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RANK ANALOGUES OF THE LIKELIHOOD RATIO TEST FOR AN ORDERED ALTERNATIVE IN A TWO-WAY LAYOUT

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Summary

Distribution-free tests for no treatment effect against the simple order alternative in a two-way layout with equal number of observations per cell are considered. The nonparametric test statistics are constructed by the rank analogues of the likelihood ratio test statistic assuming normality (i) based on within-block rankings and (ii) based on combined rankings of all the observations after alignment within each block. The exact distributions are given and large sample properties are investigated. The asymptotic power of the test (i) as the number of observations per cell tends to infinity can be satisfied enough, and in the case that the number of blocks tends to infinity, the asymptotic power of the test (ii) is almost higher than that of the test (i). Also these rank tests are compared with linear rank tests and it is shown that these proposed tests are robust by a table.

1. Introduction

Consider a randomized block design with n blocks, p treatments and N observations per cell, in which each observation is expressed as

$$(1.1) \qquad X_{ijk}=\mu+\beta_i+\tau_j+e_{ijk} \qquad (i=1,2,\cdots,n,\ j=1,2,\cdots,p,\ k=1,2,\cdots,N)\,,$$

where the block effect β_i and the treatment effect τ_j satisfy $\sum_{i=1}^{n}\beta_i=\sum_{j=1}^{p}\tau_j=0$ and error random variables $\{e_{ijk}: i=1,\cdots,n,\ j=1,\cdots,p,\ k=1,\cdots,N\}$ are independent and identically distributed to a continuous distribution function $F(x)$ with density $f(x)$.

The problem is to test the null hypothesis $H: \tau_j=0$ $(j=1,\cdots,p)$ for unknown F. By the way, in testing the alternative hypothesis that the treatment effects are not equal, there are two distribution-free test statistics given by the rank analogues of the likelihood ratio test statistic assuming normality. Friedman [7] and Mack and Skillings [9] discussed the case of the within-block rank test and showed that, assuming the normal distribution and the Wilcoxon scores, the asymptotic relative efficiency with respect to the likelihood ratio test is between $2/\pi$ and $3/\pi$, depending on the block size as the number of blocks n tends to infinity and is $3/\pi$ as the number of observations N per cell tends to infinity. In the case of the aligned rank test, Mehra and Sarangi [10] showed that the asymptotic relative efficiency is larger than $3/\pi$ when the distribution is normal and the scores function is of Wilcoxon type, and Sen [11] discussed the general scores.

So in this paper, the alternative hypothesis of interest is $K: \tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_p$ with at least one strict inequality. The proposed test statistics are constructed by the rank analogues of the likelihood ratio test statistic as the similar way to the above papers. In Section 3, we state that the exact distributions of these distribution-free tests are mixtures of distributions of the Friedman rank test and Sen's [11] aligned rank test respectively. In Section 4, using the results of Section 3, we show that the asymptotic distribution under the null hypothesis is a mixture of χ-square distributions. In Section 5, we compute the asymptotic Pitman efficiency with respect to the likelihood ratio test after we study the asymptotic distribution under a contiguous sequence of location alternatives. Also as Araki and Shirahata [1] and Sen [11] proposed linear rank tests for the ordered alternative, in Section 6, we compare these tests with linear rank tests by the asymptotic power, using the numerical computation.

2. The test statistics

If F is the normal distribution function with mean zero and known variance σ^2, the likelihood ratio test for H versus K is to reject H when the following statistic is too large,

$$\bar{\chi}^2_{nN}(X) = nN \sum_{i=1}^{p} (\hat{\tau}_i - \bar{X}_{\cdots})^2/\sigma^2 , \tag{2.1}$$

where

$$\bar{X}_{\cdots} = \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{N} X_{ijk}/(pnN) , \qquad \bar{X}_{\cdot j \cdot} = \sum_{i=1}^{n} \sum_{k=1}^{N} X_{ijk}/(nN)$$

and $\hat{\tau}_i$ $(i=1, 2, \cdots, p)$ are τ_i $(i=1, 2, \cdots, p)$ which minimize $\sum_{j=1}^{p} (\bar{X}_{\cdot j \cdot} - \tau_j)^2$ under order restriction, that is, $\tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_p$. Then from Chacko [6], $\hat{\tau}_i$ is equal to $\max_{1 \leq s \leq i} \min_{i \leq t \leq p} \sum_{j=s}^{t} \bar{X}_{\cdot j \cdot}/(t-s+1)$ and there is the Pool-Adjacent-

Violators algorithm in Barlow et al. [2] as the way of giving $\hat{\tau}_i$. It is found that $\bar{\chi}^2_{nN}(X)$ has a mixture of χ-square distributions stated in Theorem 4.1 under H from Theorem 1 of Shorack [14].

