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ON k-ORDER HARMONIC NEW BETTER THAN USED IN EXPECTATION DISTRIBUTIONS*

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Summary

Definitions of k-HNBUE and K-HNWUE are introduced and the relationship with other class of life distributions is studied. Various closure properties of k-HNBUE (k-HNWUE) are proved. Finally bounds on the moments and survival function of k-HNBUE (k-HNWUE) are given.

1. Introduction

In reliability theory various concepts of (univariate) aging or wear out have been proposed to study lifetimes of systems and components. The five most commonly studied classes of life distributions are the following: 1) The increasing failure rate class (IFR); 2) the increasing failure rate in average class (IFRA); 3) the new better than used class (NBU); 4) decreasing mean residual class (DMRL); and 5) the new better than used in expectation class (NBUE). For a description of these classes see Barlow and Proschan [2], Bryson and Siddiqui [3] and Haines [4]. Recently Rolski [7] proposed a new class of life distributions called the *harmonic new better than used in expectation* (HNBUE) class which will be defined later. Each of the above six classes have their dual with standard nomenclature. The dual of HNBUE class is said to be harmonic new worse than used in expectation (HNWUE). Klefsjö [6] has studied the properties of HNBUE (HNWUE) classes of distributions. He has proven several closure theorems for this class and the following chain of implication exists among the six classes of distributions.

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$$
\begin{tikzcd}
 & \boxed{\begin{matrix}\text{IFRA}\\ \text{(DFRA)}\end{matrix}} \arrow[r, Rightarrow] & \boxed{\begin{matrix}\text{NBU}\\ \text{(NWU)}\end{matrix}} \arrow[d, Rightarrow] & \\
\boxed{\begin{matrix}\text{IFR}\\ \text{DFR}\end{matrix}} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \\
 & \boxed{\begin{matrix}\text{DMRL}\\ \text{(IMRL)}\end{matrix}} \arrow[r, Rightarrow] & \boxed{\begin{matrix}\text{NBUE}\\ \text{(NWUE)}\end{matrix}} \arrow[r, Rightarrow] & \boxed{\begin{matrix}\text{HNBUE}\\ \text{(HNWUE)}\end{matrix}}
\end{tikzcd}
$$

Multivariate versions of HNBUE have been studied by Basu, Ebrahimi and Klefsjö [1].

The purpose of this note is to propose new classes of life distributions called *k-order harmonic new better than used in expectation* (k-HNBUE) class which will be defined later. It is shown that k-HNBUE is the largest available class of distributions with aging property. Since exponential distribution plays a major role in reliability theory, it is important to know if the underlying distribution is exponential. One motivation for proposing k-HNBUE class is that in testing for exponentiality, we deal with a larger class of available alternatives. A second reason is to see if we can develop analytical properties for as large a class as possible comprising of all known standard classes of distributions with aging properties. The dual of k-HNBUE class is called *k-order harmonic* new worse than used in expectation (k-HNWUE).

In Section 2 of this paper we have introduced the definitions of k-HNBUE and k-HNWUE and various closure properties of k-HNBUE and k-HNWUE distributions are studied in Section 3. Finally in Section 4, we give some bounds on the moments and on the survival function of a k-HNBUE (k-HNWUE) life distributions.

2. The definitions: classes k-order HNBUE and k-order HNWUE

We shall start with the definition of k-order HNBUE.

DEFINITION 2.1. Let X denote the survival (failure) time of a device having life distribution F and survival distribution $\bar{F}(x)=\mathrm{P}(X>x)$. The non-negative random variable X is said to have a *k-order harmonic new better than used in expectation* (k-HNBUE) distribution if

$$
\frac{1}{\dfrac{1}{t}\displaystyle\int_0^t \mu_F^{-k}(x)dx} \leqq \mu^k \qquad \text{for all } t \geqq 0 \tag{2.1}
$$

where $\mu_F(x)=\left(\int_x^\infty \bar{F}(y)dy\right)\Big/\bar{F}(x)$ is the mean residual life of a unit of age x. The inequality (2.1) says that the k-order integral harmonic mean value of $\mu_F(x)$ is less than or equal to the k-order integral harmonic mean value of $\mu_F(0)$, that is, of a new unit. We assume that

$\mu=\int_0^\infty \bar{F}(x)dx<\infty$ and $k\geqq 1$.

If the reversed inequality is true X is said to have *k-order harmonic new worse than used in expectation* (k-HNWUE) distribution. We should mention that for $k=1$, (2.1) is equivalent to the definition of HNBUE given by Rolski [7].

