\phi\qquad i=1,\cdots,k\,, \tag{A.4}$$

$$\mathcal{P}_{i0}^*\cap\mathcal{P}_{i|i}^*\neq\phi\qquad i=1,\cdots,k\,, \tag{A.5}$$

$$\{\bar{P}_{0|i}^*\,|\,P_0^*\in\mathcal{P}_0^*\}\subset\mathcal{P}_{0i}^*\qquad i=1,\cdots,k\,. \tag{A.6}$$

In addition, we assume that there are the least favorable pairs $(Q_{0i}^*, Q_{i0}^*)$ for $(\mathcal{P}_{0i}^*, \mathcal{P}_{i0}^*)$ $i=1,\cdots,k$, namely, $Q_{0i}^*\in\mathcal{P}_{0i}^*$, $Q_{i0}^*\in\mathcal{P}_{i0}^*$ satisfying that for all $t$

$$Q_{0i}^*(\pi_i^*>t)=\sup\{P^*(\pi_i^*>t)\,|\,P^*\in\mathcal{P}_{0i}^*\} \tag{A.7}$$

$$Q_{i0}^*(\pi_i^*>t)=\inf\{P^*(\pi_i^*>t)\,|\,P^*\in\mathcal{P}_{i0}^*\}\,, \tag{A.8}$$

where $\pi_i^*$ is a version of $dQ_{i0}^*/dQ_{0i}^*$. Note that we can take $Q_{0i}^*$, $Q_{i0}^*$, $\pi_i^*$ so as to satisfy

$$Q_{0i}^* g^{*-1} = Q_{0\tau_{g^*}i}^* \qquad i=1,\cdots,k, \text{ all } g^* \in G^*, \tag{3.4}$$

$$Q_{i0}^* g^{*-1} = Q_{\tau_{g^*}i0}^* \qquad i=1,\cdots,k, \text{ all } g^* \in G^*, \tag{3.5}$$

$$\pi_i^*(x^*) = \pi_{\tau_{g^*}i}^*(g^* x^*) \qquad i=1,\cdots,k, \text{ all } g^* \in G^*. \tag{3.6}$$

The reason is as follows. Let $(Q_{01}^*, Q_{10}^*)$ be a least favorable pair for $(\mathcal{P}_{01}^*, \mathcal{P}_{10}^*)$. Define $Q_{0i}^*$, $Q_{i0}^*$ $(i=2,\cdots,k)$ by $Q_{0\tau_{g^*}1}^* = Q_{01}^* g^{*-1}$, $Q_{\tau_{g^*}10}^* = Q_{10}^* g^{*-1}$, $g^* \in G^*$. Then we can easily see that (3.4) and (3.5) hold. Also, (3.6) follows immediately from (3.4) and (3.5). To show (A.7) and (A.8) notice first that by (3.6) we have $g^*\{\pi_i^* > t\} = \{\pi_{\tau_{g^*}i}^* > t\}$ for all $t$, $i=1, \cdots, k$ and all $g^* \in G^*$. Let $g^*$ be such that $\tau_{g^*}i = 1$ (there exists such $g^*$, because $G^*$ is transitive). Then using (A.7) (for $i=1$) and (3.2), we have

$$\begin{aligned} Q_{0i}^*(\pi_i^* > t) &= Q_{0i}^* g^{*-1}(g^*(\pi_i^* > t)) = Q_{01}^*(\pi_1^* > t) \\ &= \sup\{P^* g^*(g^{*-1}(\pi_1^* > t)) \mid P^* g^* \in \mathcal{P}_{01}^* g^*\} \\ &= \sup\{P^*(\pi_i^* > t) \mid P^* \in \mathcal{P}_{0i}^*\}. \end{aligned}$$

Thus (A.7) is verified. Similarly, (A.8) is proved.

Now we are in a position to propose our tests. Let us consider the following tests $\varphi^*$:

$$\varphi_0^*(x^*) = \begin{cases} 1 & \text{if } x^* \in \mathcal{X}_i^* \text{ and } \pi_i^*(x^*) < \lambda \\ \xi & \text{if } x^* \in \mathcal{X}_i^* \text{ and } \pi_i^*(x^*) = \lambda \\ 0 & \text{if } x^* \in \mathcal{X}_i^* \text{ and } \pi_i^*(x^*) > \lambda \end{cases}$$
$$\varphi_i^*(x^*) = \begin{cases} 1 & \text{if } x^* \in \mathcal{X}_i^* \text{ and } \pi_i^*(x^*) > \lambda \\ 1-\xi & \text{if } x^* \in \mathcal{X}_i^* \text{ and } \pi_i^*(x^*) = \lambda \\ 0 & \text{otherwise}, \end{cases} \tag{3.7}$$

where $\xi$ and $\lambda$ are constants determined $\mathrm{E}_{Q_{01}^*}[\varphi_0^*(X^*)] = 1-\alpha$.

*Remark* 1. It should be noted that since $Q_{0i}^* \pi_i^{*-1}$ $i=1,\cdots,k$ are all the same (independent of $i$) and $\varphi_0^*(x^*) + \varphi_i^*(x^*) = 1$ on $\mathcal{X}_i^*$, $\varphi^*$ is identical to the test unifying $k$ maximin size $\alpha$ tests $\varphi_{(i)}^*$ for the testing $(\mathcal{P}_{0i}^*, \mathcal{P}_{i0}^*)$ on $\mathcal{X}_i^*$, that is, $\varphi_i^*(x^*) = \varphi_{(i)}^*(x^*)$, 0 according as $x^* \in, \notin \mathcal{X}_i^*$.

LEMMA 1. *The test $\varphi^*$ is invariant under $G^*$.*

PROOF. Noting that $g^* \mathcal{X}_i^* = \mathcal{X}_{\tau_{g^*}i}^*$ and $\pi_i^*(x^*) = \pi_{\tau_{g^*}i}^*(g^* x^*)$, this lemma easily follows. Q.E.D.

LEMMA 2. *For any* $P^* \in \mathcal{M}^*$, $\bar{P}^*$ *is an invariant probability measure under* $G^*$.

PROOF. Let $B^* \in \mathcal{B}^*$ and $g^* \in G^*$. Then we have

$$\begin{aligned}\bar{P}^*(g^*B^*) &= \frac{1}{|G^*|}\sum_{g_1^* \in G^*} P^*(g_1^{*-1}(g^*B^*)) \\ &= \frac{1}{|G^*|}\sum_{g_2^* \in G^*} P^*(g_2^{*-1}B^*) \qquad (g_2^* = g^{*-1}g_1^*) \\ &= \bar{P}^*(B^*) .\end{aligned}$$

Q.E.D.

We note that by Lemma 2 $\bar{P}^*(\mathcal{X}_i^*) = 1/k$ holds for $i=1,\cdots,k$.

LEMMA 3. $\mathrm{E}_{\bar{P}^*}[\varphi_0^*(X^*)] = \mathrm{E}_{P^*}[\varphi_0^*(X^*)]$ *for all* $P^* \in \mathcal{M}^*$.

PROOF.

$$\begin{aligned}\mathrm{E}_{\bar{P}^*}[\varphi_0^*(X^*)] &= \frac{1}{|G^*|}\sum_{g^* \in G^*} \mathrm{E}_{P^*g^{*-1}}[\varphi_0^*(X^*)] \\ &= \frac{1}{|G^*|}\sum_{g^* \in G^*} \mathrm{E}_{P^*}[\varphi_0^*(g^*X^*)] \\ &= \frac{1}{|G^*|}\sum_{g^* \in G^*} \mathrm{E}_{P^*}[\varphi_0^*(X^*)] \\ &= \mathrm{E}_{P^*}[\varphi_0^*(X^*)] .\end{aligned}$$

Q.E.D.

We have now the following proposition.

PROPOSITION 3. The test $\varphi^*$ is of size $\alpha$.