For deriving the rank statistic corresponding to (2.1), we define within-block rank R_{ijk} by the rank of X_{ijk} among $\{X_{ijk}: j=1, 2, \cdots, p, k=1, 2, \cdots, N\}$ within ith block and let the scores function $a_{pN}(\cdot)$ be a mapping from $\{1, 2, \cdots, pN\}$ to real numbers such that $\sum_{k=1}^{pN} a_{pN}(k)=0$. The hypothesis H is rejected when the following statistic constructed by substituting $a_{pN}(R_{ijk})$ for X_{ijk} in the statistic given by (2.1) is too large,

$$\bar{\chi}^2_{nN}(a(R))=nN(pN-1)\sum_{i=1}^{p}\tilde{\tau}_i^2\Big/\sum_{i=1}^{pN}\{a_{pN}(i)\}^2 , \tag{2.2}$$

where

$$\bar{a}_{pN}(R_{\cdot j \cdot})=\sum_{i=1}^{n}\sum_{k=1}^{N} a_{pN}(R_{ijk})/(nN)$$

and

$$\tilde{\tau}_i=\max_{1\leq s\leq i}\min_{i\leq t\leq p}\sum_{j=s}^{t}\bar{a}_{pN}(R_{\cdot j \cdot})/(t-s+1) .$$

Next, we introduce the aligned rank and propose the test based on one. So setting

$$Y_{ijk}=X_{ijk}-\sum_{j=1}^{p}\sum_{k=1}^{N}X_{ijk}/(pN)=e_{ijk}-\sum_{j=1}^{p}\sum_{k=1}^{N}e_{ijk}/(pN)+\tau_j ,$$

Mehra and Sarangi [10] defined the aligned rank Q_{ijk} by the rank of Y_{ijk} among all observations $\{Y_{ijk}: i=1, 2, \cdots, n, j=1, 2, \cdots, p, k=1, 2, \cdots, N\}$. Here let the scores function $b_{pnN}(\cdot)$ be a mapping from $\{1, 2, \cdots, pnN\}$ to real numbers such that $\sum_{k=1}^{pnN} b_{pnN}(k)=0$. The following aligned rank statistic is constructed by substituting $b_{pnN}(Q_{ijk})$ for X_{ijk} in the likelihood ratio test statistic (2.1) except for the constant factor.

$$\bar{\chi}^2_n(b(Q))=n^2N(pN-1)\sum_{i=1}^{p}\tau_i^{*2}\Big/\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N}\{b_{pnN}(Q_{ijk})-\bar{b}_{pnN}(Q_{i\cdot\cdot})\}^2 , \tag{2.3}$$

where

$$\bar{b}_{pnN}(Q_{\cdot j \cdot})=\sum_{i=1}^{n}\sum_{k=1}^{N} b_{pnN}(Q_{ijk})/(nN) , \qquad \tau_i^*=\max_{1\leq s\leq i}\min_{i\leq t\leq p}\sum_{j=s}^{t}\bar{b}(Q_{\cdot j \cdot})/(t-s+1)$$

and

$$\bar{b}_{pnN}(Q_{i\cdot\cdot})=\sum_{j=1}^{p}\sum_{k=1}^{N} b_{pnN}(Q_{ijk})/(pN) .$$

Hence we reject H if $\bar{\chi}_n^2(b(Q))$ is too large, and the denominator of (2.3) is regarded as a constant since we think of the conditional distribution afterward.

3. Notations and expressions of exact distributions

For the later discussions, we introduce some notations used by Boswell and Brunk [5].

For $q=1, 2, \cdots, p$, define

$$U_q=\left\{\boldsymbol{u}=(u_1, \cdots, u_p):\ u_1, u_2, \cdots, u_p \text{ are nonnegative integers, } \sum_{i=1}^{p} iu_i=p \text{ and } \sum_{i=1}^{p} u_i=q\right\} \text{ and}$$

$$L(q, p)=\sum_{\boldsymbol{u}\in U_q} 1\Big/\left(\prod_{i=1}^{p} u_i!\, i^{u_i}\right) . \tag{3.1}$$

For $\boldsymbol{u} \in U_q$, define $V^{\boldsymbol{u}}=\{\boldsymbol{v}=(v_1, v_2, \cdots, v_q):\ v_1, v_2, \cdots, v_q$ are positive integers and exactly u_i of the components are equal to i for $i=1, 2, \cdots, n\}$ and $V_q=\bigcup_{\boldsymbol{u}\in U_q} V^{\boldsymbol{u}}$.

For $\boldsymbol{v} \in V^{\boldsymbol{u}}$, set $w_0=0$ and $w_i=\sum_{j=1}^{i} v_j$ $(i=1, 2, \cdots, q)$.

For $1\leqq s\leqq t\leqq p$ and a p-dimensional vector $\boldsymbol{\Xi}=(\Xi_1, \cdots, \Xi_p)$, set $\bar{a}_{[s,t]} =\sum_{j=s}^{t} \bar{a}_{pN}(R_{\cdot j \cdot})/(t-s+1)$, $\bar{b}_{[s,t]}=\sum_{j=s}^{t} \bar{b}_{pnN}(Q_{\cdot j \cdot})/(t-s+1)$ and

$$\bar{\Xi}_{[s,t]}=\sum_{i=s}^{t} \Xi_i/(t-s+1) . \tag{3.2}$$

For $\boldsymbol{v} \in V^{\boldsymbol{u}}$ and a p-dimensional vector $\boldsymbol{\Xi}=(\Xi_1, \cdots, \Xi_p)$, define

$$S_{nN}(\boldsymbol{v})=nN(pN-1) \sum_{i=1}^{q} v_i(\bar{a}_{[w_{i-1}+1, w_i]})^2\Big/\sum_{i=1}^{pN} \{a_{pN}(i)\}^2 , \tag{3.3}$$

$$T_n(\boldsymbol{v})=n^2N(pN-1) \sum_{i=1}^{q} v_i(\bar{b}_{[w_{i-1}+1, w_i]})^2\Big/\sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{N} \{b_{pnN}(Q_{ijk})-\bar{b}_{pnN}(Q_{i\cdot\cdot})\}^2 \tag{3.4}$$

and

$$S_{\boldsymbol{v}}(\boldsymbol{\Xi})=\sum_{i=1}^{q} v_i(\bar{\Xi}_{[w_{i-1}+1, w_i]})^2 . \tag{3.5}$$