Remark 1. It is clear that, X with exponential distribution, i.e., $\bar{F}(x)=\exp\{-x/\mu\}$ is both k-HNBUE and k-HNWUE.

The following theorem gives the relationship between k-HNBUE (k-HNWUE) and HNBUE (HNWUE).

THEOREM 1. (a) *A life distribution F which is k-HNBUE is also $(k+1)$-HNBUE for any $k\geqq 1$.* (b) *A life distribution F which is $(k+1)$-HNWUE is also k-HNWUE for any $k\geqq 1$.*

PROOF. If F is k-HNBUE

$$\frac{1}{\frac{1}{t}\int_0^t \mu_{\bar{F}}^{k}(x)dx}\leqq \mu^k \qquad \text{for all } t\geqq 0\,.$$

Therefore, according to Hölder's inequality

$$\begin{aligned}\frac{1}{t}\int_0^t \mu_{\bar{F}}^{(k+1)}(x)dx&=\frac{1}{t}\int_0^t (\mu_{\bar{F}}^{k}(x))^{(k+1)/k}dx\geqq \frac{1}{t}\left(\int_0^t \mu_{\bar{F}}^{k}(x)dx\right)^{(k+1)/k}\left(\frac{1}{t}\right)^{1/k}\\ &=\left[\frac{1}{t}\int_0^t \mu_{\bar{F}}^{k}(x)dx\right]^{(k+1)/k}\geqq\left(\frac{1}{\mu^k}\right)^{(k+1)/k}=\frac{1}{\mu^{k+1}}\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t\geqq 0\,.\end{aligned}$$

That is,

$$\frac{1}{\frac{1}{t}\int_0^t \mu_{\bar{F}}^{(k+1)}(x)dx}\leqq \mu^{k+1} \qquad \text{for all } t\geqq 0\,.$$

(b) If F is $(k+1)$-HNWUE we get that

$$\frac{1}{\frac{1}{t}\int_0^t \mu_{\bar{F}}^{(k+1)}(x)dx}\geqq \mu^{k+1} \qquad \text{for all } t\geqq 0\,.$$

Therefore, according to Hölder's inequality

$$\frac{1}{t}\int_0^t \mu_{\bar{F}}^{k}(x)dx\leqq\left(\frac{1}{t}\int_0^t \mu_{\bar{F}}^{(k+1)}(x)dx\right)^{k/(k+1)}\leqq\left[\left(\frac{1}{\mu}\right)^{k+1}\right]^{k/(k+1)}=\frac{1}{\mu^k}$$
$$\text{for all } t\geqq 0\,.$$

That is,

$$\frac{1}{\dfrac{1}{t}\displaystyle\int_0^t \mu_F^{-k}(x)dx} \geqq \mu^k \qquad \text{for all } t \geqq 0 .$$

The next theorem gives the relation between k-HNWUE and NWUE.

THEOREM 2. *A life distribution F which NWUE is also k-HNWUE.*

PROOF. If F is NWUE we get that

$$\mu_F(x) \geqq \mu \qquad \text{for all } x .$$

That is,

$$\mu_F^{-k}(x) \leqq \frac{1}{\mu^k} .$$

Therefore,

$$\int_0^t \left(\mu_F^{-k}(x) - \frac{1}{\mu^k}\right)dx \leqq 0 .$$

That is,

$$\int_0^t \mu_F^{-k}(x)dx < \frac{t}{\mu^k}$$

or

$$\frac{1}{\dfrac{1}{t}\displaystyle\int_0^t \mu_F^{-k}(x)dx} \geqq \mu^k .$$

Now, using Klefsjö's results [6] and Theorems 1 and 2 the following chains of implication exists among the 14 classes of distributions.

$$\begin{tikzcd}
& \boxed{\text{IFRA}} \arrow[r, Rightarrow] & \boxed{\text{NBU}} \arrow[dd, Rightarrow] & & \\
\boxed{\text{IFR}} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & \\
& \boxed{\text{DMRL}} \arrow[r, Rightarrow] & \boxed{\text{NBUE}} \arrow[r, Rightarrow] & \boxed{\text{HNBUE}} \arrow[r, Rightarrow] & \boxed{\text{k-HNBUE}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \boxed{\text{DFRA}} \arrow[r, Rightarrow] & \boxed{\text{NWU}} \arrow[dd, Rightarrow] & & \\
\boxed{\text{DFR}} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & \\
& \boxed{\text{IMRL.}} \arrow[r, Rightarrow] & \boxed{\text{NWUE}} \arrow[r, Rightarrow] & \boxed{\text{k-HNWUE}} \arrow[r, Rightarrow] & \boxed{\text{HNWUE}}
\end{tikzcd}$$

The next theorem shows that the binary random variable has 2-

HNBUE and hence k-HNBUE for $k \geqq 2$.