PROOF. For any $P_0^* \in \mathcal{P}_0^*$

$$\begin{aligned}\mathrm{E}_{P_0^*}[\varphi_0^*(X^*)] &= \mathrm{E}_{\bar{P}_0^*}[\varphi_0^*(X^*)] \\ &= \sum_{i=1}^{k} \mathrm{E}_{\bar{P}_{0|i}^*}[\varphi_0^*(X^*)]\bar{P}_0^*(\mathcal{X}_i^*) \\ &= \frac{1}{k}\sum_{i=1}^{k} [(1-\xi)\bar{P}_{0|i}^*(\pi_i^* < \lambda) + \xi\bar{P}_{0|i}^*(\pi_i^* \leqq \lambda)] \\ &\geqq \frac{1}{k}\sum_{i=1}^{k} [(1-\xi)Q_{0i}^*(\pi^* < \lambda) + \xi Q_{0i}^*(\pi_i^* \leqq \lambda)] \\ &= \frac{1}{k}\sum_{i=1}^{k} \mathrm{E}_{Q_{0i}^*}[\varphi_0^*(X^*)] = 1-\alpha .\end{aligned}$$

Q.E.D.

Our next aim is to evaluate the power of $\varphi^*$. Let $\hat{\mathcal{P}}_i^* = \{P_i^* \in \mathcal{P}_i^* \mid P_{i|i}^* \in \mathcal{P}_{i0}^*\}$. Then the following proposition can be obtained.

PROPOSITION 4. For any $P_i^* \in \hat{\mathcal{P}}_i^*$ $i=1,\cdots,k$, $\varphi^*$ satisfies

$$\sum_{i=1}^{k} \mathrm{E}_{P_i^*}[\varphi_i^*(X^*)] \geqq \mathrm{E}_{Q_{10}^*}[\varphi_1^*(X^*)] \sum_{i=1}^{k} P_i^*(\mathcal{X}_i^*) . \tag{3.8}$$

PROOF.

$$\begin{aligned}
&\sum_{i=1}^{k} \mathrm{E}_{P_i^*}[\varphi_i^*(X^*)] \\
&\quad = \sum_{i=1}^{k} \mathrm{E}_{P_{i|i}^*}[\varphi_i^*(X^*)] P_i^*(\mathcal{X}_i^*) \\
&\quad = \sum_{i=1}^{k} [\xi P_{i|i}^*(\pi_i^* > \lambda) + (1-\xi) P_{i|i}^*(\pi_i^* \geqq \lambda)] P_i^*(\mathcal{X}_i^*) \\
&\quad \geqq \sum_{i=1}^{k} [\xi Q_{i0}^*(\pi_i^* > \lambda) + (1-\xi) Q_{i0}^*(\pi_i^* \geqq \lambda)] P_i^*(\mathcal{X}_i^*) \\
&\quad = \sum_{i=1}^{k} \mathrm{E}_{Q_{i0}^*}[\varphi_i^*(X^*)] P_i^*(\mathcal{X}_i^*) \\
&\quad = \mathrm{E}_{Q_{10}^*}[\varphi_1^*(X^*)] \sum_{i=1}^{k} P_i^*(\mathcal{X}_i^*) .
\end{aligned}$$

Q.E.D.

COROLLARY 1. *If* $\hat{\mathcal{P}}_i^* = \mathcal{P}_i^*$ $(i=1, \cdots, k)$ *and if all* $P_i^* \in \mathcal{P}_i^*$ $(i=1, \cdots, k)$ *satisfy* $P_i^*(\mathcal{X}_i^*) \geqq 1/k$, *then*

$$\sum_{i=1}^{k} \mathrm{E}_{P_i}[\varphi_i^*(X^*)] > \alpha ,$$

*that is,* $\varphi^*$ *is unbiased.*

PROOF. We note that $\varphi_1^*$ is the most powerful test for the testing problem of $Q_{01}^*$ against $Q_{10}^*$ $(Q_{01}^* \neq Q_{10}^*)$ and hence $\mathrm{E}_{Q_{10}^*}[\varphi_1^*(X^*)] > \alpha$. Then this corollary immediately follows from Proposition 4. Q.E.D.

Now let us try to express $\varphi^*$ of (3.7) on the original sample space $\mathcal{X}$. Define $\pi_i$ $(i=1, \cdots, k)$ by $\pi_i(x) = \pi_i^*(\psi_i^{-1}(x))$, any according as $x \in \mathcal{X}_i$ or $x \notin \mathcal{X}_i$. Let $S_1(x) = \{i \mid \pi_i(x) < \lambda,\ i \in S(x)\}$, $S_2(x) = \{i \mid \pi_i(x) = \lambda,\ i \in S(x)\}$ and $S_3(x) = \{i \mid \pi_i(x) > \lambda,\ i \in S(x)\}$. Furthermore, let $S_{2i}(x) = \{i\} \cap S_2(x)$ and $S_{3i}(x) = \{i\} \cap S_3(x)$. Consider the test $\varphi$ given by

$$\begin{aligned}
&\varphi_0(x) = \frac{|S_1(x)| + \xi |S_2(x)|}{|S(x)|} \\
&\varphi_i(x) = \frac{(1-\xi)|S_{2i}(x)| + |S_{3i}(x)|}{|S(x)|} \qquad i=1, \cdots, k .
\end{aligned} \tag{3.9}$$

We can establish the following theorem.

THEOREM 1. *The test* $\varphi$ *of* (3.9) *is equivalent to the test* $\varphi^*$ *of* (3.7). *In particular,* $\varphi$ *is of size* $\alpha$ *and its power satisfies the inequality* (3.8).

PROOF. The first statement of this theorem easily follows as a consequence of the comparison between $\varphi$ and $\varphi^*$. The second and third statements are the immediate conclusions of Proposition 3 and Proposition 4, respectively. Q.E.D.

## 4. The testing problem of special capacities

As before, assume that we are given a sample space $\mathcal{X}$, a $\sigma$-field $\mathcal{B}$, the set $\mathcal{M}$ of all probability measures, and a slippage tuple $(\tilde{P}_0, \tilde{P}_1, \cdots, \tilde{P}_k)$. Let us consider the neighborhoods $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_k$ of $\tilde{P}_0, \tilde{P}_1, \cdots, \tilde{P}_k$ defined by

$$\begin{aligned} \mathcal{P}_0 &= \{P \in \mathcal{M} \mid P(B) \geqq (1-\varepsilon_0)\tilde{P}_0(B) - \delta_0 \text{ for all } B \in \mathcal{B}\} \\ \mathcal{P}_i &= \{P \in \mathcal{M} \mid P(B) \geqq (1-\varepsilon_1)\tilde{P}_i(B) - \delta_1 \text{ for all } B \in \mathcal{B}\} \\ & \qquad\qquad i=1,\cdots,k\,, \end{aligned} \tag{4.1}$$

where $0 \leqq \varepsilon_j,\ \delta_j < 1,\ 0 < \varepsilon_j + \delta_j < 1\ (j=0,1)$. We assume $\mathcal{P}_i \cap \mathcal{P}_j = \phi$ for $0 \leqq i \neq j \leqq k$ (this is the situation for small $\varepsilon_j$ and $\delta_j$). The neighborhoods $\mathcal{P}_i$ of (4.1) were introduced and called special capacity by Rieder [7], and regarded as a natural generalization of $\varepsilon$-contamination and total variation neighborhoods.

Suppose that we are interested in the problem (RSTP) with neighborhoods of (4.1). For the sake of applying the previous discussions we first need to give a family $\{\mathcal{X}_i\}$ of subsets of $\mathcal{X}$ such that $g\mathcal{X}_i = \mathcal{X}_{\tau_g i}$ and $\mathcal{X} = \bigcup_{i=1}^{k} \mathcal{X}_i$. As easily seen, a method to produce such a family $\{\mathcal{X}_i\}$ is to define $\mathcal{X}_i = \{x \in \mathcal{X} \mid f_i(x) = \max_{1 \leqq j \leqq k} f_j(x)\}$ where $f_i$ $i=1,\cdots,k$ are some measurable functions satisfying $f_i(x) = f_{\tau_g i}(gx)$ for $i=1,\cdots,k$ and all $g \in G$. We can refer to Proposition 4 about how to take these $f_i$ appropriately.