Setting $Q_{ij}=(Q_{ij1}, Q_{ij2}, \cdots, Q_{ijN})$ and $q_{ij}=(q_{ij1}, q_{ij2}, \cdots, q_{ijN})$, write $\Omega_n =\{(Q_{i1}, Q_{i2}, \cdots, Q_{ip})$ $i=1, 2, \cdots, n$: $(Q_{i1}, Q_{i2}, \cdots, Q_{ip})$ takes $(pN)!$ permutations of $(q_{i1}, q_{i2}, \cdots, q_{ip})$ for $i=1, 2, \cdots, n\}$.

The following two theorems are essentially a direct application of Theorem 1.2 of Boswell and Brunk [5].

THEOREM 3.1. *If H is true, for $t>0$,*

$$\Pr\{\bar{\chi}^2_{nN}(a(R))\geqq t\}=\sum_{q=1}^{p}\sum_{\boldsymbol{u}\in U_q}\Pr\{S_{nN}(\boldsymbol{v})\geqq t\}\Big/\Big(\prod_{i=1}^{p}u_i!\,i^{u_i}\Big)\,,$$

where $\boldsymbol{v}$ is selected arbitrarily from $V^{\boldsymbol{u}}$ for each $\boldsymbol{u}\in U_q$.

PROOF. Since $\Pr\{R_{ijk}=r_{ijk},\ i=1,\cdots,n,\ j=1,\cdots,p,\ k=1,\cdots,N\}=1/\{(pN)!\}^n$ under H, $(\bar{a}_{pN}(R_{\cdot1\cdot}),\cdots,\bar{a}_{pN}(R_{\cdot p\cdot}))$ is an exchangeable random vector. The remainder of the proof is similar to that of Theorem 1 of Shiraishi [13].

Since $\boldsymbol{Y}_i=(Y_{i11},Y_{i12},\cdots,Y_{i1N},Y_{i21},\cdots,Y_{ipN})'$ is an exchangeable random vector with pN elements under H and $\boldsymbol{Y}_i$'s are independent random vectors, we have the conditional probability

$$\Pr\{Q_{ijk}=q_{ijk},\ i=1,\cdots,n,\ j=1,\cdots,p,\ k=1,\cdots,N\,|\,\Omega_n\}=1/\{(pN)!\}^n \tag{3.6}$$

under H.

THEOREM 3.2. *If H is true, for $t>0$,*

$$\Pr\{\bar{\chi}^2_n(b(Q))\geqq t\,|\,\Omega_n\}=\sum_{q=2}^{p}\sum_{\boldsymbol{u}\in U_q}\Pr\{T_n(\boldsymbol{v})\geqq t\,|\,\Omega_n\}\Big/\Big(\prod_{i=1}^{p}u_i!\,i^{u_i}\Big)\,,$$

where $\boldsymbol{v}$ is selected arbitrarily from $V^{\boldsymbol{u}}$ for each $\boldsymbol{u}\in U_q$.

PROOF. $(\bar{b}_{pnN}(Q_{\cdot1\cdot}),\bar{b}_{pnN}(Q_{\cdot2\cdot}),\cdots,\bar{b}_{pnN}(Q_{\cdot p\cdot}))$ is an exchangeable random vector under the conditional probability $\Pr\{\cdot|\Omega_n\}$ from (3.6) and $\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N}\{b_{pnN}(Q_{ijk})-\bar{b}_{pnN}(Q_{i\cdot\cdot})\}^2$ is a constant for given Ω_n. The remainder of the proof is exactly parallel to Theorem 3.1.

The latter test based on the aligned rank is more troublesome than the former in computing the test statistic and we can not make a table of the upper tail probabilities of the latter test as we think of the conditional probability.

4. Asymptotic distribution under the hypothesis

In order to test H versus K approximately for the given level of significance when either the number of blocks or the number of observations per cell is too large, we give the asymptotic distribution of the statistics (2.2) and (2.3).

THEOREM 4.1. *If H is true and t is positive, we have*

$$\lim_{n\to\infty} \Pr\{\bar{\chi}^2_{nN}(a(R))\geqq t\} = \sum_{q=2}^{p} L(q, p)\Pr\{\chi^2_{q-1}\geqq t\} ,$$

where $L(q, p)$ is given by (3.1).

PROOF. Set $\bar{a}_{pN}(R_{ij\cdot})=\sum_{k=1}^{N} a_{pN}(R_{ijk})/N$, and write I_p=unit matrix of order p and a p-dimensional vector $\mathbf{1}_p=(1,\cdots,1)'$. Then the variance-covariance matrix of $(\bar{a}_{pN}(R_{i1\cdot}),\cdots,\bar{a}_{pN}(R_{ip\cdot}))'$ is $\sqrt{\sum_{i=1}^{pN}\{a_{pN}(i)\}^2/\{N(pN-1)\}}$ $\cdot(I_p-\mathbf{1}_p\mathbf{1}_p'/p)$ from Theorem II 3.1 c of Hájek and Šidák [8] and $(\bar{a}_{pN}(R_{i1\cdot}),\cdots,\bar{a}_{pN}(R_{ip\cdot}))'$ $(i=1, 2,\cdots, n)$ are independent and identically distributed random vectors, the multivariate central limit theorem implies that