THEOREM 3. *Let X be a binary random variable with survival distribution function $\bar{F}$, then $\bar{F}$ is 2-HNBUE.*

PROOF. We assume that X can take two values a and b with probabilities p and $1-p$ respectively. Therefore,

$$\bar{F}(x)=\begin{cases} 1 & x<a \\ 1-p & a \leqq x<b \\ 0 & x \geqq b\,. \end{cases}$$

We have to show that $\int_0^t \left(\bar{F}(x)\Big/\int_x^\infty \bar{F}(y)dy\right)^2 dx \geqq t/(b+p(a-b))^2$ for all $t>0$. First assume that $t<a$, then

$$\int_0^t \left(\frac{\bar{F}(x)}{\int_x^\infty \bar{F}(y)dy}\right)^2 dx=\int_0^t \left(\frac{1}{(a-x)+(b-a)(1-p)}\right)^2 dx \tag{2.2}$$

$$=\frac{1}{a+(b-a)(1-p)-t}-\frac{1}{a+(b-a)(1-p)}\,.$$

It is clear that for $t<a$, (2.2) is greater than $t/(b+p(a-b))^2$. Next assume that $a<t<b$, then

$$\int_0^t \frac{(\bar{F}(x))^2}{\left(\int_x^\infty \bar{F}(y)dy\right)^2}dx=\int_0^a \left(\frac{\bar{F}(x)}{\int_x^\infty \bar{F}(y)dy}\right)^2 dx+\int_a^t \left(\frac{\bar{F}(x)}{\int_x^\infty \bar{F}(y)dy}\right)^2 dx$$

$$\geqq \frac{t}{(b+p(a-b))^2} \qquad \text{for } a<t<b\,.$$

Finally, assume that $t>b$, then $\int_0^t \left(\bar{F}(x)\Big/\int_x^\infty \bar{F}(y)dy\right)^2 dx \geqq t/(b+p(a-b))^2$.

The following example shows that 2-HNBUE $\not\Rightarrow$ HNBUE.

Example 1. Let X be a random variable with survival distribution

$$\bar{F}(x)=\begin{cases} 1 & 0 \leqq x<1 \\ 1/4 & 1 \leqq x<5 \\ 0 & x \geqq 5\,. \end{cases}$$

Then, by Theorem 3, since any binary random variable is 2-HNBUE, $\bar{F}(x)$ is 2-HNBUE and hence by Theorem 1, k-HNBUE, $k \geqq 2$. But $\bar{F}(x)$ is not HNBUE.

The next example shows that HNWUE $\not\Rightarrow$ 2-HNWUE.

Example 2. Let X be a random variable with survival distribution

$$\bar{F}(x)=\begin{cases} .55 & \text{for } 0\leqq x<1 \\ e^{-x} & \text{for } x\geqq 1 . \end{cases}$$

Then, it is easily seen that $\bar{F}(x)$ is HNWUE but it is not 2-HNWUE.

Remark 2. It is clear that as k increases the class of k-HNBUE is getting larger. Therefore HNBUE is the smallest class among k-HNBUE class of life distributions. Conversely by increasing k the class of k-HNWUE is getting smaller, i.e., HNWUE class is the largest class among k-HNWUE class of life distributions.

3. k-HNBUE and k-HNWUE closure properties

It is known in most cases whether the distribution classes IFR, IFRA, NBU, NBUE, DMRL and HNBUE (and their duals) are closed or not under the reliability operations: (1) formation of coherent structure, (2) convolution and (3) mixtures. For references, see Haines [4], Barlow and Proschan [2], and Klefsjö [6].

In this section, we prove certain closure properties of k-HNBUE and k-HNWUE.

If $\{F_\alpha\}$ is a family of life distributions, where α is a random variable with distribution function $G(\alpha)$, the mixture F of F_α according to G is defined by

$$F(t)=\int F_\alpha(t)dG(\alpha) . \tag{3.1}$$

The following example shows that the mixture of k-HNBUE distributions F_α is not necessarily k-HNBUE.