Let $\Delta_i(x)$ be a version of $d\tilde{P}_i/d\tilde{P}_0$. By (2.1) $\Delta_i$ can be chosen so as to satisfy $\Delta_i(x) = \Delta_{\tau_g i}(gx)$. Define

$$\mathcal{X}_i = \{x \mid \Delta_i(x) = \max_{1 \leqq j \leqq k} \Delta_j(x)\} \qquad i=1,\cdots,k\,. \tag{4.2}$$

Clearly, $\{\mathcal{X}_i\}$ is not disjoint. Hence the space $\mathcal{X}$ is extended to the new space $\mathcal{X}^*$ and our problem is written as the problem (RSTP)*. Let

$$\begin{aligned} \mathcal{P}_0^{*I} &= \{P_0^* \mid P_0^* \in \mathcal{P}_0^* \text{ is invariant under } G^*\} \\ \mathcal{P}_{0|i}^{*I} &= \{P_{0|i}^* \mid P_0^* \in \mathcal{P}_0^{*I}\} && i=1,\cdots,k \\ \mathcal{P}_i^{*y} &= \{P_i^* \mid P_i^*(\mathcal{X}_i^*) \geqq y,\ P_i^* \in \mathcal{P}_i^*\} && i=1,\cdots,k \\ \mathcal{P}_{i|i}^{*y} &= \{P_{i|i}^* \mid P_i^* \in \mathcal{P}_i^{*y}\} && i=1,\cdots,k \end{aligned}$$

and define $a_i^*$ $(i=1,\cdots,k)$ by the infimum of $a$ $(>0)$ satisfying

(4.3) $\mathcal{P}_{0|i}^{*I}\subset\{P^*\in\mathcal{M}_i^*\mid P^*(B^*)\geqq(1-a\varepsilon_0)\tilde{P}_{0|i}^*(B^*)-a\delta_0$ for all $B^*\in\mathcal{B}^*\}$ .

Similarly, define $a_i^{*y}$ $(i=1,\cdots,k)$ by the infimum of $a$ $(>0)$ satisfying

(4.4) $\mathcal{P}_{i|i}^{*y}\subset\{P^*\in\mathcal{M}_i^*\mid P^*(B^*)\geqq(1-a\varepsilon_1)\tilde{P}_{i|i}^*(B^*)-a\delta_1$ for all $B^*\in\mathcal{B}^*\}$ .

Note that $a_i^*$ and $a_i^{*y}$ are well-defined and independent of $i$. We write $\varepsilon_0^*=a^*\varepsilon_0$, $\delta_0^*=a^*\delta_0$, $\varepsilon_1^{*y}=a_y^*\varepsilon_1$ and $\delta_1^{*y}=a_y^*\delta_1$, where $a^*=a_i^*$ and $a_y^*=a_i^{*y}$ $(i=1,\cdots,k)$. Let

(4.5) $\mathcal{P}_{0i}^*=\{P^*\in\mathcal{M}_i^*\mid P^*(B^*)\geqq(1-\varepsilon_0^*)\tilde{P}_{0|i}^*(B^*)-\delta_0^*$ for all $B^*\in\mathcal{B}^*\}$

(4.6) $\mathcal{P}_{i0}^{*y}=\{P^*\in\mathcal{M}_i^*\mid P^*(B^*)\geqq(1-\varepsilon_1^{*y})\tilde{P}_{i|i}^*(B^*)-\delta_1^{*y}$ for all $B^*\in\mathcal{B}^*\}$ .

Also, define

$$y_0=\inf\{P_i^*(\mathcal{X}_i^*)\mid P_i^*\in\mathcal{P}_i^*\}\quad\text{and}\quad y_1=\sup\{P_i^*(\mathcal{X}_i^*)\mid P_i^*\in\mathcal{P}_i^*\}\ .$$

It is clear that $y_0$ and $y_1$ are the same for all $i=1,\cdots,k$. We assume that $y$ is a real number on $[y_0, y_1)$ and

(A.4′) $$\mathcal{P}_{0i}^*\cap\mathcal{P}_{i0}^{*y}=\phi\ .$$

Let us check (A.1) through (A.8). It should be understood that $\mathcal{P}_i^{*y}\subsetneqq\mathcal{P}_i^*$ for $y\in(y_0, y_1)$ whereas $\mathcal{P}_i^{*y_0}=\mathcal{P}_i^*$ and that the arguments in here are concerned with $\mathcal{P}_i^{*y}$ instead of $\mathcal{P}_i^*$. Since $P_ig^{-1}=P_{\tau_g i}$ for $i=0,\cdots,k$, all $g\in G$, we have by the definition (4.1) of $\mathcal{P}_i$ that if $P\in\mathcal{P}_i$, then $Pg^{-1}\in\mathcal{P}_{\tau_g i}$. This implies (A.1). Similarly, (A.2) and (A.3) follows from the facts that $\tilde{P}_{0|i}^*g^{*-1}=\tilde{P}_{0|\tau_{g^*}i}^*$ and $\tilde{P}_{i|i}^*g^{*-1}=\tilde{P}_{\tau_{g^*}i|\tau_{g^*}i}^*$ for $i=1,\cdots,k$, all $g^*\in G^*$. It is obvious that (A.4) holds by (A.4′). Also, (A.5) follows from (4.6) and the definitions of $\varepsilon_1^{*y}$ and $\delta_1^{*y}$. To show (A.6), we shall verify that $P_0^*\in\mathcal{P}_0^*$ implies $\bar{P}_0^*\in\mathcal{P}_0^*$, because by this fact and Lemma 2 we have (A.6). Let $P_0^*\in\mathcal{P}_0^*$ (i.e., $P_0\in\mathcal{P}_0$). Then for all $B^*\in\mathcal{B}^*$

(4.7) $$\bar{P}_0^*(B^*)=\frac{1}{|G^*|}\sum_{g^*\in G^*}P_0^*g^{*-1}(B^*)=\frac{1}{|G|}\sum_{g\in G}(P_0g^{-1})^*(B^*)=(\bar{P}_0)^*(B^*)\ .$$

Moreover $\bar{P}\in\mathcal{P}_0$ holds, because for all $B\in\mathcal{B}$

$$\bar{P}_0(B)=\frac{1}{|G|}\sum_{g\in G}P_0(g^{-1}B)\geqq\frac{1}{|G|}\sum_{g\in G}[(1-\varepsilon_0)\tilde{P}_0(g^{-1}B)-\delta_0]=(1-\varepsilon_0)\tilde{P}_0(B)-\delta_0\ .$$

Hence $\bar{P}^*\in\mathcal{P}_0^*$ follows from (4.7). Thus (A.6) is proved.