$$\sqrt{nN(pN-1)\Big/\sum_{i=1}^{pN}\{a_{pN}(i)\}^2}\,(\bar{a}_{pN}(R_{\cdot 1\cdot}),\cdots,\bar{a}_{pN}(R_{\cdot p\cdot}))' \xrightarrow[L]{} N(0, I_p-\mathbf{1}_p\mathbf{1}_p'/p) \quad \text{as } n\to\infty ,$$

where $\xrightarrow[L]{}$ denotes convergence in law. Hence,

$$\sqrt{nN(pN-1)\Big/\sum_{i=1}^{pN}\{a_{pN}(i)\}^2}\,(\sqrt{v_1}a_{[w_0+1,w_1]}, \sqrt{v_2}a_{[w_1+1,w_2]},\cdots, \sqrt{v_q}a_{[w_{q-1}+1,w_q]})'$$
$$\xrightarrow[L]{} N(0, I_p-(\sqrt{v_1}, \sqrt{v_2},\cdots, \sqrt{v_q})'(\sqrt{v_1}, \sqrt{v_2},\cdots, \sqrt{v_q})/p) \quad \text{as } n\to\infty .$$

Since $I_q-(\sqrt{v_1}, \sqrt{v_2},\cdots, \sqrt{v_q})'(\sqrt{v_1}, \sqrt{v_2},\cdots \sqrt{v_q})/p$ is idempotent and its rank is $q-1$, using a standard result from the multivariate analysis,

$$S_{nN}(\boldsymbol{v})\xrightarrow[L]{}\chi^2_{q-1} \qquad \text{for } q\geqq 2 \text{ as } n\to\infty ,$$

where $S_{nN}(\boldsymbol{v})$ is given by (3.3). This, combined with Theorem 3.1, completes the proof of Theorem 4.1.

In order to derive the asymptotic distribution of $\bar{\chi}^2_{nN}(a(R))$ defined by (2.2) for a large number of replications N, we impose Assumption (I).

ASSUMPTION (I). The underlying distribution F has the finite Fisher information number and $a_l(\cdot)$ is a scores function satisfying

$$\lim_{l\to\infty}\int_0^1\{a_l(1+[ul])-\phi(u)\}^2du=0$$

for some square integrable function $\phi(u)$ such that

$$\int_0^1\{\phi(u)\}^2du>0 \qquad \text{and} \qquad \int_0^1\phi(u)du=0 ,$$

where $[ul]$ is the largest integer not exceeding ul.

THEOREM 4.2. *If Assumption* (I) *is satisfied and t is positive, under H, we have*

$$\lim_{N\to\infty} \Pr\{\bar{\chi}^2_{nN}(a(R))\geqq t\} = \sum_{q=2}^{p} L(q, p) \Pr\{\chi^2_{q-1}\geqq t\} .$$

PROOF. From Theorem V 2.2 of Hájek and Šidák [8], setting

$$\boldsymbol{\xi}_i = \sqrt{N(pN-1)\Big/\sum_{k=1}^{pN}\{a_{pN}(k)\}^2}\,(\bar{a}_{pN}(R_{i1\cdot}), \cdots, \bar{a}_{pN}(R_{ip\cdot}))' ,$$

we have $\boldsymbol{\xi}_i \xrightarrow[L]{} N(0, I_p - 1_p1_p'/p)$ and that $\{\boldsymbol{\xi}_i : i=1, 2, \cdots, n\}$ are independent random vectors. Hence, using Theorem 3.2 of Billingsley [4], we get

$$\sqrt{nN(pN-1)\Big/\sum_{k=1}^{pN}\{a_{pN}(k)\}^2}\,(\bar{a}_{pN}(R_{\cdot 1\cdot}), \cdots, \bar{a}_{pN}(R_{\cdot p\cdot}))' \xrightarrow[L]{} N(0, I_p - 1_p1_p'/p)$$
$$\text{as } N\to\infty .$$

The remainder of the proof is similar to that of Theorem 4.1. Also we impose Assumption (II) in order to derive the asymptotic theory of $\bar{\chi}^2_n(b(Q))$.

ASSUMPTION (II). There exists a square integrable function $\phi(\cdot)$ such that

$$\lim_{l\to\infty}\int_0^1 \{b_l(1+[ul])-\phi(u)\}^2 du = 0 , \qquad \int_0^1 \phi(u)du = 0$$
$$\text{and} \qquad \int_0^1 \{\phi(u)\}^2 du > 0 .$$

THEOREM 4.3. *Suppose that the scores function $b_{pnN}(\cdot)$ satisfies Assumption* (II) *and the assumptions* (2.3) *and* (3.4) *of Tardif* [15]. *If H is true and t is positive, we have*

$$\lim_{n\to\infty} \Pr\{\bar{\chi}^2_n(b(Q))\geqq t\} = \sum_{q=2}^{p} L(q, p) \Pr\{\chi^2_{q-1}\geqq t\} .$$

PROOF. From Theorem 3.1 of Tardif [15], we have

$$\lim_{n\to\infty} \Pr\{T_n(v)\geqq t\} = \Pr\{\chi^2_{q-1}\geqq t\} ,$$

where $T_n(v)$ is given by (3.4). Hence, using Theorem 3.2, we get the result.

The critical values of the asymptotic distribution given by Theorems 4.1, 4.2 and 4.3 are shown in tables A.1, A.2 and A.3 of Barlow et al. [2].