Example 3. Suppose that every F_α is one parameter exponential with different means and therefore DFR. A mixture of distributions, all of which are DFR, is itself DFR (see Theorem 4.7 in Barlow and Proschan [2]). Therefore F given by (3.1) is DFR and also k-HNWUE.

Remark 2. It is clear that, if every F_α has the same mean then the mixture F is k-HNBUE (as well as k-HNWUE).

On the other hand, the mixture F is k-HNWUE if every F_α is k-HNWUE. To prove this first we have to prove the following lemma.

LEMMA 1. *For a given k*

$$(3.2)\qquad t\Big/\int_0^t \left(\int \bar{F}_\alpha(x)dG(\alpha)\Big/\int \left(\int_x^\infty \bar{F}_\alpha(y)dy\right)dG(\alpha)\right)^k dx$$
$$\geqq \left(\int \left(t\Big/\int_0^t \mu_{F_\alpha}^{-k}(x)dx\right)^{1/k} dG(\alpha)\right)^k .$$

PROOF. We use induction to prove the lemma. Let $k=1$, then

$$\begin{aligned}
&t\Big/\int_0^t \left(\int \bar{F}_\alpha(x)dG(\alpha)\Big/\int \left(\int_x^\infty \bar{F}_\alpha(y)dy\right)dG(\alpha)\right)dx \\
&=t\Big/\int_0^t -\left(d/dx \log \int \left(\int_x^\infty \bar{F}_\alpha(y)dy\right)dG(\alpha)\right)dx \\
&=t\Big/\log \int \mu_\alpha dG(\alpha)-\log \int \left(\int_t^\infty \bar{F}_\alpha(y)dy\right)dG(\alpha) \\
&\geqq \int \left(t\Big/\left(\log \mu_\alpha-\log \int_t^\infty \bar{F}_\alpha(y)dy\right)\right)dG(\alpha) \\
&=\int \left(t\Big/\int_0^t \mu_F^{-1}(x)dx\right)dG(\alpha) .
\end{aligned}$$

We should mention that the inequality comes from the fact that $\log y$ is concave function. Now, assume that the inequality (3.2) holds for k then we have to show that it is also true for $k+1$.

$$\begin{aligned}
&t\Big/\int_0^t \left(\int \bar{F}_\alpha(x)dG(\alpha)\Big/\int \left(\int_x^\infty \bar{F}_\alpha(y)dy\right)dG(\alpha)\right)^{k+1} dx \\
&\geqq t\Big/\int_0^t \left(\int \bar{F}_\alpha(x)\Big/\int \int_x^\infty \bar{F}_\alpha(y)dy\right)^k dx \\
&\geqq \left(\int \left(t\Big/\int_0^t \mu_{F_\alpha}^{-k}(x)dx\right)^{1/k} dG(\alpha)\right)^k \\
&\geqq \left(\int \left(t\Big/\int_0^t \mu_{F_\alpha}^{-k-1}(x)dx\right)^{1/k+1} dG(\alpha)\right)^{k+1} .
\end{aligned}$$

To get the last inequality we use the Jensen inequality and the fact that $y \log y$ is a convex function.

THEOREM 4. *If every F_α is k-HNWUE, then the mixture F is also k-HNWUE.*

PROOF. To prove this we have to show that

$$t\Big/\int_0^t (\mu_F^{-k}(x))dx \geqq \mu^k \qquad \text{for all } t\geqq 0$$

if

$$t\Big/\int_t^\infty \mu_{F_\alpha}^{-1}(x)dx \geqq \mu_\alpha^k \qquad \text{for all } t\geqq 0$$

where

$$\mu_\alpha=\int_0^\infty \bar{F}_\alpha(x)dx \quad \text{and} \quad \mu=\int_0^\infty \bar{F}(x)dx=\int \mu_\alpha dG(\alpha) .$$

From Lemma 1 we have

$$t\Big/\int_0^t \mu_{\bar{F}}^k(x)dx=t\Big/\int_0^t \left(\int \bar{F}_\alpha(x)dG(\alpha)\Big/\int \left(\int \bar{F}_\alpha(y)dy\right)dG(\alpha)\right)^k$$
$$\geq\left(\int \left(t\Big/\int_0^t \mu_{\bar{F}_\alpha}^{-k}(x)dx\right)^{1/k} dG(\alpha)\right)^k \geq \left(\int \mu_\alpha dG(\alpha)\right)^k=\mu^k .$$

The following examples show that the k-HNWUE and k-HNBUE are not closed under formation of coherent structures.