According to Theorem 5.2 and the discussions in Section 6 of Rieder [7], there exist the least favorable pairs $(Q_{0i}^{*y}, Q_{i0}^{*y})$ between $\mathcal{P}_{0i}^*$ and $\mathcal{P}_{i0}^{*y}$ $(i=1,\cdots,k)$ and a version $\pi_i^{*y}$ of $dQ_{i0}^{*y}/dQ_{0i}^{*y}$ is given by

$$(4.8)\qquad \pi_i^{*y}=\frac{1-\varepsilon_1^{*y}}{1-\varepsilon_0^*}(C_{i0}^y\vee \Delta_{i|i}^*\wedge C_{i1}^y)\qquad (\tilde{P}_{0|i}^*+\tilde{P}_{i|i}^*)\ \text{a.e.}\,,$$

where $\Delta_{i|i}^*\in d\tilde{P}_{i|i}^*/d\tilde{P}_{0|i}^*$, $a\vee b\wedge c=\max(a,\min(b,c))$, and $C_{i0}^y$ and $C_{i1}^y$ are constants determined by the following two equations:

$$(4.9)\qquad C_{i0}^y\tilde{P}_{0|i}^*(\Delta_{i|i}^*<C_{i0}^y)-\tilde{P}_{i|i}^*(\Delta_{i|i}^*<C_{i0}^y)=\nu_1^{*y}+\omega_0^*C_{i0}^{*y}\,,$$

$$(4.10)\qquad \tilde{P}_{i|i}^*(\Delta_{i|i}^*>C_{i1}^y)-C_{i1}^y\tilde{P}_{0|i}^*(\Delta_{i|i}^*>C_{i1}^y)=\nu_0^*C_{i1}^{*y}+\omega_1^{*y}\,,$$

$$\nu_0^*=(\varepsilon_0^*+\delta_0^*)/(1-\varepsilon_0^*),\qquad \omega_0^*=\delta_0^*/(1-\varepsilon_0^*)\,,$$

$$\nu_1^{*y}=(\varepsilon_1^{*y}+\delta_1^{*y})/(1-\varepsilon_1^{*y}),\quad \omega_1^{*y}=\delta_1^{*y}/(1-\varepsilon_1^{*y})\,.$$

Thus (A.7) and (A.8) hold. Notice that we can take $Q_{0i}^{*y}$, $Q_{i0}^{*y}$, $\pi_i^{*y}$ $i=1,\cdots,k$ so that (3.4), (3.5) and (3.6) are satisfied. In addition, it is readily observed that $\Delta_{i|i}^*$, $C_{i0}^y$, $C_{i1}^y$ $i=1,\cdots,k$ satisfy

$$(4.11)\qquad \Delta_{i|i}^*(x^*)=\Delta_{\tau_{g^*}i|\tau_{g^*}i}^*(g^*x^*)\qquad i=1,\cdots,k,\ \text{all}\ g^*\in G^*$$

$$(4.12)\qquad C_{i0}^y=C_{10}^y\,,\quad C_{i1}^y=C_{11}^y\qquad i=1,\cdots,k\,.$$

We consider the test $\varphi^{*y}$ presented by (3.7) (equivalently, (3.9)) with $\pi_i^*$ replaced by $\pi_i^{*y}$ $i=1,\cdots,k$. Of course $\varphi^{*y}$ satisfies Proposition 3 and Proposition 4. For this case, the following corollary holds.

COROLLARY 2. *If* $P_i^*\in\mathcal{P}_i^{*y}$ $i=1,\cdots,k$, *then*

$$(4.13)\qquad \sum_{i=1}^k \mathrm{E}_{P_i^*}[\varphi_i^{*y}(X^*)]\geqq ky\beta(y)\,,$$

*where* $\beta(y)=\mathrm{E}_{Q_{10}^{*y}}[\varphi_i^{*y}(X^*)]$.

PROOF. This follows immediately from the definition of $\mathcal{P}_i^{*y}$ and Proposition 4. Q.E.D.

We note that $\beta(y)>\alpha$ (because of $Q_{01}^{*y}\neq Q_{10}^{*y}$) and $\beta(y)$ is nondecreasing in $y$ and that if $y=y_0>1/k$, then $\varphi^{*y}$ is unbiased. Let us express $\varphi^*$ in terms of $x$ instead of $x^*$. Let $\varphi^y$ be defined (on $\mathcal{X}$) by

$$(4.14)\qquad \begin{aligned}&\varphi_0^y(x)=1,\ \xi_y,\ 0\qquad \text{if}\ \max_{1\leq j\leq k}\Delta_j^y(x)<,=,>\lambda_y\\ &\varphi_i^y(x)=0,\ (1-\varphi_0^y(x))/m(x)\qquad \text{if}\ \Delta_i(x)<,=\max_{1\leq j\leq k}\Delta_j(x)\,,\\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,k\,,\end{aligned}$$

where $m(x)$ $(=|S(x)|)$ is the number of times $\max_{1\leq j\leq k}\Delta_j(x)$ is attained, $\lambda_y$ and $\xi_y$ are constants determined by $\mathrm{E}_{Q_{01}^y}[\varphi_0^y(X)]=1-\alpha$, $Q_{01}^y=Q_{01}^{*y}\psi^{-1}$ and $\Delta_i^y=C_0^y\vee\Delta_i\wedge C_1^y$ is a truncated version of $\Delta_i$, $C_0^y=((1-\varepsilon_1^{*y})/(1-\varepsilon_0^*))(C_{10}^y/a)$, $C_1^y=((1-\varepsilon_1^{*y})/(1-\varepsilon_0^*))(C_{11}^y/a)$, $a=kb$, $b=\tilde{P}_1^*(\mathcal{X}_1^*)$. Then we can have the following theorem.

THEOREM 2. *The test $\varphi^{*y}$ is equivalent to the test $\varphi^{y}$.*

PROOF. Let $\Delta_i^*$ $(i=1,\cdots,k)$ be a version of $d\tilde{P}_i^*/d\tilde{P}_0^*$ such that $\Delta_i^*(x^*)=\Delta_{\tau_{g^*}i}^*(g^*x^*)$ hold for $i=1,\cdots,k$ and all $g^*\in G^*$. Then it is easily seen that

$$\Delta_i^*=a\Delta_{i|i}^* \qquad \text{a.e. } (\tilde{P}_{0|i}^*+\tilde{P}_{i|i}^*)\,. \tag{4.15}$$

Also we have

$$\Delta_i^*(x^*)=\Delta_i(\psi(x^*)) \qquad \text{a.e. } (\tilde{P}_{0|i}^*+\tilde{P}_{i|i}^*)\,. \tag{4.16}$$

To show (4.16), let $\Delta_i^*=q_{ii}^*/q_{0i}^*$ and $\Delta_i=q_{ii}/q_{0i}$, where $q_{0i}^*\in d\tilde{P}_0^*/d(\tilde{P}_0^*+\tilde{P}_i^*)$, $q_{ii}^*\in d\tilde{P}_i^*/d(\tilde{P}_0^*+\tilde{P}_i^*)$, $q_{0i}\in d\tilde{P}_0/d(\tilde{P}_0+\tilde{P}_i)$ and $q_{ii}\in d\tilde{P}_i/d(\tilde{P}_0+\tilde{P}_i)$. Then for all $B_i^*\in\mathcal{B}_i^*$

$$\begin{aligned}\int_{B_i^*} q_{ii}^* d(\tilde{P}_0^*+\tilde{P}_i^*) &= \sum_{j=1}^{k}\sum_{S\in\mathcal{S}}\frac{1}{|S|}\tilde{P}_j(\psi(B_i^*\cap\mathcal{X}_j^*)\cap\mathcal{X}_S)\\ &= \sum_{i\in S\in\mathcal{S}}\frac{1}{|S|}\int_{\psi(B_i^*)\cap\mathcal{X}_S} q_{ii}d(\tilde{P}_0+\tilde{P}_i)\\ &= \int_{\psi(B_i^*)} q_{ii}d((\tilde{P}_0+\tilde{P}_i)/|S|)\\ &= \int_{B_i^*} q_{ii}(\psi)d(\tilde{P}_0^*+\tilde{P}_i^*)\,.\end{aligned}$$

This implies

$$q_{ii}^*(x^*)=q_{0i}(\psi(x^*)) \qquad \text{a.e. } (\tilde{P}_{0|i}^*+\tilde{P}_{i|i}^*)\,.$$

Similarly,

$$q_{0i}^*(x^*)=q_{0i}(\psi(x^*)) \qquad \text{a.e. } (\tilde{P}_{0|i}^*+\tilde{P}_{i|i}^*)\,.$$

Thus (4.16) is verified. Therefore, taking $\Delta_i^*$, $\Delta_i$ so as to satisfy (4.15) and (4.16) for all $x^*\in\mathcal{X}_i^*$, it follows from (4.8) that $\varphi^{*y}$ can be written as (4.14). Q.E.D.