5. Asymptotic relative efficiency

Since the tests based on (2.1), (2.2) and (2.3) are all similar to the grand mean μ and the block effects β_i $(i=1, 2, \cdots, n)$ for fixed F in the equation (1.1), we may assume that μ and β_i $(i=1, \cdots, n)$ equal zero in the rest of this paper. In order to compute the asymptotic Pitman efficiency, we consider the contiguous sequence of location alternatives specified by

$$Q_{g(n)h(N)}(\boldsymbol{x})=\prod_{i=1}^{g(n)}\prod_{j=1}^{p}\prod_{k=1}^{h(N)} f\left(x_{ijk}-\frac{\Delta_j}{\sqrt{nN}}\right)$$

where $g(\cdot)$ and $h(\cdot)$ are mappings from positive integers to themselves, $\sum_{j=1}^{p}\Delta_j=0$ and $\sum_{j=1}^{p}\Delta_j^2>0$.

THEOREM 5.1. *If F has the finite Fisher information number and a finite variance σ^2, the asymptotic relative efficiency of the within-block rank test based on $\bar{\chi}^2_{nN}(a(R))$ with respect to the likelihood ratio test based on $\bar{\chi}^2_{nN}(X)$ for testing H versus K as n tends to infinity is*

$$\text{ARE}\,(\bar{\chi}^2_{nN}(a(R)), \bar{\chi}^2_{nN}(X))=\sigma^2\left\{\sum_{i=1}^{pN} a_{pN}(i)\,\mathrm{E}\,\psi(X^{(i)})\right\}^2\Big/\left[(pN-1)\sum_{i=1}^{pN}\{a_{pN}(i)\}^2\right]$$

where $\psi(X^{(i)})=-f'(X^{(i)})/f(X^{(i)})$ and $X^{(i)}$ is the ith order statistic in a sample of size pN from $F(x)$.

PROOF. If we let $\lim_{n\to\infty} g(n)/n=b$ for $0<b<1$, from Corollary 3.4 of Schach [12],

$$\sqrt{g(n)(pN-1)\Big/\sum_{k=1}^{pN}\{a_{pN}(k)\}^2}\,(\bar{a}_{pN}(R_{\cdot 11}), \bar{a}_{pN}(R_{\cdot 12}), \cdots, \bar{a}_{pN}(R_{\cdot 1N}),$$
$$\bar{a}_{pN}(R_{\cdot 21}), \cdots, \bar{a}_{pN}(R_{\cdot pN}))' \xrightarrow[L]{} N(\boldsymbol{\mu}_1, I_{pN}-1_{pN}1'_{pN}/pN)$$

with

$$\boldsymbol{\mu}_1=\sqrt{b\Big/\left[N(pN-1)\sum_{k=1}^{pN}\{a_{pN}(k)\}^2\right]}$$
$$\cdot\left\{\sum_{i=1}^{pN} a_{pN}(i)\,\mathrm{E}\,(X^{(i)})\right\}(\Delta_1, \cdots, \Delta_1, \Delta_2, \cdots, \Delta_p)',$$

under $\{Q_{g(n)N}\}$. Therefore if we set

$$\xi_j=\sqrt{g(n)N(pN-1)\Big/\sum_{k=1}^{pN}\{a_{pN}(k)\}^2}\,\bar{a}_{pN}(R_{\cdot j\cdot})$$

(5.1) $$\boldsymbol{\xi}=(\xi_1, \xi_2, \cdots, \xi_p)' \xrightarrow[L]{} \boldsymbol{Y} \qquad \text{where } \boldsymbol{Y}\sim N(\boldsymbol{\mu}, I_p-1_p1'_p/p)$$

with

$$\mu=\sqrt{b\Big/\Big[(pN-1)\sum_{k=1}^{pN}\{a_{pN}(k)\}^2\Big]}\sum_{k=1}^{pN}a_{pN}(k)\,\mathrm{E}\,\phi(X^{(k)})\cdot\Delta$$

under $\{Q_{g(n)N}\}$, where $\Delta=(\Delta_1, \Delta_2,\cdots, \Delta_p)'$.

On the other hand, the central limit theorem implies that

$$\eta=\sqrt{nN}/\sigma(\bar{X}_{.1.}-\bar{X}_{...}, \bar{X}_{.2.}-\bar{X}_{...},\cdots, \bar{X}_{.p.}-\bar{X}_{...})'\xrightarrow[L]{}Z \tag{5.2}$$

under $\{Q_{nN}\}$, where $Z\sim N(\Delta/\sigma, I_p-1_p1_p'/p)$.

Now Shorack [14] showed that, for p-dimensional vector Ξ,

$$\begin{aligned}(5.3)\qquad &\{\Xi:\ \max_{1\le s\le i}\min_{i\le t\le p}\bar{\bar{\Xi}}_{[s,t]}=\bar{\bar{\Xi}}_{[w_{j-1}+1,w_j]}\\ &\qquad \text{for } i=v_{j-1}+1,\cdots, v_j \text{ and } j=1,2,\cdots,q\}\\ &=\{\Xi:\ \bar{\bar{\Xi}}_{[w_{j-1}+1,w_j]}-\bar{\bar{\Xi}}_{[w_j+1,w_{j+1}]}<0\\ &\qquad \text{for } j=1,2,\cdots,q-1 \text{ and}\\ &\quad \bar{\bar{\Xi}}_{[w_{j-1}+1,w_{j-1}+i]}-\bar{\bar{\Xi}}_{[w_{j-1}+1,w_j]}>0\\ &\qquad \text{for } i=1,2,\cdots,v_j-1 \text{ and } j=1,2,\cdots,q\}\ ,\end{aligned}$$