Example 4. Consider a parallel system with two independent components whose life distributions are exponential with different means. Then the survival function of the system is IFRA and therefore k-HNBUE and not k-HNWUE.

Example 5. Consider a system with two independent components. Let X_1 and X_2 be the lifetimes of the components with the same survival distribution function

$$\bar{F}(x)=\begin{cases} 1 , & 0\leqq x<1 \\ .6 , & 1\leqq x<\sqrt{5} \\ \exp(-x/4) , & x\geqq\sqrt{5} \end{cases}$$

and let $X=[\mathrm{Min}\,(X_1^2, X_2^2)]$ be the lifetime of the system. Then $\bar{F}$ is 2-HNBUE, but the distribution function of X is not 2-HNBUE. Here X represents the lifetime of a coherent system.

The following example shows that k-HNWUE is not closed under convolution.

Example 6. If X_1 and X_2 are both exponentially distributed with expectation 1 then X_1+X_2 has a gamma distribution with the density $f(t)=te^{-t}$, $t\geqq 0$. This distribution is IFR and hence k-HNBUE and not k-HNWUE.

We have not been able to prove whether k-HNBUE is closed under convolution or not. But we know that if $k=1$, then the class of life distributions is closed under convolution.

4. Bounds on the survival function and moments of k-HNBUE (k-HNWUE) life distribution

In this section we give some bounds on the moments and survival function of a k-HNBUE (k-HNWUE) life distribution.

We start with the following theorem.

THEOREM 6. *Let F be a life distribution and let $\mu_r=\int_0^\infty x^r dF(x)$. If F is k-HNBUE then*

$$\left\{\begin{array}{ll} \mu_r \leqq \mu_1^{rk+1-k}\Gamma(r+1)\,, & r\geqq 1 \\ \mu_r \geqq \mu_1^{rk+1-k}\Gamma(r+1)\,, & 0<r<1\,. \end{array}\right. \tag{4.1}$$

PROOF. *The case $r\geqq 1$.* Since $h(x)=x^r$ is increasing and convex for $r\geqq 1$

$$\begin{aligned}\mu_r&=\int_0^\infty x^r dF(x)\leqq\int_0^\infty x^r \mu_1^{1-2k}\exp\left(-\frac{x}{\mu_1^k}\right)dx\\ &=\int_0^\infty t^r \mu_1^{rk-k+1}e^{-t}dt=\mu_1^{rk-k+1}\Gamma(r+1)\,.\end{aligned}$$

The first inequality comes from (2.1) and the fact that

$$\int_0^t\left(\frac{\bar F(x)}{\int_x^\infty \bar F(y)dy}\right)^k dx\geqq\frac{t}{\mu_1^k}\Longrightarrow\int_t^\infty \bar F(x)dx\leqq\mu_1\exp\left(-\frac{t}{\mu_1^k}\right)\,.$$

The case $0<r<1$. Since

$$\mu_r=r\int_0^\infty x^{r-1}\bar F(x)dx \quad\text{and}\quad r\int_0^\infty x^{r-1}\exp\left(\frac{-x}{\mu_1^k}\right)dx=r\mu_1^{rk}\Gamma(r)$$

$=\Gamma(r+1)\mu_1^{rk}$, it is sufficient to prove that

$$\int_0^\infty x^{r-1}\bar F(x)dx\geqq\mu_1^{1-k}\int_0^\infty x^{r-1}\exp\left(-\frac{x}{\mu_1^k}\right)dx\,.$$

But if F is k-HNBUE then

$$\Psi(t)=\int_0^t\left\{\bar F(x)-\mu_1^{1-k}\exp\left(-\frac{x}{\mu_1^k}\right)\right\}dx\geqq 0 \qquad\text{for every } t\geqq 0\,.$$

Thus $\int_0^\infty I_t(x)d\Psi(x)\geqq 0$, where $I_t(x)$ is the characteristic function of the set $[0, t)$. Since $g(x)=x^{r-1}$, $0<r<1$, is decreasing and therefore can be approximated from below by functions of the form

$$\sum_{j=1}^k c_j I_{t_j}(x) \qquad\text{where } c_j\geqq 0\,.$$

The result follows from the Lebesque monotone convergence theorem.