We can observe from (4.9), (4.10) and Lemma 4 of Rieder [7] that $C_{10}^y$ $(C_{11}^y)$ is strictly increasing (strictly decreasing) in $\varepsilon_0^*$, $\delta_0^*$, $\varepsilon_1^{*y}$, $\delta_1^{*y}$, and that $C_{10}^y\to 0$ and $C_{11}^y\to\infty$ hold as $\varepsilon_0^*\to 0$, $\delta_0^*\to 0$, $\varepsilon_1^{*y}\to 0$, $\delta_1^{*y}\to 0$. This fact implies that if $y\leqq b$ (i.e., $\tilde{P}_{i|i}^*\in\mathcal{P}_{i|i}^{*y}$) and if $\varepsilon_j\to 0$ and $\delta_j\to 0$ $(j=0,1)$, then $C_0^y\to 0$ and $C_1^y\to\infty$, and hence $\Delta_i^y\to\Delta_i$. That is, $\varphi^y$ converges to the test $\varphi^M$ of (4.14) with $\Delta_i^y$ replaced by $\Delta_i$, which is one of the most powerful size $\alpha$ tests for the testing problem of $\tilde{P}_0$ against $\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_k$ (see (2.4)). We also notice that $\varphi^y$ is a partially truncated version of $\varphi^M$.

Now, by changing $\Delta_i$ in (4.14) to $\bar{\Delta}_i$, we can obtain various robust tests $\bar{\varphi}^y$, that is,

$$
(4.17)\qquad \begin{aligned}
&\bar{\varphi}_0^y(x)=1,\ \xi_y,\ 0 \qquad \text{if } \max_{1\leq j\leq k} \Delta_j^y(x)<,=,>\lambda_y \\
&\bar{\varphi}_i^y(x)=0,\ (1-\bar{\varphi}_0^y(x))/m(x) \qquad \text{if } \bar{\Delta}_i(x)<,=\max_{1\leq j\leq k} \bar{\Delta}_j(x) \\
&\qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,k\,,
\end{aligned}
$$

where $\lambda_y$ and $\xi_y$ are the same constants as in (4.14), and $m(x)$ is the number of times $\max_{1\leq j\leq k} \bar{\Delta}_j(x)$ is attained.

In order to achieve the tests with good robustness property, we can take

$$
(4.18)\qquad \bar{\Delta}_i=d_0\vee\Delta_i\wedge d_1 \qquad i=1,\cdots,k\,,
$$

$$
(4.19)\qquad \bar{\Delta}_i=\prod_{j=1}^{k}\left(d_0^{ij}\vee\left(\frac{\Delta_i}{\Delta_j}\right)\wedge d_1^{ij}\right) \qquad i=1,\cdots,k\,,
$$

where $d_0$, $d_1$, $d_0^{ij}$ and $d_1^{ij}$ are some constants such that $d_0\to 0$, $d_1\to\infty$, $d_0^{ij}\to 0$, $d_1^{ij}\to\infty$ as $\varepsilon_1\to 0$, $\delta_1\to 0$.

We note that $\Delta_i=\max_{1\leq j\leq k}\Delta_j$ if and only if $\prod_{l=1}^{k}\left(\frac{\Delta_i}{\Delta_l}\right)=\max_{1\leq j\leq k}\prod_{l=1}^{k}\left(\frac{\Delta_j}{\Delta_l}\right)$. Hence $\bar{\varphi}^y$ with $\bar{\Delta}_i$ of (4.18) and (4.19) converge to $\varphi^y$ as $\varepsilon_1\to 0$, $\delta_1\to 0$. Thus when all the neighborhoods $\mathcal{P}_i$ shrink to $\tilde{P}_i$ $i=0,\cdots,k$, these tests $\bar{\varphi}^y$ also converge to $\varphi^M$. As for $d_0^{ij}$ and $d_1^{ij}$, we especially recommend the ones determined by the following two equations:

$$
(4.20)\qquad \begin{aligned}
&d_0^{ij}\tilde{P}_j(\Delta_{ij}<d_0^{ij})-\tilde{P}_i(\Delta_{ij}<d_0^{ij})=\nu_1+\omega_1 d_0^{ij}\,, \\
&\tilde{P}_i(\Delta_{ij}>d_1^{ij})-d_1^{ij}\tilde{P}_j(\Delta_{ij}>d_1^{ij})=\nu_1 d_1^{ij}+\omega_1\,, \\
&\qquad \nu_1=(\varepsilon_1+\delta_1)/(1-\nu_1),\ \omega_1=\delta_1/(1-\varepsilon_1),\ \Delta_{ij}\in d\tilde{P}_i/d\tilde{P}_j\,.
\end{aligned}
$$

The reason is that $\bar{\varphi}^y$ with $d_0^{ij}$ and $d_1^{ij}$ of (4.20) uses the least favorable pairs for $(\mathcal{P}_i, \mathcal{P}_j)$ $1\leqq i\neq j\leqq k$. However, needless to say, $\bar{\varphi}^y$ is less powerful than $\varphi^y$ near $(\tilde{P}_1,\cdots,\tilde{P}_k)$.

Of course we are particularly interested in the case of $y=y_0$ (i.e. $\mathcal{P}_i^{*y}=\mathcal{P}_i^*$ $i=1,\cdots,k$). In this case, the test $\bar{\varphi}$ $(=\bar{\varphi}^{y_0}$ with $d_0^{ij}$ and $d_1^{ij}$ of (4.20)) is just the one that was conjectured at the first stage as a test with good robustness property.

Next let us consider the situation that we have $n$ mutually independent random elements $X_1, X_2,\cdots, X_n$ taking values in $\mathcal{X}$. Let $X^n=(X_1, X_2,\cdots, X_n)$ and denote by $(\mathcal{X}^n, \mathcal{B}^n)$ the $n$-fold product measurable space of $(\mathcal{X}, \mathcal{B})$. Also let $\mathcal{P}_i^n=\mathcal{P}_i\times\cdots\times\mathcal{P}_i$ ($n$-fold product of $\mathcal{P}_i$) $i=0,\cdots,k$. Suppose that we wish to test

$$
(\text{RSTP})^n\quad H_0^n:\ \mathcal{L}(X^n)\in\mathcal{P}_0^n \text{ against } H_i^n:\ \mathcal{L}(X^n)\in\mathcal{P}_i^n \qquad i=1,\cdots,k\,.
$$

For this problem we can use the following test $\bar{\varphi}_n^y$:

$$\text{(4.21)}\quad \begin{aligned} &\bar{\varphi}_{n0}^y(x^n)=1,\ \xi_n,\ 0 \qquad \text{if } \max_{1\leq j\leq k} \Delta_{nj}^y(x^n)<,=,>\lambda_n \\ &\bar{\varphi}_{ni}^y(x^n)=0,\ (1-\bar{\varphi}_{n0}^y(x^n))/m(x^n) \qquad \text{if } \bar{\Delta}_{ni}(x^n)<,=\max_{1\leq j\leq k} \bar{\Delta}_{nj}^y(x^n) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,k\,, \end{aligned}$$

where

$$\Delta_{ni}^y(x^n)=\prod_{l=1}^n \Delta_i^y(x_l)\,, \qquad \bar{\Delta}_{ni}^y(x^n)=\prod_{l=1}^n \bar{\Delta}_i^y(x_l)\,,$$

$m(x^n)$ is the number of times $\max_{1\leq j\leq k} \bar{\Delta}_{nj}^y(x^n)$ is attained, $\lambda_n$ and $\xi_n$ are constants such that

$$Q_y^n\left[\prod_{l=1}^n \max_{1\leq j\leq k} \Delta_j^y(x_l)<\lambda_n\right]+\xi_n Q_y^n\left[\prod_{l=1}^n \max_{1\leq j\leq k} \Delta_j^y(x_l)=\lambda_n\right]=1-\alpha\,,$$

$Q_y$ is the probability measure (on $(\mathcal{X}, \mathcal{B})$) defined by

$$\text{(4.22)}\qquad Q_y(B)=\sum_{S\in\mathcal{S}} \frac{1}{|S|} \sum_{i\in S} Q_{0i}^y(B\cap\mathcal{X}_S) \qquad B\in\mathcal{B}\,,$$

$Q_{0i}^y=Q_{0i}^{*y}\psi^{-1}$ $i=1,\cdots,k$ and $Q_y^n$ is the $n$-fold product of $Q_y$.