where $\bar{\bar{\Xi}}_{[s,t]}$ is defined by (3.2). Therefore if we let the set $A_v(t)$ be defined by the following relation, $\Xi\in A_v(t)$ implies $S_v(\Xi)\geqq t$,

$$\bar{\bar{\Xi}}_{[w_{j-1}+1,w_j]}-\bar{\bar{\Xi}}_{[w_j+1,w_{j+1}]}<0 \qquad \text{for } i=1,2,\cdots,q-1$$

and

$$\begin{aligned}&\bar{\bar{\Xi}}_{[w_{j-1}+1,w_{j-1}+i]}-\bar{\bar{\Xi}}_{[w_{j-1}+1,w_j]}>0\\ &\qquad \text{for } i=1,2,\cdots,v_j-1 \text{ and } j=1,2,\cdots,q\ ,\end{aligned}$$

where $\bar{\bar{\Xi}}_{[s,t]}$ and $S_v(\Xi)$ are respectively defined by (3.2) and (3.5). Using (5.1), (5.2) and (5.3), we get

$$\begin{aligned}&\lim_{n\to\infty}\Pr\{\bar{\chi}^2_{g(n)N}(a(R))\geqq t\}\\ &=\lim_{n\to\infty}\sum_{q=2}^{p}\sum_{v\in V_q}\Pr\{\eta\in A_v(t)\}=\sum_{q=2}^{p}\sum_{v\in V_q}\Pr\{Y\in A_v(t)\}\\ &=\Pr\{\bar{\chi}^2_{11}(Y)\geqq t\}\ ,\end{aligned}$$

where $\bar{\chi}^2_{11}$ is $\bar{\chi}^2_{nN}$ given by setting $n=N=1$. Also, from the similar way,

$$\lim_{n\to\infty}\Pr\{\bar{\chi}^2_{nN}(X)\geqq t\}=\lim_{n\to\infty}\sum_{q=2}^{p}\sum_{v\in V_q}\Pr\{\eta\in A_v(t)\}=\Pr\{\bar{\chi}^2_{11}(Z)\geqq t\}\ .$$

Here since $\Pr\{\bar{\chi}^2_{11}(Y)\geqq t\}$ is strictly increasing in b from Lemma 2 of Shiraishi [13], we have

$$\lim_{n\to\infty}\Pr\{\bar{\chi}^2_{g(n)N}(a(R))\geqq t\}=\lim_{n\to\infty}\Pr\{\bar{\chi}^2_{nN}(X)\geqq t\}\ ,$$

if and only if

$$\lim_{n\to\infty} n/g(n)=1/b=\sigma^2\left\{\sum_{i=1}^{pN} a(i)\,\mathrm{E}\,\phi(X^{(i)})\right\}^2\Big/\left[(pN-1)\sum_{i=1}^{pN}\{a_{pN}(i)\}^2\right].$$

Thus the conclusion follows.

The above asymptotic relative efficiencies for certain distributions and Wilcoxon scores and normal scores are given in Table 1.

THEOREM 5.2. *If Assumption* (I) *is satisfied, then the asymptotic relative efficiency of the test based on* $\bar{\chi}^2_{nN}(a(R))$ *with respect to the test based on* $\bar{\chi}^2_{nN}(X)$ *for testing* H *versus* K *as* N *tends to infinity is*

$$\mathrm{ARE}\,(\bar{\chi}^2_{nN}(a(R)),\,\bar{\chi}^2_{nN}(X))=\sigma^2\left\{\int_0^1\phi(u)\phi(u,f)du\right\}^2\Big/\left[\int_0^1\{\phi(u)\}^2du\right],$$

where $\phi(u,f)=-f'(F^{-1}(u))/f(F^{-1}(u))$.

PROOF. If we let $\lim_{N\to\infty} h(N)/N=c$ for $0<c<1$ and set

$$\xi_i=\sqrt{h(N)(ph(N)-1)\Big/\sum_{k=1}^{ph(N)}\{a_{ph(N)}(k)\}^2}\cdot(\bar{a}_{ph(N)}(R_{i1\cdot}),\,\bar{a}_{ph(N)}(R_{i2\cdot}),\cdots,\bar{a}_{ph(N)}(R_{ip\cdot}))',$$

using Theorem VI 3.1 of Hájek and Šidák [8],

$$\boldsymbol{\xi}_i\xrightarrow[L]{}N(\boldsymbol{\mu}_1,\,I_p-1_p1_p'/p)$$

with

$$\boldsymbol{\mu}_1=\int_0^1\phi(u)\phi(u,f)du\cdot\Delta\sqrt{c\Big/\left[n\int_0^1\{\phi(u)\}^2du\right]}$$

under $\{Q_{nh(N)}\}$, and $\{\boldsymbol{\xi}_i:\ i=1,2,\cdots,n\}$ are independent random vectors. Here, from Theorem 3.2 of Billingsley [4],

$$\sqrt{nh(N)(ph(N)-1)\Big/\sum_{k=1}^{ph(N)}\{a_{ph(N)}(k)\}^2}\cdot(\bar{a}_{ph(N)}(R_{\cdot1\cdot}),\cdots,\bar{a}_{ph(N)}(R_{\cdot p\cdot}))'\xrightarrow[L]{}N(\boldsymbol{\mu},\,I_p-1_p1_p'/p)$$

with

$$\boldsymbol{\mu}=\int_0^1\phi(u)\phi(u,f)du\Delta\sqrt{c\Big/\int_0^1\{\phi(u)\}^2du}$$

under $\{Q_{nh(N)}\}$. On the other hand, $\boldsymbol{\eta}\xrightarrow[L]{}\boldsymbol{Z}$ under $\{Q_{nN}\}$, where $\boldsymbol{\eta}$ and $\boldsymbol{Z}$ are defined by (5.2). Thus the above two asymptotic properties and the similar argument to the proof of Theorem 5.1 yield the conclusion.