THEOREM 7. *Suppose F is a life distribution which is k-HNBUE with mean μ. Then*

$$\bar{F}(t) \leqq \begin{cases} 1 & \text{for } t \leqq \mu^k \\ \dfrac{1}{\mu^{k-1}} \exp\left(\dfrac{\mu^k - t}{\mu^k}\right), & t > \mu^k. \end{cases} \tag{4.2}$$

PROOF. Let $t>0$. By using the definition of k-HNBUE we get that

$$\int_s^t \bar{F}(x)dx \leqq \int_x^\infty \bar{F}(x)dx \leqq \mu \exp\left(-\frac{s}{\mu^k}\right) \qquad \text{for every } 0<s<t.$$

But

$$\int_s^t \bar{F}(x)dx \geqq (t-s)\bar{F}(t).$$

Hence we obtain that

$$\bar{F}(t) \leqq \operatorname*{Inf}_{0<s<t} \frac{\mu \exp(-s/\mu^k)}{t-s} = \begin{cases} 1 & \text{for } t \leqq \mu^k \\ \dfrac{1}{\mu^{k-1}} \exp\left(\dfrac{\mu^k - t}{\mu^k}\right) & \text{for } t > \mu^k. \end{cases}$$

Remark 3. In the HNBUE case Klefsjö [6] gave the upper bound

$$\bar{F}(t) = \begin{cases} 1 & \text{for } t \leqq \mu \\ \exp\left(\dfrac{\mu - t}{\mu}\right) & \text{for } t > \mu. \end{cases}$$

THEOREM 8. *Suppose F is a life distribution which is k-HNBUE with mean μ. Then*

$$\bar{F}(t) \geqq \begin{cases} \dfrac{\exp(-\alpha/\mu^k)}{\alpha - t}\{\alpha - t + \mu^k - \mu\} & \text{for } t < \mu^k \\ 0 & t \geqq \mu^k \end{cases} \tag{4.3}$$

where $\alpha=\alpha(t)$ is the largest non-negative number for which

$$\frac{1}{\mu^{k-1}}(\alpha - t + \mu^k) \exp\left(-\frac{\alpha}{\mu^k}\right) - \mu + t = 0. \tag{4.4}$$

PROOF. Let $t \geqq 0$. Then

$$\int_0^s \bar{F}(x)dx \leqq t + \bar{F}(t)(s-t) \qquad \text{for } s>t.$$

Furthermore,

$$\int_0^s \bar{F}(x)dx \geqq \mu\left(1 - \exp\left(-\frac{s}{\mu^k}\right)\right).$$

Accordingly

$$\bar{F}(t) \geqq \left\{\mu\left(1-\exp\left(-\frac{s}{\mu^k}\right)\right)-t\right\} \Big/ (s-t) \qquad \text{for every } s>t$$

i.e.,

$$\bar{F}(t) \geqq \sup_{s>t} \left\{\mu\left(1-\exp\left(-\frac{s}{\mu^k}\right)\right)-t\right\} \Big/ (s-t) .$$

Standard calculus then gives that for $t<\mu^k$ the supremum is attained for $s=\alpha$ given by (4.4).

Now suppose that in addition to μ we also know $\bar{F}(t^*)$ for some $t^*>0$. Then we can improve the bounds in Theorems 7 and 8.

THEOREM 9. *Suppose that F is a life distribution which is k-HNBUE with mean μ. Let*

$$C=\begin{cases} \dfrac{(\mu^k-\mu^k \log \mu^{k-1}\bar{F}(t^*)-t^*)\bar{F}(t^*)}{t-t^*} & \textit{if } \bar{F}(t^*) \geqq \dfrac{1}{\mu^{k-1}} \exp\left\{-\dfrac{t^*}{\mu^k}\right\} \\[2ex] \dfrac{\mu \exp(-t^*/\mu^k)}{t-t^*} & \textit{if } \bar{F}(t^*) < \dfrac{1}{\mu^{k-1}} \exp\left\{-\dfrac{t^*}{\mu^k}\right\} \end{cases}$$

and

$$h(t)=(C-t^*)\bar{F}(t^*)/(t-t^*) .$$

If $t^*>\mu^k$ then

$$\bar{F}(t) \leqq \begin{cases} 1 & \textit{for } 0 \leqq t \leqq \dfrac{\mu^{k-1}(\mu-t^*\bar{F}(t^*))}{1-\mu^{k-1}\bar{F}(t^*)} \\[2ex] \dfrac{\mu \exp(-\gamma t/\mu^k)-(t^*-t)F(t^*)}{(1-\gamma)t} & \textit{for } \dfrac{\mu^{k-1}(\mu-t^*\bar{F}(t^*))}{1-\mu^{k-1}\bar{F}(t^*)} < t < t^* \\[2ex] \bar{F}(t^*) & \textit{for } t^* < t < C \\ h(t) & \textit{for } t \geqq C \end{cases}$$

where $\gamma=\gamma(t)$ satisfies the following condition

$$\frac{1}{\mu^{k-1}}(\mu^k-t+\gamma t) \exp\left(\frac{-\gamma t}{\mu^k}\right)-(t^*-t)\bar{F}(t^*)=0 . \tag{4.5}$$