For defining $Q_y$ by (4.22), we have to require the condition that $T$ is $k$-fold transitive, that is, $T$ is the symmetric group (in this case, we have $Q_{0i}^y(\mathcal{X}_S)=Q_{0j}^y(\mathcal{X}_S)$ for all $i, j\in S$ and all $S\in\mathcal{S}$). It can be easily seen that the test $\bar{\varphi}_n^y$ is of size $\alpha$.

## 5. The $k$-sample testing problem

Let $\mathcal{X}^\circ$ be a sample space, $\mathcal{B}^\circ$ a $\sigma$-field on $\mathcal{X}^\circ$, $\mathcal{M}^\circ$ the set of all probability measures on $(\mathcal{X}^\circ, \mathcal{B}^\circ)$. Let $\tilde{P}_0^\circ$, $\tilde{P}_1^\circ$ denote two distinct elements of $\mathcal{M}^\circ$, and $\mathcal{P}_0^\circ$, $\mathcal{P}_1^\circ$ two disjoint neighborhoods of $\tilde{P}_0^\circ$, $\tilde{P}_1^\circ$, respectively. Also let $X_i$ $i=1,\cdots,k$ be mutually independent random elements taking values in $\mathcal{X}^\circ$. Then we are interested in testing

$$H_0:\ \mathcal{L}(X_j)\in\mathcal{P}_0^\circ \qquad j=1,\cdots,k\,.$$

(RSTP)$_0$ against

$$H_i:\ \mathcal{L}(X_i)\in\mathcal{P}_1^\circ,\ \mathcal{L}(X_j)\in\mathcal{P}_0^\circ \qquad j=1,\cdots,k,\ j\neq i,\ i=1,\cdots,k\,.$$

Let us show below that this problem (RSTP)$_0$ is a special case of the robust slippage testing problem (RSTP) formulated in Section 2. Denote $X=(X_1,\cdots,X_k)$, $\mathcal{X}=\mathcal{X}^\circ\times\cdots\times\mathcal{X}^\circ$ ($k$-fold product of $\mathcal{X}^\circ$), $\mathcal{B}=\mathcal{B}^\circ\times\cdots\times\mathcal{B}^\circ$ ($k$-fold product of $\mathcal{B}^\circ$), $\mathcal{P}_0=\mathcal{P}_0^\circ\times\cdots\times\mathcal{P}_0^\circ$ ($k$-fold product of $\mathcal{P}_0^\circ$)

and $\mathcal{P}_i=\mathcal{P}_0^\circ\times\cdots\times\mathcal{P}_1^\circ\times\cdots\times\mathcal{P}_0^\circ$ ($\mathcal{P}_1^\circ$ in $i$th place, $\mathcal{P}_0^\circ$ in other places) $i=1,\cdots,k$.

Let $T$ be the symmetric group of all permutations on $\{1,2,\cdots,k\}$ and define transformations $g_\tau$ on $\mathcal{X}$ by $g_\tau: x_i\rightarrow x_{\tau^{-1}i}$ $i=1,\cdots,k$, where $\tau(1,\cdots,k)=(\tau 1,\cdots,\tau k)$, $\tau\in T$. It is easily seen that $G=\{g_\tau|\tau\in T\}$ is a transformation group with the operation $g_{\tau_1}\circ g_{\tau_2}=g_{(\tau_1\circ\tau_2)}$ $(\tau_1,\tau_2\in T)$ and isomorphic to $T$ by the correspondence $g_\tau\rightarrow\tau$. Denote such pairs $(g_\tau,\tau)$ by $(g,\tau_g)$ and define $\tau_g 0=0$ for all $g\in G$. Noting that if $P=P_1^\circ\times\cdots\times P_k^\circ$, then $Pg^{-1}=P_{\tau_g 1}^\circ\times\cdots\times P_{\tau_g k}^\circ$, we can readily observe that (A.1) holds. It follows immediately that the present problem $(\mathrm{RSTP})_0$ is written as the problem (RSTP).

Our main aim in this section is to establish a theorem which yields a least favorable slippage tuple for $(\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_k)$ from a least favorable pair for $(\mathcal{P}_0^\circ,\mathcal{P}_1^\circ)$. To begin with, we give the following lemma.

LEMMA 4. *If $(\hat{P}_0,\hat{P}_1,\cdots,\hat{P}_k)$, $\hat{P}_i\in\mathcal{P}_i$, is a slippage tuple such that for all $P_i\in\mathcal{P}_i$ $i=0,\cdots,k$ and all $t$,*

$$P_0(\hat{\pi}_i>t)\leqq\hat{P}_0(\hat{\pi}_i>t)\qquad i=1,\cdots,k\,, \tag{5.1}$$

$$P_i(\hat{\pi}_i>t)\geqq\hat{P}_i(\hat{\pi}_i>t)\qquad i=1,\cdots,k\,, \tag{5.2}$$

$$P_i(\hat{\pi}_j>t)\leqq\hat{P}_i(\hat{\pi}_j>t)\qquad i,j=1,\cdots,k,\ i\neq j\,, \tag{5.3}$$

$$\hat{\pi}_i\ \ i=1,\cdots,k \text{ are mutually independent}\,, \tag{5.4}$$

*then $(\hat{P}_0,\hat{P}_1,\cdots,\hat{P}_k)$ is the least favorable for $(\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_k)$, where $\hat{\pi}_i$ is a version of $d\tilde{P}_i/d\tilde{P}_0$.*

PROOF. Let $\hat{Q}=(\hat{P}_0,\hat{P}_1,\cdots,\hat{P}_k)$, and take $\xi(x)=\xi$ (a constant) and $\eta_i(x)=(1-\varphi_0^{\hat{Q}}(x))/m(x)$ in (2.4), where $m(x)$ is the number of times $\max_{1\leqq j\leqq k}\hat{\pi}_j(x)$ is attained. It is obviously sufficient to show that $\varphi^{\hat{Q}}$ satisfies (2.5) and (2.6). First, to show (2.5) note that for any $P_0\in\mathcal{P}_0$ $\hat{P}_0\hat{\pi}_i^{-1}$ is stochastically larger than $P_0\hat{\pi}_i^{-1}$ by (5.1), and that $\varphi_0^{\hat{Q}}$ is nonincreasing in $\hat{\pi}_i$. Then, using (5.4) and Theorem 1 of Lehmann [6] we have (2.5).

Secondly, to show (2.6) we note that (5.2) ((5.3)) implies that for any $P_i\in\mathcal{P}_i$ $\hat{P}_i\hat{\pi}_i^{-1}$ $(\hat{P}_i\hat{\pi}_j^{-1})$ is stochastically smaller (larger) than $\hat{P}_i\hat{\pi}_i^{-1}$ $(\hat{P}_i\hat{\pi}_j^{-1})$. Moreover notice that $\varphi_i^{\hat{Q}}$ is nondecreasing (nonincreasing) in $\hat{\pi}_i$ $(\hat{\pi}_j)$ $i,j=1,\cdots,k$, $i\neq j$. Hence by (5.4) and Theorem 1 of Lehmann [6] we have $\mathrm{E}_{P_i}[\varphi_i^{\hat{Q}}(X)]\geqq\mathrm{E}_{\hat{P}_i}[\varphi_i^{\hat{Q}}(X)]$ $i=1,\cdots,k$. This implies (2.6). Thus the proof of this lemma is completed. Q.E.D.