This asymptotic relative efficiency is the same as the asymptotic relative efficiency of the Kruskall-Wallis type test based on general scores with respect to the F-test for the alternative hypothesis of unequal treatment effects in the one-way layout and is stated in Table 2 of Shiraishi [13]. For instance, if F is normal and $a_{pN}(\cdot)$ is of Wilcoxon-type, its efficiency is $3/\pi$.

THEOREM 5.3. *Suppose that conditions* (3.4) *and* (4.1)-(4.3) *of Tardif* [15] *and Assumption* (II) *are satisfied. Define* $G(x)$ *and* $G^*(x, y)$ *by the marginal distribution of* $(e_{111}-\bar{e}_{1\cdot\cdot})$ *and the joint distribution of* $(e_{111}-\bar{e}_{1\cdot\cdot}, e_{112}-\bar{e}_{1\cdot\cdot})$. *Then the asymptotic relative efficiency of the test based on* $\bar{\chi}_n^2\cdot(b(Q))$ *with respect to the test based on* $\bar{\chi}_{nN}^2(X)$ *for testing H versus K as n tends to infinity is*

$$\mathrm{ARE}\,(\bar{\chi}_n^2(b(Q)),\ \bar{\chi}_{nN}^2(X))=\sigma^2\left\{\int_0^1 \phi(u)g'(G^{-1}(u))/g(G^{-1}(u))du\right\}^2\Big/ \left[\int_0^1 \{\phi(u)\}^2du-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(G(x))\phi(G(y))dG^*(x, y)\right]$$

where $\phi(u)$ *is defined in Assumption* (II).

PROOF. From Theorem 5.1 of Tardif [15], we have

$$\sqrt{n^2N(pN-1)\Big/\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N}\{b_{pnN}(Q_{ijk})-\bar{b}_{pnN}(Q_{i\cdot\cdot})\}^2}\cdot(\bar{b}_{pnN}(Q_{\cdot1\cdot}), \bar{b}_{pnN}(Q_{\cdot2\cdot}), \cdots, \bar{b}_{pnN}(Q_{\cdot p\cdot}))' \xrightarrow[L]{} N(\boldsymbol{\mu}_1, I_p-1_p1_p'/p)$$

with

$$\boldsymbol{\mu}_1=\int_0^1 \phi(u)g'(G^{-1}(u))/g(G^{-1}(u))du\Delta\Big/ \sqrt{\left[\int_0^1 \{\phi(u)\}^2du-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(G(x))\phi(G(y))dG^*(x, y)\right]}$$

under $\{Q_{nN}\}$. Hence we get

$$\gamma=\sqrt{n^2N(pN-1)\Big/\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N}\{b_{pnN}(Q_{ijk})-\bar{b}_{pnN}(Q_{i\cdot\cdot})\}^2}\cdot(\bar{b}_{pnN}(Q_{\cdot1\cdot}), \bar{b}_{pnN}(Q_{\cdot2\cdot}), \cdots, \bar{b}_{pnN}(Q_{\cdot p\cdot}))' \xrightarrow[L]{} N(\boldsymbol{\mu}, I_p-1_p1_p'/p) \tag{5.4}$$

with

$$\boldsymbol{\mu}=-\int_0^1 \phi(u)g'(G^{-1}(u))/g(G^{-1}(u))du\Delta\Big/ \sqrt{\left[\int_0^1 \{\phi(u)\}^2du-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(G(x))\phi(G(y))dG^*(x, y)\right]}$$

under $\{Q_{nN}\}$.

Thus by the similar argument to the proof of Theorem 5.1, the conclusion follows.

This asymptotic relative efficiency is the same as the asymptotic relative efficiency of the aligned rank test with respect to the F-test for the null hypothesis H versus the alternative hypothesis that treatment effects are not equal, and is discussed in Mehra and Sarangi [10] and Sen [11]. If F is normal and $b_l(i)=2i/(l+1)-1$, the efficiency attains the maximum value 0.9662 at $pN=3$ and decreases monotonically to $3/\pi$ as $pN\to\infty$. Also if F is normal and $b_l(\cdot)$ is normal scores, it takes one.

Thus, in these cases, it is found that the asymptotic local power of $\bar{\chi}_n^2(b(Q))$ as $n\to\infty$ is higher than that of $\bar{\chi}_{nN}^2(a(R))$ as compared with Table 1.

Table 1. Table of the Asymptotic Relative Efficiency (ARE) of the test based on $\bar{\chi}_{nN}^2(a(R))$ with respect to the one based on $\bar{\chi}_{nN}^2(X)$ as n tends to infinity.