If $t^*<\mu^k$ then

$$\bar{F}(t) \leqq \begin{cases} 1 & \textit{for } 0 \leqq t < t^* \\ \bar{F}(t^*) & \textit{for } t^* < t < C \\ h(t) & \textit{for } t \geqq C . \end{cases}$$

PROOF. Follow the same argument as in Klefsjö [6] Theorem 6.6 and the fact that F is k-HNBUE implies $\int_t^\infty F(x)dx \leqq \mu \exp\{-t/\mu^k\}$.

The next theorem gives a lower bound on $F(t)$ based on the knowledge of $\bar{F}(t^*)$ for some $t^*>0$.

THEOREM 10. *Suppose that F is a life distribution which is k-HNBUE with mean μ. Let $\alpha=\alpha(t)$ be given by (4.4) and let*

$$D=\min(E, t^*)$$

where

$$E=\text{the } t \text{ for which } \bar{F}(t^*)=\frac{\exp(-\alpha/\mu^k)}{\alpha-t}\{\alpha-t+\mu^k-\mu\}\ .$$

If $t^*<\mu^k$ then

$$\bar{F}(t)\geqq\begin{cases}\dfrac{\exp(-\alpha/\mu^k)}{\alpha-t}\{\alpha-t+\mu^k-\mu\} & \text{for } 0<t<D\\ \bar{F}(t^*) & \text{for } D\leqq t\leqq t^*\\ \dfrac{\exp(-\gamma/\mu^k)}{\gamma-t}\{\gamma-t+\mu^k-\mu\} & \text{for } t^*<t<t^*+(\mu-t^*)/\bar{F}(t^*)\\ 0 & t\geqq t^*+(\mu-t^*)/F(t^*)\end{cases}$$

where $\gamma=\gamma(t)$ is the positive solution to

$$\frac{1}{\mu^{k-1}}(\exp(-\gamma/\mu^k))(\gamma-t)-\mu+t^*+(t-t^*)\bar{F}(t^*)\ .$$

If $t^*>\mu^k$

$$\bar{F}(t)\geqq\begin{cases}\dfrac{\exp(-\alpha/\mu^k)}{\alpha-t}\{\alpha-t+\mu^k-\mu\} & \text{for } 0\leqq t<E\\ \bar{F}(t^*) & \text{for } E\leqq t<t^*\\ 0 & \text{for } t>t^*\ .\end{cases}$$

PROOF. Following the same way as in Klefsjö [6], we will get the result.

The following theorems give bounds on the moments and survival function of k-HNWUE.

THEOREM 11. *Let F be a life distribution and let $\mu_r=\int_0^\infty x^r dF(x)$*

and $\lambda_r=\mu_r/\Gamma(r+1)$ *for* $r>0$. *If* F *is k-HNWUE then*

$$\lambda_{r+1/k+1}\geqq\lambda_1\lambda_r\ .$$

PROOF. Since F is k-HNWUE, therefore

$$t^{1/k}\int_t^\infty \bar{F}(y)dy\geqq\mu_1 F(t)$$

$$\int_0^\infty \frac{t^{r+1/k-1}}{\Gamma(r+1/k)}\left(\int_t^\infty \bar{F}(y)dy\right)\geqq\int_0^\infty \frac{\mu_1 t^{r-1}}{\Gamma(r+1)}\bar{F}(t)dt=\lambda_1\lambda_r\ .$$

But the left-hand side is equal to $\lambda_{r+1/k+1}$, therefore $\lambda_{r+1/k+1}\geqq\lambda_1\lambda_r$.