THEOREM 3. *If $(\hat{P}_0^\circ,\hat{P}_1^\circ)$ is a least favorable pair for $(\mathcal{P}_0^\circ,\mathcal{P}_1^\circ)$, then $(\hat{P}_0,\hat{P}_1,\cdots,\hat{P}_k)$ is a least favorable slippage tuple for $(\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_k)$,* *where* $\hat{P}_0=\hat{P}_0^\circ\times\cdots\times\hat{P}_0^\circ$ *(k-fold product of* $\hat{P}_0^\circ$*) and* $\hat{P}_i=\hat{P}_0^\circ\times\cdots\times\hat{P}_1^\circ\times\cdots\times\hat{P}_0^\circ$ *(*$\hat{P}_1^\circ$ *in ith place,* $P_0^\circ$ *in other places)* $i=1,\cdots,k$.

PROOF. It is obvious that $(\hat{P}_0,\hat{P}_1,\cdots,\hat{P}_k)$ is a slippage tuple. We show that (5.1) through (5.4) hold for all $P_i\in\mathcal{P}_i$ $i=0,\cdots,k$. Since $(\hat{P}_0^\circ,\hat{P}_1^\circ)$ is the least favorable for $(\mathcal{P}_0^\circ,\mathcal{P}_1^\circ)$, it is satisfied by the definition that for all $t$

$$\hat{P}_0^\circ(\hat{\pi}^\circ>t)=\sup\{P_0^\circ(\hat{\pi}^\circ>t)\,|\,P_0^\circ\in\mathcal{P}_0^\circ\}\ , \tag{5.5}$$

$$\hat{P}_1^\circ(\hat{\pi}^\circ>t)=\inf\{P_1^\circ(\hat{\pi}^\circ>t)\,|\,P_1^\circ\in\mathcal{P}_1^\circ\}\ , \tag{5.6}$$

where $\hat{\pi}^\circ$ is a version of $d\hat{P}_1^\circ/d\hat{P}_0^\circ$. Define $\hat{\pi}_i$ $(i=1,\cdots,k)$ by $\hat{\pi}_i(x)=\hat{\pi}^\circ(x_i)$, where $x=(x_1,\cdots,x_k)$. Then we can see that $\hat{\pi}_i$ is a version of $d\hat{P}_i/d\hat{P}_0$. Let $P_i\in\mathcal{P}_i$ be denoted by $P_i=P_{i1}^\circ\times\cdots\times P_{ik}^\circ$, where $P_{ii}^\circ\in\mathcal{P}_1^\circ$ and $P_{ij}^\circ\in\mathcal{P}_0^\circ$ for $i\neq j$. Then from (5.5) it follows that for $i=0,\cdots,k$, $j=1,\cdots,k$, $i\neq j$ and all $t$

$$P_i(\hat{\pi}_j>t)=P_{ij}^\circ(\hat{\pi}^\circ>t)\leqq\hat{P}_0^\circ(\hat{\pi}^\circ>t)=\hat{P}_i(\hat{\pi}_j>t)\ .$$

Thus (5.1) and (5.3) hold. Similarly, by (5.6) we have

$$P_i(\hat{\pi}_i>t)=P_{ii}^\circ(\hat{\pi}^\circ>t)\geqq\hat{P}_1^\circ(\hat{\pi}^\circ>t)=\hat{P}_i(\hat{\pi}_i>t)\ .$$

This is (5.2). It is clear from the definition of $\hat{\pi}_i$ that (5.4) holds. Therefore, applying Lemma 4 completes the proof of this theorem.

Q.E.D.

Next let us consider the situation that all the sizes of $k$ samples are equal to $n$. Suppose that we have mutually independent random elements $X_{il}$ $i=1,\cdots,k$, $l=1,\cdots,n$ taking values in $\mathcal{X}^\circ$. We now wish to test

$$H_0^n:\ \mathcal{L}(X_{jl})\in\mathcal{P}_0^\circ \qquad j=1,\cdots,k,\ l=1,\cdots,n$$

$(\mathrm{RSTP})_0^n$ against

$$H_i^n:\ \mathcal{L}(X_{il})\in\mathcal{P}_1^\circ,\ \mathcal{L}(X_{jl})\in\mathcal{P}_0^\circ \qquad j\neq i,\ l=1,\cdots,n,\ i=1,\cdots,k\ .$$

We note here that the $i$th sample $X_{il}$ $l=1,\cdots,n$ need not be identically distributed. This $(\mathrm{RSTP})_0^n$ is a robust version of the $k$ sample slippage problem of testing $H_0:\mathcal{L}(X_{jl})=\tilde{P}_0^\circ$, $j=1,\cdots,k$, $l=1,\cdots,n$ against $H_i:\mathcal{L}(X_{il})=\tilde{P}_1^\circ$, $\mathcal{L}(X_{jl})=\tilde{P}_0^\circ$ $j\neq i$, $l=1,\cdots,n$; $i=1,\cdots,k$.

Let $X_l=(X_{1l},X_{2l},\cdots,X_{kl})$ $l=1,\cdots,n$ and $X^n=(X_1,X_2,\cdots,X_n)$. We denote by $(\mathcal{X}^n,\mathcal{B}^n)$ the $n$-fold product measurable space of $(\mathcal{X},\mathcal{B})$ and let $\mathcal{P}_i^n=\mathcal{P}_i\times\cdots\times\mathcal{P}_i$ ($n$-fold product of $\mathcal{P}_i$) $i=0,\cdots,k$. Then the problem $(\mathrm{RSTP})_0^n$ is written in a simple form:

$$H_0^n:\ \mathcal{L}(X^n)\in\mathcal{P}_0^n \text{ against } H_i^n:\ \mathcal{L}(X^n)\in\mathcal{P}_i^n \qquad i=1,\cdots,k\,.$$

Let transformations $g_n$ on $\mathcal{X}^n$ be defined by $g_n(x^n)=(gx_1, gx_2,\cdots, gx_n)$. Clearly, $G_n=\{g_n|g\in G\}$ is a group with the operation $g_n\circ g'_n=(g\circ g')_n$. Then the following result can be obtained.

COROLLARY 3. *If there exists a least favorable pair* $(\hat{P}_0^\circ, \hat{P}_1^\circ)$ *for* $(\mathcal{P}_0^\circ, \mathcal{P}_1^\circ)$, *then for any sample size* $n$ *and any size* $\alpha\in(0,1)$ *maximin tests* $\hat{\varphi}_n$ *for the problem* $(\mathrm{RSTP})_0^n$ *are given by*

$$(5.7)\quad \begin{aligned} &\hat{\varphi}_{n0}(x^n)=1,\ \xi_n,\ 0 \qquad \text{if } \max_{1\leq j\leq k}\prod_{l=1}^n \hat{\pi}^\circ(x_{jl})<,=,>\lambda_n \\ &\hat{\varphi}_{ni}(x^n)=0,\ (1-\hat{\varphi}_{n0}(x^n))/m(x) \qquad \text{if } \prod_{l=1}^n \hat{\pi}^\circ(x_{il})<,=\max_{1\leq j\leq k}\prod_{l=1}^n \hat{\pi}^\circ(x_{jl}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,k\,, \end{aligned}$$

*where* $\hat{\pi}^\circ$ *is a version of* $d\hat{P}_1^\circ/d\hat{P}_0^\circ$, $m(x^n)$ *is the number of times* $\max_{1\leq j\leq k}\prod_{l=1}^n \hat{\pi}^\circ(x_{jl})$ *is attained, and* $\lambda_n$ *and* $\xi_n$ *are constants determined by* $\mathrm{E}_{\hat{P}_0^n}[\hat{\varphi}_{n0}\cdot(X^n)]=1-\alpha$, $\hat{P}_0^n=\hat{P}_0\times\cdots\times\hat{P}_0$ (*n-fold product of* $\hat{P}_0$).