(1) Wilcoxon scores $a_{pN}(i)=2i/(pN+1)-1$

			F is normal				
pN	3	4	5	10	20	30	50
ARE	.716	.764	.796	.868	.910	.924	.936
			F is logistic				
pN	3	4	5	10	20	30	50
ARE	.822	.877	.914	.997	1.044	1.061	1.075

(2) Normal scores $a_{pN}(i)=\mathrm{E}Z^{(i)}$

			F is normal				
pN	3	4	5	10	20	30	50
ARE	.716	.765	.799	.879	.930	.950	.968
			F is logistic				
pN	3	4	5	10	20	30	50
ARE	.822	.876	.910	.984	1.021	1.032	1.040

$\mathrm{E}Z^{(i)}$ is the expected value of the ith order statistic in a sample of size pN from the standard normal population.

6. Comparison with linear rank tests

For ordered alternatives, Araki and Shirahata [1] compared many other distribution-free tests, using the Pitman asymptotic relative efficiency, and showed that the within-block rank test based on the following statistic

$$S(a(R))=\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N} j\cdot a_{pN}(R_{ijk}) . \tag{6.1}$$

Also Sen [11] proposed the test based on the statistic

$$S(b(Q))=\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{N} j\cdot b_{pmN}(Q_{ijk}) . \tag{6.2}$$

So if $\boldsymbol{\xi}$ or $\boldsymbol{\gamma}$ by (5.1) or (5.4) has asymptotically a normal distribution with mean vector $\boldsymbol{\mu}$ under the contiguous sequence of location alternatives as n or N tends to infinity, the power function with level α of significance of the test based on $S(a(R))$ or $S(b(Q))$ is

$$\beta(\boldsymbol{\mu})=1-\Phi\left(t_\alpha-\sum_{i=1}^{p} i\mu_i/\sqrt{p(p+1)(p-1)/12}\right) ,$$

where t_α is the upper 100α-percentage point of the standard normal distribution. Here we give the asymptotic power of $\bar{\chi}^2_{nN}(a(R))$ or $\bar{\chi}^2_n(b(Q))$ as n or N tends to infinity in Table 2 when the levels of significance and $\beta(\boldsymbol{\mu})$ are fixed. Then we use the expression of the power function of $\bar{\chi}^2_{nN}(X)$ stated in Bartholomew [3] for $p=3$, and also use the Monte-

Table 2. The asymptotic powers of the $\bar{\chi}^2_{nN}(a(R))$ and $\bar{\chi}^2_n(b(Q))$ when those of linear rank tests are fixed for any distribution function $F(t)$ and any scores functions $a_l(k)$ and $b_l(k)$.

Number of treatments p	Configuration of μ_i's	Level of significance α	Asymptotic power of the linear rank tests	Asymptotic power of the proposed tests
3	$\mu_1=-\mu,\ \mu_2=0,\ \mu_3=\mu$	.05	.50	.4697
			.80	.7698
		.01	.50	.4617
			.80	.7655
	$\mu_1=\mu_2=-\mu,\ \mu_3=2\mu$	.05	.50	.5309
			.80	.8458
		.01	.50	.5676
			.80	.8696
	$\mu_1=-2\mu,\ \mu_2=-\mu,\ \mu_3=3\mu$	.05	.50	.4927
			.80	.8010
		.01	.50	.5025
			.80	.8104
5	$\mu_1=-2\mu,\ \mu_2=-\mu,\ \mu_3=0,\ \mu_4=\mu,\ \mu_5=2\mu$	.05	.50	.4507
			.80	.7517
		.01	.50	.4394
			.80	.7464
	$\mu_1=-\mu,\ \mu_2=\mu_3=\mu_4=0,\ \mu_5=\mu$	.05	.50	.4996
			.80	.8086
		.01	.50	.5083
			.80	.8194
	$\mu_1=-4\mu,\ \mu_2=\mu_3=\mu_4=\mu_5=\mu$	.05	.50	.6217
			.80	.9277
		.01	.50	.7154
			.80	.9575

Table 2. (Continued)

Number of treatments p	Configuration of μ_i's	Level of significance α	Asymptotic power of the linear rank tests	Asymptotic power of the proposed tests
10	$\mu_1=-9\mu$, $\mu_2=-7\mu, \cdots, \mu_5=-\mu$, $\mu_6=\mu$, $\mu_7=3\mu, \cdots, \mu_{10}=9\mu$	.05	.50	.4481
			.80	.7454
		.01	.50	.4355
			.80	.7400
	$\mu_1=-9\mu$, $\mu_2=\mu_3=\mu_4=\cdots=\mu_{10}=\mu$	.05	.50	.8131
			.80	.9930
		.01	.50	.9291
			.80	.9992
	$\mu_1=\mu_2=\cdots=\mu_5=-\mu$, $\mu_6=\mu_7=\cdots=\mu_{10}=\mu$	.05	.50	.8461
			.80	.9945
		.01	.50	.9432
			.80	.9995

Carlo simulation for $p=5, 10$ and the estimates of the power of the test are obtained from 10,000 repetitions of sampling.

Hence from Table 2, we get that the powers of linear rank tests are higher than those of the proposed tests about 5 percent in the case of the linear trend alternative, that is, $\mu_i=i\mu$ $(i=1, 2, \cdots, p)$ but the formers are almost lower than or equal to the latters in the other cases. Especially for $p=10$ and configurations: $\mu_1=\cdots=\mu_i<\mu_{i+1}=\cdots=\mu_p$ for some i, the latters are greatly high. Also we feel that the proposed tests are robust with respect to the ordered alternative as the powers of those are not low for any configurations of μ_i's.

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