THEOREM 12. *Suppose that* F *is a life distribution which is k-HNWUE with mean* μ. *Then*

$$\bar{F}(t)\leqq\frac{\mu t^{1/k}}{t^{(1/k)+1}+\mu}\ .$$

PROOF. Since F is k-HNWUE, therefore

$$t^{1/k}\left(\mu-\int_0^t \bar{F}(y)dy\right)\geqq\mu\bar{F}(t)$$

$$\Longrightarrow t^{1/k}\int_0^t \bar{F}(y)dy\leqq\mu(t^{1/k}-\bar{F}(t))$$

$$\Longrightarrow t^{1/k}(tF(t))\leqq\mu(t^{1/k}-F(t))$$

$$\Longrightarrow \bar{F}(t)\leqq\frac{\mu t^{1/k}}{t^{(1/k)+1}+\mu}\qquad \text{for } t\geqq 0\ .$$

Now suppose that besides μ we also know $\bar{F}(t^*)$ for some $t^*>0$. The following theorem gives an upper bound in terms of $\bar{F}(t^*)$.

THEOREM 13. *Suppose that* F *is a life distribution which is k-HNWUE with mean* μ. *Then*

$$\bar{F}(t)\leqq\begin{cases}\mu t^{1/k}/\mu+t^{(1/k)+1} & 0\leqq t<t^*\\ \bar{F}(t^*) & t^*<t<C\\ \dfrac{t^{1/k}(\mu-t^*\bar{F}(t^*))}{(\mu+(t-t^*)t^{1/k})} & t>C\end{cases}$$

where $C=\sup\{t\colon\ \mu t^{1/k}-t^{(1/k)+1}\bar{F}(t^*)>\mu\bar{F}(t^*)\}$.

PROOF. Let $t^*<t$, then

$$t^{1/k}(t^*\bar{F}(t^*)+(t-t^*)\bar{F}(t))\leqq t^{1/k}\int_0^t \bar{F}(x)dx\leqq\mu(t^{1/k}-\bar{F}(t)) \tag{4.6}$$

$$\Longrightarrow \bar{F}(t)\leqq\frac{t^{1/k}(\mu-t^*\bar{F}(t^*))}{(\mu+(t-t^*)t^{1/k})}\ .$$

Since for $t<C$, $t^{1/k}(\mu-t^*\bar{F}(t^*))/(\mu+(t-t^*)t^{1/k})>\bar{F}(t^*)$, therefore the result follows. We should mention that the first inequality in (4.6) follows from the fact that, for any life distribution F with

$$\mu_r=r\int_0^\infty x^{r-1}\bar{F}(x)dx<\infty \qquad \text{for some } r\geqq 1\,,$$

$$t_1^r\bar{F}(t_1)+(t_2^r-t_1^r)\bar{F}(t_2)\leqq r\int_0^{t_2} x^{r-1}\bar{F}(x)dx\leqq t_1^r+(t_2^r-t_1^r)\bar{F}(t_1) \qquad \text{for } 0\leqq t_1\leqq t_2\,.$$

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CORRECTIONS TO

"ON K-ORDER HARMONIC NEW BETTER THAN USED IN EXPECTATION DISTRIBUTIONS"

A. P. BASU AND NADER EBRAHIMI

(Annals of the Institute of Statistical Mathematics, A, (1984), Vol. 36, pp. 87-100)

1. Theorem 3 is not true in general. A counter example by Klefsjö [1] is as follows:
   Let

$$F(x)=\begin{cases} 1\,, & x\leqq a \\ 1-p\,, & a<x\leqq b \\ 0\,, & x\geqq b\,. \end{cases}$$

   Then with $a=1$ and $b=6$, straight forward calculation shows that $\bar{F}$ is 2-HNBUE if $1-p<(3-\sqrt{5})/10$ or if $1-p>(3+\sqrt{5})/10$.

   Proceeding along the same line one obtains conditions on $1-p$ such that $\bar{F}$ is 2-HNBUE for arbitrary a and b.

2. Page 94, line 5. K-HNBUE should be 2-HNBUE. Using the example in paragraph 1 above one can also show that K-HNBUE distributions are not closed under formation of coherent structure for $k=3$. The result for general k is not known.

3. In theorems 6–10 we have made the tacit assumption that

$$\int_0^t \left(\bar{F}(x)\Big/\int_x^\infty \bar{F}(y)dy\right)^k dx \leqq \int_0^t \left(\bar{F}(x)\Big/\int_x^\infty \bar{F}(y)dy\right) dx\,.$$

   An example where this is true is exponential distribution with mean >1.

4. Theorems 11, 12 and 13 are not correct.

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[1] Klefsjö, Bengt (1984). Comments and corrections to a paper on k-order harmonic new better than used in expectation distributions by A. P. Basu and N. Ebrahimi, *Research Report* 1984-3, Department of Mathematics, Luleå, Sweden.