PROOF. Let $\hat{\pi}_i^n\in d\hat{P}_i^n/d\hat{P}_0^n$, where $\hat{P}_i^n$ is the $n$-fold product of $\hat{P}_i$. Note that we can take $\hat{\pi}_i^n(x^n)=\prod_{l=1}^n \pi_i(x_l)=\prod_{l=1}^n \hat{\pi}^\circ(x_{il})$, where $\hat{\pi}_i\in d\hat{P}_i/d\hat{P}_0$. Since $\hat{\pi}_i(X_l)$ $i=1,\cdots,k$, $l=1,\cdots,n$ are mutually independent and satisfy (5.1) through (5.4) for each $l$, we have (5.1) through (5.4) with $P_i$, $\hat{P}_i$, $\hat{\pi}_i$ replaced by $W_i$ $(\in\mathcal{P}_i^n)$, $\hat{P}_i^n$, $\hat{\pi}_i^n$, respectively. Also it easily follows that $(\hat{P}_0^n, \hat{P}_1^n,\cdots,\hat{P}_k^n)$ is a slippage tuple with respect to $G_n$. These facts and Lemma 4 imply that $(\hat{P}_0^n, \hat{P}_1^n,\cdots,\hat{P}_k^n)$ is a least favorable slippage tuple for $(\mathcal{P}_0^n, \mathcal{P}_1^n,\cdots,\mathcal{P}_k^n)$. Therefore this corollary is an immediate consequence of Theorem 3 and Proposition 1. Q.E.D.

*Remark* 2. The test $\hat{\varphi}_n$ is invariant under $G_n$.

*Examples*. We consider Huber's ([3], [4]) results. Let $\mathcal{X}^\circ$ be the real line, $\mathcal{B}^\circ$ the Borel $\sigma$-field on $\mathcal{X}^\circ$ and $\mathcal{M}^\circ$ the set of all probability measures on $(\mathcal{X}^\circ, \mathcal{B}^\circ)$. Let $\tilde{P}_0^\circ$ and $\tilde{P}_1^\circ$ be two distinct probability measures on $(\mathcal{X}^\circ, \mathcal{B}^\circ)$ with their densities $\tilde{p}_0^\circ$ and $\tilde{p}_1^\circ$ with respect to some measure $\mu$. We assume that the likelihood ratio $\tilde{p}_1^\circ(x)/\tilde{p}_0^\circ(x)$ is monotone in $x$ a.e. $\mu$. Let

$$(5.8)\quad \begin{aligned} &\mathcal{P}_0^\circ=\{P^\circ\in\mathcal{M}^\circ\,|\,P^\circ(X<t)\geqq(1-\varepsilon_0)\tilde{P}_0^\circ(X<t)-\delta_0 \text{ for all } t\}\,, \\ &\mathcal{P}_1^\circ=\{P^\circ\in\mathcal{M}^\circ\,|\,P^\circ(X>t)\geqq(1-\varepsilon_1)\tilde{P}_1^\circ(X>t)-\delta_1 \text{ for all } t\}\,, \end{aligned}$$

where $0\leqq\varepsilon_0, \varepsilon_1, \delta_0, \delta_1<1$ are some given numbers. We assume that $\mathcal{P}_0^\circ$

and $\mathcal{P}_1^\circ$ are disjoint (i.e., that $\varepsilon_j$ and $\delta_j$ are sufficiently small). Then there exists a least favorable pair $(\hat{P}_0^\circ, \hat{P}_1^\circ)$ for $(\mathcal{P}_0^\circ, \mathcal{P}_1^\circ)$. Hence, according to Corollary 3 the maximin tests for the problem $(\mathrm{RSTP})_0^n$ are given by (5.7) with

$$\hat{\pi}^\circ(x)=C_0\vee\frac{\tilde{p}_1^\circ(x)}{\tilde{p}_0^\circ(x)}\wedge C_1\,, \tag{5.9}$$

where $C_0$ and $C_1$ are constants determined by the equations:

$$\begin{aligned}&C_0\tilde{P}_0^\circ[(\tilde{p}_1^\circ(x)/\tilde{p}_0^\circ(x))<C_0]-\tilde{P}_1^\circ[(\tilde{p}_1^\circ(x)/\tilde{p}_0^\circ(x))<C_0]=\nu_1+\omega_0C_0\,,\\ &\tilde{P}_1^\circ[(\tilde{p}_1^\circ(x)/\tilde{p}_0^\circ(x))>C_1]-C_1\tilde{P}_0^\circ[(\tilde{p}_1^\circ(x)/\tilde{p}_0^\circ(x))>C_1]=\nu_0C_1+\omega_1\,,\end{aligned} \tag{5.10}$$

$$\nu_j=\frac{\varepsilon_j+\delta_j}{1-\varepsilon_j},\quad \omega_j=\frac{\delta_j}{1-\varepsilon_j}\quad j=0,1\,.$$

We note that $\mathcal{P}_0^\circ$ contains each of neighborhoods of $\tilde{P}_0^\circ$ in terms of $\varepsilon$-contamination, total variation, Prohorov distance, Kolmogorov distance and Lévy distance, and that $\hat{P}_0^\circ$ is contained in each of them. These facts imply that the maximin tests (5.7) with $\hat{\pi}^\circ$ of (5.9) are also maximin for the above five types of neighborhoods. Moreover it is emphasized that if $\{\tilde{P}_\theta\,|\,\theta\in R\}$ is a monotone likelihood ratio family, then the maximin tests (5.7) constructed for neighborhoods $\mathcal{P}_j^\circ$ of $\tilde{P}_{\theta_j}^\circ$ $(j=0,1)$ are maximin for the problem $(\mathrm{RSTP})_0^n$ not only of $\tilde{P}_{\theta_0}^\circ$ and $\tilde{P}_{\theta_1}^\circ$ but also of $\{\tilde{P}_\theta^\circ\,|\,\theta\leqq\theta_0\}$ and $\{\tilde{P}_\theta^\circ\,|\,\theta\geqq\theta_1\}$ where $\theta_0<\theta_1$.

In particular, consider the case that $\tilde{P}_j^\circ$ $(j=0,1)$ are normal distributions $N(\theta_j,1)$ where $\theta_0<\theta_1$. Suppose that $\varepsilon_0=\varepsilon_1=\varepsilon$ and $\delta_0=\delta_1=\delta$. As easily seen, letting $Y=X-(\theta_0+\theta_1)/2$, the problem $(\mathrm{RSTP})_0^n$ is reduced to the case of $N(-\Delta,1)$ and $N(\Delta,1)$, where $\Delta=(\theta_1-\theta_0)/2$. From symmetry $C_0=1/C_1$ follows. Write $C_0=e^{-2\Delta b}$. Then (5.10) reduces to

$$e^{-2\Delta b}\Phi(\Delta-b)-\Phi(-\Delta-b)=\frac{\varepsilon+\delta+\delta e^{-2\Delta b}}{1-\varepsilon}\,,$$

where $\Phi$ denotes the distribution function of $N(0,1)$. Let $b$ be a solution of this equation. Then the maximin tests $\varphi_n$ of (5.7) are equivalent to

$$\varphi_{n0}(x^n)=1,\ \xi_n,\ 0\qquad\text{if }\max_{1\leqq j\leqq k}\sum_{l=1}^n\psi\Big(x_{jl}-\frac{\theta_0+\theta_1}{2}\Big)<,=,>\lambda_n$$

$$\varphi_{ni}(x^n)=0,\ (1-\varphi_{n0}(x^n))/m(x^n)$$
$$\text{if }\sum_{l=1}^n\psi\Big(x_{il}-\frac{\theta_0+\theta_1}{2}\Big)<,=\max_{1\leqq j\leqq k}\sum_{l=1}^n\psi\Big(x_{jl}-\frac{\theta_0+\theta_1}{2}\Big),$$
$$i=1,\cdots,k\,,$$

where

$$\psi(x)=(-b)\vee x\wedge b .$$

This is a robust version of a primitive form of tests for outliers. Of course these tests are also maximin for the problem $(\mathrm{RSTP})_0^n$ of $\{N(\theta, 1)|\theta\leqq\theta_0\}$ and $\{N(\theta, 1)|\theta\geqq\theta_1\}$.

The other examples of $(\mathcal{P}_0^\circ, \mathcal{P}_1^\circ)$ for which there exists a least favorable pair are found in Huber [4], Huber and Strassen [5] and Rieder [7].

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